Journal of Mathematical Inequalities
Volume 11, Number 1 (2017), 27–39 doi:10.7153/jmi-11-03

# INEQUALITIES FOR GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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(*Communicated by S. Koumandos*)

*Abstract.* In this paper, we first shall present some inequalities for Gaussian hypergeometric functions, which generalize an identity involving the inverse hyperbolic tangent function. Further, the monotonicity of general hypergeometric function is proved. The obtained results of this paper improve some known results.

## 1. Introduction

For real numbers $a,b$ and $c$ such that $c \neq 0,-1,-2,\cdots$, the Gaussian hypergeometric functions $F(a,b;c;x):(-1,1)\to\mathbb{R}$ are defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n\geqslant 0}\frac{(a,n)(b,n)}{(c,n)}\cdot\frac{x^n}{n!}, \tag{1.1}$$

where $(a,0)=1$ for $a\neq 0$, and $(a,n)=a(a+1)(a+2)\cdots(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)}$ for each $n\in\{1,2,\cdots\}$ denotes the Pochammer (or Appell) symbol. The Gaussian hypergeometric functions $F(a,b;c;x)$ admit the Euler integral representation [1, 11], as follows:

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{c-1}(1-t)^{c-b-1}(1-xt)^{-a}dt,$$

where $\Gamma(z)$ is Gamma function.

The Gaussian hypergeometric function has attracted the interest of authors and perhaps been widely used in [12–20]. Such as, some well-known class of mathematical physics are particular or limiting cases of it [8–10].

It is clear that for many rational values $(a,b,c)$ of the Gaussian hypergeometric function $F(a,b;c;x)$ reduces to many well-known special elementary functions. The readers can see reference [1], such as,

$$F\left(\frac{1}{2},1;\frac{3}{2};r\right) = \frac{1}{2r}\operatorname{arctanh} r. \tag{1.2}$$

*Mathematics subject classification* (2010): 39B62, 33C05.
*Keywords and phrases*: Gaussian hypergeometric function, power series, monotonicity, inequality.
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This work is supported by the Doctoral Programs Foundation of Education Ministry of China (20113401110009); Universities Natural Science Foundation of Anhui Province (KJ2016A310).

It is not difficult to find that for $r \in (0,1)$, the inverse hyperbolic tangent function $\operatorname{arctanh} x$ satisfies the following identity

$$2\operatorname{arctanh}\sqrt{r} = \operatorname{arctanh}\left(\frac{2\sqrt{r}}{1+r}\right). \tag{1.3}$$

Combing (1.2) and (1.3), we get

$$F\left(\frac{1}{2},1;\frac{3}{2};\frac{4r}{(1+r)^2}\right) = (1+r)F\left(\frac{1}{2},1;\frac{3}{2};r\right), \tag{1.4}$$

$$F\left(\frac{1}{2},1;\frac{3}{2};\frac{1-r}{1+r}\right) = \frac{1+r}{2}F\left(\frac{1}{2},1;\frac{3}{2};1-r^2\right). \tag{1.5}$$

For the zero-balance hypergeometric function $F(a,b;a+b;x)$, Qiu and Vuorinen in [3, Thm 1.2] and Simić and Vuorinen in [4, Thm 2.1] obtained some meaningful results, respectively.

Afterwards, Baricz [5] extended some important results of the zero-balance hypergeometric function $F(a,b;a+b;x)$ to the general hypergeometric function $F(a,b;c;x)$.

Recently, Wang, Song and Chu [6] studied (1.10) and (1.11), and presented several inequalities for the zero-balance hypergeometric function $F(a,b;a+b;x)$, as follows.

THEOREM A. ([6]) *Let $a,b,c \in \mathbb{R}$ such that $c \neq 0,-1,-2,\cdots$, and for all $r \in (0,1)$.*

*If $(a,b) \in \{(a,b)|a,b>0, ab \leqslant \frac{1}{2}, 3ab-(a+b) \leqslant 0\}$, then*

$$F\left(a,b;a+b;\frac{4r}{(1+r)^2}\right) \leqslant (1+r)F(a,b;a+b;r), \tag{1.6}$$

$$\frac{1+r}{2}F(a,b;a+b;1-r^2) \leqslant F\left(a,b;a+b;\frac{1-r}{1+r}\right). \tag{1.7}$$

*If $(a,b) \in \{(a,b)|a,b>0, ab \geqslant \frac{1}{2}, 3ab-(a+b) \geqslant 0\}$, then*

$$F\left(a,b;a+b;\frac{4r}{(1+r)^2}\right) \geqslant (1+r)F(a,b;a+b;r), \tag{1.8}$$

$$F\left(a,b;a+b;\frac{1-r}{1+r}\right) \leqslant \frac{1+r}{2}F(a,b;a+b;1-r^2). \tag{1.9}$$

In this paper, motivated by [3, 4, 5, 6], we make a contribution to this subject. The results can be extended the zero-balance hypergeometric functions $F(a,b;a+b;x)$ to the general Gaussian hypergeometric functions $F(a,b;c;x)$. In a word, we first shall present some inequalities for Gaussian hypergeometric functions, which generalize an identity involving the inverse hyperbolic tangent function. Further, the monotonicity of general hypergeometric function is proved. The obtained results of this paper improve some known results.

## 2. Main results

Throughout this paper, for $a,b,c \in \mathbb{R}$ and $c \neq 0,-1,-2,\cdots$ we set

$$D_1 = \left\{(a,b,c)|a+b \leqslant c, ab \leqslant \min\left\{\frac{1}{2}-(a+b), \frac{c+1}{5}\right\}\right\};$$
$$D_2 = \left\{(a,b,c)|a+b \geqslant c, ab \geqslant \max\left\{\frac{1}{2}, \frac{c+1}{5}\right\}\right\};$$
$$D_3 = \left\{(a,b,c)|a+b \leqslant c, ab \leqslant \min\left\{\frac{1}{2}, \frac{c}{3}\right\}\right\};$$
$$D_4 = \left\{(a,b,c)|a+b \geqslant c, ab \geqslant \max\left\{\frac{1}{2}, \frac{c}{3}\right\}\right\};$$
$$D_5 = \left\{(a,b,c)|a,b,c>0, c \leqslant \min\{a,b\}, ab \geqslant \max\left\{\frac{1}{2}, \frac{c}{3}, \frac{c+1}{5}\right\}\right\};$$
$$D_6 = \left\{(a,b,c)|a,b,c>0, c \geqslant a+b, ab \leqslant \min\left\{\frac{1}{2}-(a+b), \frac{c}{3}, \frac{c+1}{5}\right\}\right\},$$

where

$$D_5 = (a,b,c) \in D_2 \cap D_4 \cap \{(a,b,c)|a,b,c>0, c \leqslant \min\{a,b\}\};$$
$$D_6 = D_1 \cap D_3 \cap \{(a,b,c)|a,b,c>0, c = a+b\}.$$

Our first main result is stated as follows.

THEOREM 2.1. *Let $a,b,c \in \mathbb{R}$ such that $c \neq 0,-1,-2,\cdots$, and for all $r \in (0,1)$. The following assertions are valid.*

(**1**). *If $a,b,c \in D_4$, then*

$$F\left(a,b;c;\frac{4r}{(1+r)^2}\right) \geqslant (1+r)F(a,b;c;r), \tag{2.1}$$
$$F\left(a,b;c;\frac{1-r}{1+r}\right) \leqslant \frac{1+r}{2}F(a,b;c;1-r). \tag{2.2}$$

(**2**). *If $a,b,c \in D_3$, then*

$$F\left(a,b;c;\frac{4r}{(1+r)^2}\right) \leqslant (1+r)F(a,b;c;r), \tag{2.3}$$
$$F\left(a,b;c;\frac{1-r}{1+r}\right) \geqslant \frac{1+r}{2}F(a,b;c;1-r). \tag{2.4}$$

Now, we consider the following Gaussian hypergeometric transformation [1]

$$F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; a-b+1; \frac{4r}{(1+r)^2}\right) = (1+r)^a F\left(a, ab; a-b+1; r\right), \tag{2.5}$$

which may be regarded as the generalization of the identity (1.3). By applying this transformation we can get the following result.

THEOREM 2.2. *Let $a,b>0$, $c\in\mathbb{R}$ with $c\neq 0,-1,-2,\cdots$. The following assertions are valid.*

(**1**). *If $\frac{1}{2}a+\frac{1}{2}\leqslant b\leqslant \min\{a+1,a+1-c\}$, then*

$$F\left(a,b;c;\frac{4r}{(1+r)^2}\right)\geqslant (1+r)^a F(a,b;c;r). \tag{2.6}$$

(**2**). *If $a-c+1\leqslant b\leqslant \min\{\frac{a}{2},a+1\}$ and $a\leqslant 1$, then*

$$F\left(a,b;c;\frac{4r}{(1+r)^2}\right)\leqslant (1+r)^a F(a,b;c;r). \tag{2.7}$$

Noting that by changing $r$ to $\frac{1-r}{1+r}$ in inequalities (2.6) and (2.7), we obtain the following inequalities.

COROLLARY 2.3. *Let $a,b>0$, $c\in\mathbb{R}$ with $c\neq 0,-1,-2,\cdots$. The following assertions are valid.*

(**1**). *If $\frac{1}{2}a+\frac{1}{2}\leqslant b\leqslant \min\{a+1,a+1-c\}$, then*

$$F\left(a,b;c;\frac{1-r}{1+r}\right)\leqslant \left(\frac{1+r}{2}\right)^a F(a,b;c;r). \tag{2.8}$$

(**2**). *If $a-c+1\leqslant b\leqslant \min\{\frac{a}{2},a+1\}$ and $a\leqslant 1$, then*

$$F\left(a,b;c;\frac{1-r}{1+r}\right)\geqslant \left(\frac{1+r}{2}\right)^a F(a,b;c;r). \tag{2.9}$$

THEOREM 2.4. *For all $r\in(0,1)$, the function*

$$J(r)=(1+r)F(a,b;c;r)-F\left(a,b;c;\frac{4r}{(1+r)^2}\right)$$

*is monotone decreasing on $D_5$ and monotone increasing on $D_6$.*

## 3. Some lemmas

We first introduce some lemmas, which play an important role in the proof of main results.

Let us recall the following assertion of Biernacki and Krzyż [7], which we shall use in the sequel.

LEMMA 3.1. *Let $a_n \in \mathbb{R}$ and $b_n > 0$, and for all $n \in \{0,1,\cdots\}$, the power series $f(x) = \sum_{n\geqslant 0} a_n x^n$ and $g(x) = \sum_{n\geqslant 0} b_n x^n$, and both converge on $(-r,r)$, $r > 0$. Suppose that*

$$h(x) = \frac{f(x)}{g(x)} = \frac{\sum_{n\geqslant 0} a_n x^n}{\sum_{n\geqslant 0} b_n x^n}.$$

*If the sequence $\{a_n/b_n\}_{n\geqslant 0}$ is increasing (decreasing), then the function $h(x)$ is increasing (decreasing) too on $(0,r)$.*

For different proofs and various applications of Lemma 2.1, the readers are referred to the references [4, 5, 13].

LEMMA 3.2. ([11]) *For the Gaussian hypergeometric function $F(a,b;c;x)$, we have*

$$F(a,b;c;x) = F(b,a;c;x), \tag{3.1}$$

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x), \tag{3.2}$$

$$(1-x)\frac{d}{dx}F(a,b;c;x) = \frac{(c-a)(c-b)}{c}F(a,b;c+1;x) - (c-a)(c-b)F(a,b;c;x). \tag{3.3}$$

We denote

$$F(r) = F(a,b;c;r), \quad G(r) = F(a,b;c+1;r), \tag{3.4}$$

and

$$\widetilde{F}(r) = F\left(\frac{1}{2},1;\frac{3}{2};r\right) \qquad \widetilde{G}(r) = F\left(\frac{1}{2},1;\frac{5}{2};r\right), \tag{3.5}$$

where $a,b > 0$, $(a,b) \neq (\frac{1}{2},1)$ and $(a,b) \neq (1,\frac{1}{2})$.

LEMMA 3.3. *Let $r \in (0,1)$.*
($\mathbf{1}$). *The function $g(r) = \frac{G(r)}{\widetilde{G}(r)}$ is decreasing on $D_1$ and increasing on $D_2$.*
($\mathbf{2}$). *The function $f(r) = \frac{F(r)}{\widetilde{F}(r)}$ is decreasing on $D_3$ and increasing on $D_4$.*

*Proof.* ($\mathbf{1}$). Since

$$g(r) = \frac{G(r)}{\widetilde{G}(r)} = \frac{F(a,b;c+1;r)}{F\left(\frac{1}{2},1;\frac{5}{2};r\right)} = \frac{\sum_{n\geqslant 0}\frac{(a,n)(b,n)}{(c+1,n)}\cdot\frac{r^n}{n!}}{\sum_{n\geqslant 0}\frac{(\frac{1}{2},n)(1,n)}{(\frac{5}{2},n)}\cdot\frac{r^n}{n!}}.$$

Therefore, by Lemma 3.1, in order to prove the monotonicity of $g(r)$, we only need to obtain the monotonicity of the sequence

$$u_n = \frac{(a,n)(b,n)}{(c+1,n)}\cdot\frac{(\frac{5}{2},n)}{(\frac{1}{2},n)(1,n)}.$$

By a simple calculation, one has

$$\frac{u_{n+1}}{u_n}=\frac{(a+n)(b+n)}{(c+n+1)}\frac{(\frac{5}{2}+n)}{(\frac{1}{2}+n)(1+n)}\geqslant 1(\leqslant 1)$$

if and only if

$$\Delta_n^1=(a+b-c)n^2+\left[(a+b)+\frac{3}{2}(a+b-c)+ab-\frac{1}{2}\right]n+\frac{5}{2}ab-\frac{1}{2}c-\frac{1}{2}\geqslant 0(\leqslant 0).$$

Hence, if $(a,b,c)\in D_1$, then $\Delta_n^1\leqslant 0$ for all $n\in\{0,1,\cdots\}$, that is $\{u_n\}_{n\geqslant 0}$ is decreasing. So, by Lemma 3.1 the function $g(r)$ is decreasing. On the other hand, if $(a,b,c)\in D_2$, then $\Delta_n^1\geqslant 0$ for all $n\in\{0,1,\cdots\}$, that is $\{u_n\}_{n\geqslant 0}$ is increasing. So, by Lemma 3.1 the function $g(r)$ is increasing.

(**2**). Since

$$f(r)=\frac{F(r)}{\widetilde{F}(r)}=\frac{F(a,b;c;r)}{F\left(\frac{1}{2},1;\frac{3}{2};r\right)}=\frac{\sum_{n\geqslant 0}\frac{(a,n)(b,n)}{(c,n)}\cdot\frac{r^n}{n!}}{\sum_{n\geqslant 0}\frac{(\frac{1}{2},n)(1,n)}{(\frac{3}{2},n)}\cdot\frac{r^n}{n!}}.$$

Therefore, by Lemma 3.1, the monotonicity of $f(r)$ depends on the monotonicity of the sequence

$$\{\alpha_n\}_{n\geqslant 0}=\left\{\frac{(a,n)(b,n)}{(c,n)}\cdot\frac{(\frac{3}{2},n)}{(\frac{1}{2},n)(1,n)}\right\}_{n\geqslant 0}.$$

A simple calculation yields that

$$\frac{\alpha_{n+1}}{\alpha_n}=\frac{(a+n)(b+n)(\frac{3}{2}+n)}{(c+n)(\frac{1}{2}+n)(1+n)}\geqslant 1(\leqslant 1),$$

if and only if

$$\Delta_n'=(a+b-c)n^2+\left[\frac{3}{2}(a+b-c)+ab-\frac{1}{2}\right]n+\frac{3}{2}ab-\frac{1}{2}c\geqslant 0(\leqslant 0).$$

Thus, if $a+b\geqslant c$ and $ab\geqslant\max\{\frac{1}{2},\frac{c}{3}\}$, then $\Delta_n'\geqslant 0$ for all $n\in\{0,1,\cdots\}$, that is $\{\alpha_n\}_{n\geqslant 0}$ is increasing. So, by Lemma 3.1 the function $f(r)$ is increasing. On the other hand, if $a+b\leqslant c$ and $ab\leqslant\min\{\frac{1}{2},\frac{c}{3}\}$, then $\Delta_n'\leqslant 0$ for all $n\in\{0,1,\cdots\}$, that is $\{\alpha_n\}_{n\geqslant 0}$ is decreasing. So, by Lemma 3.1 the function $f(r)$ is decreasing. □

## 4. Proofs of theorems

*Proof of Theorem 2.1.* By Lemma 3.3 (2), for $(a,b,c)\in D_4$ and $r\in(0,1)$, the function $f(r)$ is increasing. Hence, for each $0<x<y<1$, we have $f(x)<f(y)$. Now choosing $x=x(r)=r$ and $y=y(r)=\frac{4r}{(1+r)^2}$, we get that

$$\frac{F(a,b;c;r)}{F(\frac{1}{2},1;\frac{3}{2};r)}\leqslant\frac{F(a,b;c;\frac{4r}{(1+r)^2})}{F(\frac{1}{2},1;\frac{3}{2};\frac{4r}{(1+r)^2})},$$

i.e.

$$F(a,b;c;r) \leqslant F\left(a,b;c;\frac{4r}{(1+r)^2}\right)\frac{F(\frac{1}{2},1;\frac{3}{2};r)}{F(\frac{1}{2},1;\frac{3}{2};\frac{4r}{(1+r)^2})}. \tag{4.1}$$

From (1.3) and (4.1), a simple substitution yields (2.1).

Similarly, if $(a,b,c)\in D_3$ and $r\in(0,1)$, the function $f(r)$ is decreasing. Hence

$$\frac{F(a,b;c;r)}{F(\frac{1}{2},1;\frac{3}{2};r)} \geqslant \frac{F(a,b;c;\frac{4r}{(1+r)^2})}{F(\frac{1}{2},1;\frac{3}{2};\frac{4r}{(1+r)^2})},$$

i.e.

$$F(a,b;c;r) \geqslant F\left(a,b;c;\frac{4r}{(1+r)^2}\right)\frac{F(\frac{1}{2},1;\frac{3}{2};r)}{F(\frac{1}{2},1;\frac{3}{2};\frac{4r}{(1+r)^2})}. \tag{4.2}$$

From (1.3) and (4.2), a simple substitution yields (2.3).

The proof for (2.2) and (2.4)is similar. We need only choose $x=x(r)=\frac{1-r}{1+r}$ and $y=y(r)=1-r^2$, and thus we omit the details. So the proof is complete. $\square$

*Proof of Theorem 2.2.* The proof is similar as in the proof of Theorem 2.1. Firstly, we consider the function $V:(0,1)\to(0,+\infty)$, defined by

$$V(x)=\frac{F(a,b;c;x)}{F(a,b;a-b+1;x)}=\frac{\sum_{n\geqslant 0}\frac{(a,n)(b,n)}{(c,n)}\cdot\frac{x^n}{n!}}{\sum_{n\geqslant 0}\frac{(a,n)(b,n)}{(a-b+1,n)}\cdot\frac{x^n}{n!}}.$$

In according to Lemma 3.1, to discuss the monotonicity of $V(x)$ we need to study the monotonicity of the sequence $\{\beta_n\}_{n\geqslant 0}$, defined by $\beta_n=\frac{(a-b+1,n)}{(c,n)}$. Since

$$\frac{\beta_{n+1}}{\beta_n}=\frac{a-b+1+n}{c+n}.$$

Obviously, for $a-b+1\geqslant c$ the sequence $\{\beta_n\}_{n\geqslant 0}$ is increasing; for $a-b+1\leqslant c$ the sequence $\{\beta_n\}_{n\geqslant 0}$ is decreasing.

(**1**). We now consider the case $a-b+1\geqslant c$ and $a-b+1>0$, then the sequence $\{\beta_n\}_{n\geqslant 0}$ is increasing, and using Lemma 3.1 the function $V(x)$ is increasing. That is, for each $0<x<y<1$, we have $V(x)<V(y)$. Now choosing $x=x(r)=r$ and $y=y(r)=\frac{4r}{(1+r)^2}$, we get that

$$\frac{F(a,b;c;r)}{F(a,b;a-b+1;r)} \leqslant \frac{F(a,b;c;\frac{4r}{(1+r)^2})}{F(a,b;a-b+1;\frac{4r}{(1+r)^2})}. \tag{4.3}$$

By (2.5) and (4.3), we have

$$
\begin{aligned}
F(a,b;c;r) &\leqslant F\Big(a,b;c;\frac{4r}{(1+r)^2}\Big)\frac{F(a,b;a-b+1;r)}{F(a,b;a-b+1;\frac{4r}{(1+r)^2})}\\
&= F\Big(a,b;c;\frac{4r}{(1+r)^2}\Big)\frac{1}{(1+r)^a}\frac{F\left(\frac12 a,\frac12 a+\frac12;a-b+1;\frac{4r}{(1+r)^2}\right)}{F(a,b;a-b+1;\frac{4r}{(1+r)^2})}. \qquad (4.4)
\end{aligned}
$$

Observing that if $b\geqslant\frac{a+1}{2}$, then $(\frac12 a+\frac12,n)\geqslant(b,n)$ for all $n\in\{0,1,2,\cdots\}$. Thus, we have

$$
\frac{(\frac12 a,n)(\frac12 a+\frac12,n)}{(a-b+1,n)\cdot n!}\leqslant\frac{(a,n)(b,n)}{(a-b+1,n)\cdot n!},
$$

and hence for all $r\in(0,1)$ one has

$$
F\left(\frac12 a,\frac12 a+\frac12;a-b+1;\frac{4r}{(1+r)^2}\right)\leqslant F(a,b;a-b+1;\frac{4r}{(1+r)^2}). \qquad (4.5)
$$

From (4.4) and (4.5), a simple substitution yields (2.6).

(**2**). Secondly, we prove (2.7) is valid. If $a-b+1\leqslant c$, then the sequence $\{\beta_n\}_{n\geqslant 0}$ is decreasing, and applying Lemma 3.1 the function $V(x)$ is decreasing. That is, for each $0<x<y<1$, we have $V(x)>V(y)$. Now choosing $x=x(r)=r$ and $y=y(r)=\frac{4r}{(1+r)^2}$, we get that

$$
\frac{F(a,b;c;r)}{F(a,b;a-b+1;r)}\geqslant\frac{F(a,b;c;\frac{4r}{(1+r)^2})}{F(a,b;a-b+1;\frac{4r}{(1+r)^2})}. \qquad (4.6)
$$

By (2.5) and (4.6), and further using (3.1), we have

$$
\begin{aligned}
F(a,b;c;r) &\geqslant F\Big(a,b;c;\frac{4r}{(1+r)^2}\Big)\frac{F(a,b;a-b+1;r)}{F(a,b;a-b+1;\frac{4r}{(1+r)^2})}\\
&= F\Big(a,b;c;\frac{4r}{(1+r)^2}\Big)\frac{1}{(1+r)^a}\frac{F\left(\frac12 a,\frac12 a+\frac12;a-b+1;\frac{4r}{(1+r)^2}\right)}{F(a,b;a-b+1;\frac{4r}{(1+r)^2})}\\
&= F\Big(a,b;c;\frac{4r}{(1+r)^2}\Big)\frac{1}{(1+r)^a}\frac{F\left(\frac12 a+\frac12,\frac12 a;a-b+1;\frac{4r}{(1+r)^2}\right)}{F(a,b;a-b+1;\frac{4r}{(1+r)^2})}. \qquad (4.7)
\end{aligned}
$$

Observing that if $2b\leqslant a<1$, then $(\frac12 a+\frac12,n)\leqslant(a,n)$ and $(\frac12 a,n)\leqslant(b,n)$ for all $n\in\{0,1,2,\cdots\}$. Thus, we have

$$
\frac{(\frac12 a,n)(\frac12 a+\frac12,n)}{(a-b+1,n)\cdot n!}\geqslant\frac{(a,n)(b,n)}{(a-b+1,n)\cdot n!},
$$

and hence for all $r \in (0,1)$ one has

$$F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; a-b+1; \frac{4r}{(1+r)^2}\right) \geqslant F\left(a,b;a-b+1;\frac{4r}{(1+r)^2}\right). \tag{4.8}$$

From (4.7) and (4.8), it yields (2.7). So the proof is complete. □

*Proof of Theorem 2.4.* Suppose $z = \frac{4r}{(1+r)^2}$. Then

$$1-z = \left(\frac{1-r}{1+r}\right); \quad \frac{dz}{dr} = \frac{4(1-r)}{(1+r)^3}.$$

Hence

$$\begin{aligned}
(1-r)J'(r) &= (1-r)F(a,b;c;r) + (1-r^2)\frac{dF(a,b;c;r)}{dr} - \frac{4(1-r)^2}{(1+r)^3}\frac{dF(a,b;c;z)}{dz} \\
&= (1-r)F(a,b;c;r) + (1-r^2)\frac{dF(a,b;c;r)}{dr} - \frac{4(1-z)}{1+r}\frac{dF(a,b;c;z)}{dz} \\
&= (1-r)F(a,b;c;r) \\
&\quad + (1+r)\left[\frac{(c-a)(c-b)}{c}F(a,b;c+1;r) - (c-a-b)F(a,b;c;r)\right] \\
&\quad - \frac{4}{1+r}\left[\frac{(c-a)(c-b)}{c}F(a,b;c+1;z) - (c-a-b)F(a,b;c;z)\right] \\
&= (1-r)F(r) + (1+r)\left[\frac{(c-a)(c-b)}{c}G(r) - (c-a-b)F(r)\right] \\
&\quad - \frac{4}{1+r}\left[\frac{(c-a)(c-b)}{c}G(z) - (c-a-b)F(z)\right].
\end{aligned} \tag{4.9}$$

On the other hand, differentiating for (1.3) with respect to $r$, we get

$$\frac{4}{3}\cdot\frac{\widetilde{G}(z)}{1+r} = (1-r)\widetilde{F}(r) + \frac{1+r}{3}\widetilde{G}(r). \tag{4.10}$$

Firstly, we prove $J(r)$ is monotone decreasing on $D$.

By Lemma 3.3 (1), for $(a,b,c) \in D_2$ and $0 < r < z < 1$, then $g(r) < g(z)$, and

$$G(z) \geqslant \frac{G(r)}{\widetilde{G}(r)}\widetilde{G}(z).$$

This, together with (4.9) for $c \leqslant \min\{a,b\}$ and using (4.10), it yields that

$$\begin{aligned}
(1-r)J'(r) &< (1-r)F(r)+(1+r)\left[\frac{(c-a)(c-b)}{c}G(r)-(c-a-b)F(r)\right]\\
&\quad -\frac{4}{1+r}\left[\frac{(c-a)(c-b)}{c}\frac{G(r)}{\widetilde{G}(r)}\widetilde{G}(z)-(c-a-b)F(z)\right]\\
&= (1-r)F(r)+(1+r)\left[\frac{(c-a)(c-b)}{c}G(r)-(c-a-b)F(r)\right]\\
&\quad -\frac{3(c-a)(c-b)}{c}\left[(1-r)\frac{\widetilde{F}(r)}{\widetilde{G}(r)}G(r)+\frac{1+r}{3}G(r)\right]+\frac{4}{1+r}(c-a-b)F(z)\\
&= (1-r)\left[F(r)-\frac{3(c-a)(c-b)}{c}\frac{\widetilde{F}(r)}{\widetilde{G}(r)}G(r)\right]\\
&\quad +\left[\frac{4}{1+r}F(z)-(1-r)F(r)\right](c-a-b). \qquad (4.11)
\end{aligned}$$

Since

$$\frac{F'(r)}{\widetilde{F}'(r)}=\frac{\frac{(c-a)(c-b)}{c}G(r)-(c-a-b)F(r)}{\frac{1}{3}\widetilde{G}(r)}=\frac{3(c-a)(c-b)}{c}\frac{G(r)}{\widetilde{G}(r)}-3(c-a-b)\frac{F(r)}{\widetilde{G}(r)}.$$

Then

$$\frac{3(c-a)(c-b)}{c}\frac{G(r)}{\widetilde{G}(r)}=\frac{F'(r)}{\widetilde{F}'(r)}+3(c-a-b)\frac{F(r)}{\widetilde{G}(r)}. \qquad (4.12)$$

By (4.11) and (4.12), it yields that

$$\begin{aligned}
J'(r) &< \left[F(r)-\frac{F'(r)}{\widetilde{F}'(r)}\widetilde{F}(r)\right]-3(c-a-b)\frac{F(r)}{\widetilde{G}(r)}+\left[\frac{4}{1+r}F(z)-(1+r)F(r)\right]\frac{(c-a-b)}{1-r}\\
&= \left[F(r)-\frac{F'(r)}{\widetilde{F}'(r)}\widetilde{F}(r)\right]+\frac{(c-a-b)}{1-r}\left[\frac{4}{1+r}F(z)-(1+r)F(r)-3(1-r)\frac{F(r)}{\widetilde{G}(r)}\right]. \qquad (4.13)
\end{aligned}$$

Since

$$\widetilde{G}(r)>\lim_{r\to 0^+}\widetilde{G}(r)=1, \qquad (4.14)$$

and $c \leqslant a+b$, then

$$\begin{aligned}
J'(r) &< \left[F(r)-\frac{F'(r)}{\widetilde{F}'(r)}\widetilde{F}(r)\right]+\frac{(c-a-b)}{1-r}\left[\frac{4}{1+r}F(z)-(1+r)F(r)-3(1-r)F(r)\right]\\
&= \left[F(r)-\frac{F'(r)}{\widetilde{F}'(r)}\widetilde{F}(r)\right]+\frac{(c-a-b)}{1-r}\left[\frac{4}{1+r}F(z)-(4-2r)F(r)\right]\\
&= \frac{(F(r))^2}{\widetilde{F}'(r)}\left(\frac{\widetilde{F}(r)}{F(r)}\right)'+\frac{(c-a-b)}{1-r}\left[\frac{4}{1+r}F(z)-(4-2r)F(r)\right]. \qquad (4.15)
\end{aligned}$$

On the other hand, by $(a,b,c)\in D_4\cap\{(a,b,c)|a,b,c>0\}$, and using Lemma 3.3 (2) and (3.1), then $\left(\frac{\widetilde{F}(r)}{F(r)}\right)'\leqslant 0$, and

$$\frac{(c-a-b)}{1-r}\left[\frac{4}{1+r}F(z)-(4-2r)F(r)\right]\leqslant\frac{(c-a-b)}{1-r}2rF(r)\leqslant 0.$$

Therefore, we have $J'(r)<0$.

In summary, for $(a,b,c)\in D_2\cap D_4\cap\{(a,b,c)|a,b,c>0\}$ and $c\leqslant\min\{a,b\}$, we have $J'(r)<0$.

Secondly, we show that $J(r)$ is monotone increasing on $D$.

By Lemma 3.3 (1), for $(a,b,c)\in D_1$ and $0<r<z<1$, then $g(r)>g(z)$, and

$$G(z)\leqslant\frac{G(r)}{\widetilde{G}(r)}\widetilde{G}(z).$$

This, together with (4.9) and using (4.10), it yields that

$$\begin{aligned}
(1-r)J'(r)>{}&(1-r)F(r)+(1+r)\left[\frac{(c-a)(c-b)}{c}G(r)-(c-a-b)F(r)\right]\\
&-\frac{4}{1+r}\left[\frac{(c-a)(c-b)}{c}\frac{G(r)}{\widetilde{G}(r)}\widetilde{G}(z)-(c-a-b)F(z)\right]\\
={}&(1-r)F(r)+(1+r)\left[\frac{(c-a)(c-b)}{c}G(r)-(c-a-b)F(r)\right]\\
&-\frac{3(c-a)(c-b)}{c}\left[(1-r)\frac{\widetilde{F}(r)}{\widetilde{G}(r)}G(r)+\frac{1+r}{3}G(r)\right]+\frac{4}{1+r}(c-a-b)F(z)\\
={}&(1-r)\left[F(r)-\frac{3(c-a)(c-b)}{c}\frac{\widetilde{F}(r)}{\widetilde{G}(r)}G(r)\right]\\
&+\left[\frac{4}{1+r}F(z)-(1-r)F(r)\right](c-a-b).
\end{aligned}\tag{4.16}$$

By (4.12), (4.16) and (4.14), and $c\geqslant a+b$, it yields that

$$\begin{aligned}
J'(r)>{}&\left[F(r)-\frac{F'(r)}{\widetilde{F}'(r)}\widetilde{F}(r)\right]-3(c-a-b)\frac{F(r)}{\widetilde{G}(r)}+\left[\frac{4}{1+r}F(z)-(1+r)F(r)\right]\frac{(c-a-b)}{1-r}\\
={}&\left[F(r)-\frac{F'(r)}{\widetilde{F}'(r)}\widetilde{F}(r)\right]+\frac{(c-a-b)}{1-r}\left[\frac{4}{1+r}F(z)-(1+r)F(r)-3(1-r)\frac{F(r)}{\widetilde{G}(r)}\right]
\end{aligned}$$

$$> \left[F(r) - \frac{F'(r)}{\widetilde{F}'(r)}\widetilde{F}(r)\right] + \frac{(c-a-b)}{1-r}\left[\frac{4}{1+r}F(z) - (1+r)F(r) - 3(1-r)F(r)\right] \tag{4.17}$$

$$= \frac{(F(r))^2}{\widetilde{F}'(r)}\left(\frac{\widetilde{F}(r)}{F(r)}\right)' + \frac{(c-a-b)}{1-r}\left[\frac{4}{1+r}F(z) - (4-2r)F(r)\right]. \tag{4.18}$$

Hence, for $(a,b,c) \in D_1 \cap D_3 \cap \{(a,b,c)|a,b,c > 0\}$ and and $c = a+b$, and using Lemma 3.3 (2), we have $J'(r) > 0$.

In conclusion, for $(a,b,c) \in D_1 \cap D_3 \cap \{(a,b,c)|a,b,c > 0\}$ and $c = a+b$, we have $J'(r) > 0$. $\square$

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(Received October 22, 2015)

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