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A NOTE ON A WIELANDT TYPE NORM INEQUALITY

XIAOHUI FU AND JUNJIAN YANG

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Abstract. As a continuation of recent study on a Wielandt type norm inequality due to Lin [13, Conjecture 3.4], we prove the following result: Let $A \in M_n(\mathbb{C})$ satisfying $0 < m \leqslant A \leqslant M$, and let X and Y be $n \times k$ matrices such that $X^*X = Y^*Y = I_k$ and $X^*Y = 0$. Then for every 2-positive unital linear map Φ, we have

$$\|(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX))^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}\| \leqslant \begin{cases} \left(\frac{M-m}{M+m}\right)^p \frac{(M^{\frac{p}{2}}+m^{\frac{p}{2}})^2}{4M^{\frac{p}{2}}m^{\frac{p}{2}}} & 1 < p < 2 \\ \frac{(M-m)^p}{4M^{\frac{p}{2}}m^{\frac{p}{2}}} & p \geqslant 2. \end{cases}$$

1. Introduction

Let M, m be scalars. $M_n(\mathbb{C})$ denotes the set of all $n \times n$ complex matrices. $A^* \in M_n(\mathbb{C})$ stands for the adjoint of A. For a Hermitian matrix $A \in M_n(\mathbb{C})$, we use the notation $A \geqslant 0$ to mean that A is positive semidefinite, and $A > 0$ to mean it is positive definite. A linear map $\Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C})$ is called (strictly) positive if $\Phi(A) \geqslant 0$ $(\Phi(A) > 0)$ whenever $A \geqslant 0$ $(A > 0)$. It is said to be unital if $\Phi(I_n) = I_k$. We say that Φ is 2-positive if whenever the 2×2 matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive, then so is $\begin{bmatrix} \Phi(A) & \Phi(B) \\ \Phi(B^*) & \Phi(C) \end{bmatrix}$. We use $\|\cdot\|$ for operator norm.

The Wielandt inequality [10, p. 443] is as follows: if $0 < mI \leqslant A \leqslant MI$, and $x, y \in \mathscr{H}$ with $x \perp y$, then

$$|\langle x, Ay\rangle| \leqslant \left(\frac{M-m}{M+m}\right)^2 \langle x, Ay\rangle\langle y, Ay\rangle. \tag{1.1}$$

Wielandt's inequality plays an important role in different contexts. For example, it has a variety of applications in numerical methods, especially eigenvalue estimation [6]. It is also applied in multivariate analysis [2, 5, 7, 10]. For the latest study on the Wielandt and generalized Wielandt inequality, readers are referred to [12].

The operator version of (1.1) was proved by Bhatia and Davis [3] (independently by Wang and Ip [16]) as follows: Let $0 < m \leqslant A \leqslant M$, and let X, Y be two partial

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isometries on a Hilbert space $\mathscr{H}$ whose final spaces are orthogonal to each other. Then for every 2-positive linear map Φ,

$$\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \leqslant \left(\frac{M-m}{M+m}\right)^2 \Phi(X^*AX). \tag{1.2}$$

Under the same condition, Lin [13, Conjecture 3.4] conjectured the following assertion could be true:

$$\|\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\Phi(X^*AX)^{-1}\| \leqslant \left(\frac{M-m}{M+m}\right)^2. \tag{1.3}$$

Recently, the authors [6] obtained the following result in the finite-dimensional case: Let $A \in M_n(\mathbb{C})$ satisfying $0 < m \leqslant A \leqslant M$, and let X and Y be $n \times k$ matrices such that $X^*X = Y^*Y = I_k$ and $X^*Y = 0$. Then for every 2-positive unital linear map Φ, we have

$$\begin{aligned} &\left\|\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\Phi(X^*AX)^{-1}\right\| \\ &\leqslant \frac{1}{4}\left(\left(\frac{M-m}{M+m}\right)^2 M + \frac{1}{m}\right)^2, \end{aligned} \tag{1.4}$$

which was a step closer to the conjecture (1.3).

In this note, we obtain the following result in the finite-dimensional case: Let $A \in M_n$ with $0 < mI_n \leqslant A \leqslant MI_n$, and let X and Y be $n \times k$ matrices such that $X^*X = Y^*Y = I_k$ (i.e. isometries) and $X^*Y = 0$. Then for every 2-positive unital linear map Φ, we have

$$\begin{aligned} &\|(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX))^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}\| \\ &\leqslant \begin{cases} \left(\frac{M-m}{M+m}\right)^p \frac{(M^{\frac{p}{2}}+m^{\frac{p}{2}})^2}{4M^{\frac{p}{2}}m^{\frac{p}{2}}} & 1<p<2 \\ \frac{(M-m)^p}{4M^{\frac{p}{2}}m^{\frac{p}{2}}} & p \geqslant 2, \end{cases} \end{aligned}$$

which is tighter than (1.4).

2. Main result

We need two lemmas which play a very important role in the proof of the main theorem of this paper. The first Lemma is Ando-Zhan's celebrated result.

LEMMA 1. [1] *Let A and B be positive operators. Then for $1 \leqslant r < \infty$*

$$\|A^r + B^r\| \leqslant \|(A+B)^r\|. \tag{2.1}$$

The next lemma holds for positive definite matrices but a careful observation shows that it is true for positive definite operators on a Hilbert space.

LEMMA 2. [4] *Let $A, B > 0$. Then the following norm inequality holds:*

$$\|AB\| \leqslant \frac{1}{4}\|A+B\|^2. \tag{2.2}$$

Now we are devoted to presenting the main result which is a refinement of (1.4) in the finite-dimensional case.

THEOREM 3. *Let $A \in M_n$ with $0 < mI_n \leqslant A, B \leqslant MI_n$ and let X and Y be $n \times k$ matrices such that $X^*X = Y^*Y = I_k$ and $X^*Y = 0$. Then for every 2-positive unital linear map Φ,*

$$\|(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX))^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}\| \leqslant \begin{cases} \left(\frac{M-m}{M+m}\right)^p \frac{(M^{\frac{p}{2}}+m^{\frac{p}{2}})^2}{4M^{\frac{p}{2}}m^{\frac{p}{2}}} & 1 < p < 2 \\ \frac{(M-m)^p}{4M^{\frac{p}{2}}m^{\frac{p}{2}}} & p \geqslant 2. \end{cases} \tag{2.3}$$

Proof. Firstly, consider the case of $p \geqslant 2$. Compute

$$\begin{aligned}
&\left\|(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX))^{\frac{p}{2}}\left(\left(\frac{M-m}{M+m}\right)^2 Mm\Phi(X^*AX)^{-1}\right)^{\frac{p}{2}}\right\| \\
&\leqslant \frac{1}{4}\left\|(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX))^{\frac{p}{2}} + \left(\left(\frac{M-m}{M+m}\right)^2 Mm\Phi(X^*AX)^{-1}\right)^{\frac{p}{2}}\right\|^2 \\
&\qquad\qquad\text{(by (2.2))} \\
&\leqslant \frac{1}{4}\left\|\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) + \left(\frac{M-m}{M+m}\right)^2 Mm\Phi(X^*AX)^{-1}\right\|^p \quad \text{(by (2.1))} \\
&\leqslant \frac{1}{4}\left\|\left(\frac{M-m}{M+m}\right)^2 \Phi(X^*AX) + \left(\frac{M-m}{M+m}\right)^2 Mm\Phi(X^*AX)^{-1}\right\|^p \quad \text{(by (1.2))} \\
&= \frac{1}{4}\left(\frac{M-m}{M+m}\right)^{2p} \|\Phi(X^*AX) + Mm\Phi(X^*AX)^{-1}\| \\
&\leqslant \frac{1}{4}\frac{(M-m)^{2p}}{(M+m)^p}.
\end{aligned}$$

The last inequality above is obtained: Since $0 < mI_n \leqslant A \leqslant MI_n$, $mI_k \leqslant \Phi(X^*AX) \leqslant MI_k$ and $\frac{1}{M} \leqslant \Phi(X^*AX)^{-1} \leqslant \frac{1}{m}$, we have

$$(M - \Phi(X^*AX))(m - \Phi(X^*AX))\Phi(X^*AX)^{-1} \leqslant 0,$$

which implies

$$Mm\Phi(X^*AX)^{-1} + \Phi(X^*AX) \leqslant M+m.$$

So

$$\|(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX))^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}\| \leqslant \frac{(M-m)^p}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}.$$

Next consider the case of $1 < p < 2$. Compute

$$\begin{aligned}
&\left\|(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX))^{\frac{p}{2}}\left(\left(\frac{M-m}{M+m}\right)^2 Mm\Phi(X^*AX)^{-1}\right)^{\frac{p}{2}}\right\| \\
&\leqslant \frac{1}{4}\left\|(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX))^{\frac{p}{2}} + \left(\frac{M-m}{M+m}\right)^p M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}\right\|^2 \\
&\qquad \text{(by (2.2))} \\
&\leqslant \frac{1}{4}\left\|\left(\frac{M-m}{M+m}\right)^p \Phi(X^*AX)^{\frac{p}{2}} + \left(\frac{M-m}{M+m}\right)^p M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}\right\|^2 \\
&\qquad \text{(by Löwner-Heinz inequality and (1.2))} \\
&= \frac{1}{4}\left\|\left(\frac{M-m}{M+m}\right)^p \left(\Phi(X^*AX)^{\frac{p}{2}} + M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}\right)\right\|^2 \\
&= \frac{1}{4}\left(\frac{M-m}{M+m}\right)^{2p}(M^{\frac{p}{2}}+m^{\frac{p}{2}})^2.
\end{aligned}$$

The last inequality above holds as follows: By using $0 < mI_n \leqslant A \leqslant MI_n$, $m^{\frac{p}{2}} \leqslant \Phi(X^*AX)^{\frac{p}{2}} \leqslant M^{\frac{p}{2}}$ and $M^{-\frac{p}{2}} \leqslant \Phi(X^*AX)^{-\frac{p}{2}} \leqslant m^{-\frac{p}{2}}$, we have

$$(M^{\frac{p}{2}} - \Phi(X^*AX)^{\frac{p}{2}})(m^{\frac{p}{2}} - \Phi(X^*AX)^{\frac{p}{2}})\Phi(X^*AX)^{-\frac{p}{2}} \leqslant 0,$$

which means

$$M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}} + \Phi(X^*AX)^{\frac{p}{2}} \leqslant M^{\frac{p}{2}} + m^{\frac{p}{2}}.$$

That is,

$$\begin{aligned}
&\|(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX))^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}\| \\
&\leqslant \left(\frac{M-m}{M+m}\right)^p \frac{(M^{\frac{p}{2}}+m^{\frac{p}{2}})^2}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}. \quad \square
\end{aligned}$$

REMARK 4. If $p = 2$, the right side of the inequality (2.3) is $\frac{(M-m)^2}{4Mm}$. Obviously, the below inequality holds

$$\frac{(M-m)^2}{4Mm} \leqslant \frac{M}{m}\left(\frac{M-m}{M+m}\right)^2 \leqslant \frac{1}{4}\left(\left(\frac{M-m}{M+m}\right)^2 + \frac{1}{m}\right)^2,$$

which shows that the bound of (2.3) is smaller than that of (1.4). Thus, (2.3) is a refinement of (1.4) for $p = 2$.

REMARK 5. When $p = 2$, the author [9, (2.7)] obtained a stronger result than the inequality (2.3). However, if we present p $(p > 2)$ power of (2.7) in [9] through the similar method of the proof of Theorem 3, we will find that the result for $p > 2$ is very complicated and not continuous at $p = 2$.

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Xiaohui Fu
School of Mathematics and Statistics
Hainan Normal University
Haikou, 571158, P. R. China
e-mail: fxh6662@sina.com

Junjian Yang
School of Mathematics and Statistics
Hainan Normal University
Haikou, 571158, P. R. China