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A RECIPROCAL SUM RELATED TO THE RIEMANN ζ–FUNCTION

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Abstract. This paper, we using the elementary method and several new inequalities to study the computational problem of one kind reciprocal sums related to the Riemann zeta-function at the point $s = 4$, and give an explicit computational formula for it.

1. Introduction

Let complex number $s = \sigma + it$, if $\sigma > 1$, then the famous Riemann zeta-function $\zeta(s)$ is defined as the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and by analytic continuation to the whole complex plane except for a simple pole at $s = 1$ with residue 1.

About the various properties of $\zeta(s)$, many mathematicians had studied them, and obtained abundant research results, for example, see [1], [2] and [3]. Very recently, the first author [4] studied the computational problem of the reciprocal sum related to the Riemann ζ-function, and used the elementary method and some new inequalities to prove that for any positive integer n, one has the identities

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{k^2}\right)^{-1}\right] = n - 1$$

and

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{k^3}\right)^{-1}\right] = 2n(n-1),$$

where function $[x]$ denotes the greatest integer $\leqslant x$.

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Paper JMI-11-20

At the same time, Lin Xin [4] also proposed the following open problem: Whether there exists an explicit computational formula for

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^s}\right)^{-1}\right], \tag{1}$$

where s is an integer with $s \geqslant 4$.

This problem is interesting, because it is actually an effective approximation for the partial sum of the Riemann zeta-function $\zeta(s)$.

More contents related to other second-order linear recurrence sequences and polynomials had also been studied by many authors, see references [5]–[14]. For example, H. Ohtsuka and S. Nakamura [5] studied the properties of the sum of reciprocal Fibonacci numbers, and proved the identities

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{F_k}\right)^{-1}\right] = \begin{cases} F_{n-2}, & \text{if } n \geqslant 2 \text{ is an even number;} \\ F_{n-2}-1, & \text{if } n \geqslant 1 \text{ is an odd number,} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{F_k^2}\right)^{-1}\right] = \begin{cases} F_{n-1}F_n-1, & \text{if } n \geqslant 2 \text{ is an even number;} \\ F_nF_{n-1}, & \text{if } n \geqslant 1 \text{ an odd number.} \end{cases}$$

Zhang Wenpeng and Wang Tingting [8] considered the computational problem of Pell numbers, and proved the identity

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{P_k}\right)^{-1}\right] = \begin{cases} P_{n-1}+P_{n-2}, & \text{if } n \geqslant 2 \text{ is an even number;} \\ P_{n-1}+P_{n-2}-1, & \text{if } n \geqslant 1 \text{ is an odd number.} \end{cases}$$

where Pell numbers P_n are defined as $P_0 = 0$, $P_1 = 1$, and $P_{n+1} = 2P_n + P_{n-1}$ for all integers $n \geqslant 1$.

The main purpose of this paper is to study the computational problem of (1) for $s = 4$, and use the elementary method and some new inequalities to give an interesting computational formula for (1) with $s = 4$. That is, we shall prove the following conclusion:

THEOREM. *For any positive integer n, we have the identity*

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^4}\right)^{-1}\right] = \begin{cases} 24m^3 - 18m^2 + \left[\frac{3(5m-1)}{2}\right], & \text{if } n = 2m; \\ 24m^3 - 54m^2 + \left[\frac{3(58m-17)}{4}\right], & \text{if } n = 2m-1. \end{cases}$$

For general integers $s \geqslant 5$, whether there exists an explicit computational formula for (1) is an open problem. Perhaps using our method one can study it for the cases $s \geqslant 5$, but it is very complicated to construct those new inequalities when s is larger. With the aid of computer and mathematical software "Mathematica", I believe that we can solve this problem completely.

2. Several Lemmas

In this section, we shall give some lemmas which are necessary in the proof of our theorem. First we have the following:

LEMMA 1. *For any integer $n > 1$, we have the inequality*

$$\frac{1}{24n^3-18n^2+\frac{15}{2}n-\frac{9}{8}} - \frac{1}{24(n+1)^3-18(n+1)^2+\frac{15}{2}(n+1)-\frac{9}{8}} < \frac{1}{(2n)^4}+\frac{1}{(2n+1)^4}.$$

Proof. In fact, the inequality in Lemma 1 is equivalent to the inequality

$$\frac{1}{3\left(8n^3-6n^2+\frac{5}{2}n-\frac{3}{8}\right)} - \frac{1}{3\left(8n^3+18n^2+\frac{29}{2}n+\frac{33}{8}\right)} < \frac{32n^4+32n^3+24n^2+8n+1}{256n^8+512n^7+384n^6+128n^5+16n^4}$$

or

$$\frac{8n^2+4n+\frac{3}{2}}{\left(8n^3-6n^2+\frac{5}{2}n-\frac{3}{8}\right)\left(8n^3+18n^2+\frac{29}{2}n+\frac{33}{8}\right)} < \frac{32n^4+32n^3+24n^2+8n+1}{256n^8+512n^7+384n^6+128n^5+16n^4}. \quad (2)$$

Let

$$f(n) = \left(8n^2+4n+\frac{3}{2}\right)\left(256n^8+512n^7+384n^6+128n^5+16n^4\right),$$

and

$$g(n) = \left(8n^3-6n^2+\frac{5}{2}n-\frac{3}{8}\right)\left(8n^3+18n^2+\frac{29}{2}n+\frac{33}{8}\right)\left(32n^4+32n^3+24n^2+8n+1\right).$$

Then

$$f(n) = 2048n^{10}+5120n^9+5504n^8+3328n^7+1216n^6+256n^5+24n^4. \quad (3)$$

$$\begin{aligned} g(n) &= \left(64n^6+96n^5+28n^4-12n^3+\frac{19}{4}n^2+\frac{39}{8}n-\frac{99}{64}\right) \\ &\quad \times\left(32n^4+32n^3+24n^2+8n+1\right) \\ &= 2048n^{10}+5120n^9+5504n^8+3328n^7+1272n^6+340n^5+\frac{305}{2}n^4. \quad (4) \end{aligned}$$

From (3) and (4) we know that $f(n) < g(n)$. So inequality (2) is correct.

This proves Lemma 1. □

LEMMA 2. *For any integer $n \geqslant 1$, we also have the inequality*

$$\frac{1}{(2n)^4}+\frac{1}{(2n+1)^4} < \frac{1}{24n^3-18n^2+\frac{15}{2}n-\frac{3}{2}} - \frac{1}{24(n+1)^3-18(n+1)^2+\frac{15}{2}(n+1)-\frac{3}{2}}.$$

Proof. First we have the identity

$$\begin{aligned} &\frac{1}{24n^3-18n^2+\frac{15}{2}n-\frac{3}{2}} - \frac{1}{24(n+1)^3-18(n+1)^2+\frac{15}{2}(n+1)-\frac{3}{2}} \\ =&\frac{8n^2+4n+\frac{3}{2}}{\left(8n^3-6n^2+\frac{5}{2}n-\frac{1}{2}\right)\left(8n^3+18n^2+\frac{29}{2}n+4\right)} \\ =&\frac{8n^2+4n+\frac{3}{2}}{64n^6+96n^5+28n^4-14n^3+\frac{13}{4}n^2+\frac{11}{4}n-2}. \end{aligned} \tag{5}$$

Let $h(n)=\left(64n^6+96n^5+28n^4-14n^3+\frac{13}{4}n^2+\frac{11}{4}n-2\right)(32n^4+32n^3+24n^2+8n+1)$, then we have

$$\begin{aligned} h(n) =& 2048n^{10}+5120n^9+5504n^8+3264n^7+1160n^6+176n^5 \\ &+18n^4+14n^3-\frac{91}{4}n^2-\frac{53}{4}n-2. \end{aligned} \tag{6}$$

From (3) and (6) we know that $f(n)>h(n)$. Thus, from the definition of $f(n)$ we have the inequality

$$\frac{1}{(2n)^4}+\frac{1}{(2n+1)^4} < \frac{1}{24n^3-18n^2+\frac{15}{2}n-\frac{3}{2}} - \frac{1}{24(n+1)^3-18(n+1)^2+\frac{15}{2}(n+1)-\frac{3}{2}}.$$

This proves Lemma 2. □

LEMMA 3. *For any integer $n \geqslant 1$, we have the inequality*

$$\begin{aligned} &\frac{1}{3\left(8n^3-18n^2+\frac{29}{2}n-\frac{17}{4}\right)} - \frac{1}{3\left(8(n+1)^3-18(n+1)^2+\frac{29}{2}(n+1)-\frac{17}{4}\right)} \\ >&\frac{1}{(2n-1)^4}+\frac{1}{(2n)^4}. \end{aligned}$$

Proof. It is clear that

$$\frac{1}{(2n-1)^4}+\frac{1}{(2n)^4} = \frac{32n^4-32n^3+24n^2-8n+1}{16\left(16n^8-32n^7+24n^6-8n^5+n^4\right)}. \tag{7}$$

$$\begin{aligned}
&\frac{1}{3\left(8n^3-18n^2+\frac{29}{2}n-\frac{17}{4}\right)}-\frac{1}{3\left(8(n+1)^3-18(n+1)^2+\frac{29}{2}(n+1)-\frac{17}{4}\right)}\\
&=\frac{8n^2-4n+\frac{3}{2}}{\left(8n^3-18n^2+\frac{29}{2}n-\frac{17}{4}\right)\left(8n^3+6n^2+\frac{5}{2}n+\frac{1}{4}\right)}\\
&=\frac{8n^2-4n+\frac{3}{2}}{64n^6-96n^5+28n^4+10n^3+\frac{25}{4}n^2-7n-\frac{17}{16}}. \qquad (8)
\end{aligned}$$

For convenience, we let

$$\begin{aligned}
U(n)=&\left(64n^6-96n^5+28n^4+10n^3+\frac{25}{4}n^2-7n-\frac{17}{16}\right)\\
&\times\left(32n^4-32n^3+24n^2-8n+1\right),
\end{aligned}$$

$$V(n)=16\left(16n^8-32n^7+24n^6-8n^5+n^4\right)\left(8n^2-4n+\frac{3}{2}\right).$$

Then by computations, we have

$$V(n)-U(n)=64n^7-168n^6+248n^5-264n^4+174n^3-\frac{147}{4}n^2-\frac{3}{2}n-\frac{17}{16}.$$

It is easy to prove that $V(n)-U(n)>0$ for all integers $n\geqslant 1$. So combining (7) and (8) we may immediately deduce Lemma 3. □

LEMMA 4. *For any integer $n\geqslant 1$, we have the inequality*

$$\begin{aligned}
\frac{1}{(2n-1)^4}+\frac{1}{(2n)^4}>&\frac{1}{3\left(8n^3-18n^2+\frac{29}{2}n-\frac{25}{6}\right)}\\
&-\frac{1}{3\left(8(n+1)^3-18(n+1)^2+\frac{29}{2}(n+1)-\frac{25}{6}\right)}.
\end{aligned}$$

Proof. From the method of proving Lemma 3 we can easily deduce this inequality. □

3. Proof of the theorem

In this section, we shall complete the proof of our theorem. If $n=2m$ is an even number, then from Lemma 1 and Lemma 2 we have

$$\begin{aligned}
&\frac{1}{24m^3-18m^2+\frac{15}{2}m-\frac{9}{8}}\\
&=\sum_{k=m}^{\infty}\left(\frac{1}{24k^3-18k^2+\frac{15}{2}k-\frac{9}{8}}-\frac{1}{24(k+1)^3-18(k+1)^2+\frac{15}{2}(k+1)-\frac{9}{8}}\right)\\
&<\sum_{k=m}^{\infty}\left(\frac{1}{(2k)^4}+\frac{1}{(2k+1)^4}\right)=\sum_{k=2m}^{\infty}\frac{1}{k^4}. \qquad (9)
\end{aligned}$$

Similarly, from Lemma 2 we can deduce that

$$\begin{aligned}\sum_{k=2m}^{\infty}\frac{1}{k^4} &= \sum_{k=m}^{\infty}\left(\frac{1}{(2k)^4}+\frac{1}{(2k+1)^4}\right)\\ &< \sum_{k=m}^{\infty}\left(\frac{1}{24k^3-18k^2+\frac{15}{2}k-\frac{3}{2}}-\frac{1}{24(k+1)^3-18(k+1)^2+\frac{15}{2}(k+1)-\frac{3}{2}}\right)\\ &= \frac{1}{24m^3-18m^2+\frac{15}{2}m-\frac{3}{2}}. \end{aligned} \tag{10}$$

Combining (9) and (10) we may immediately deduce that

$$24m^3-18m^2+\frac{15}{2}m-\frac{3}{2} < \left(\sum_{k=2m}^{\infty}\frac{1}{k^4}\right)^{-1} < 24m^3-18m^2+\frac{15}{2}m-\frac{9}{8},$$

which implies (according to the parity of m):

$$\left[\left(\sum_{k=2m}^{\infty}\frac{1}{k^4}\right)^{-1}\right] = 24m^3-18m^2+\left[\frac{3(5m-1)}{2}\right]. \tag{11}$$

If $n=2m-1$ be an odd number, then from Lemma 3, Lemma 4 and the method of proving (9) and (10) we have

$$24m^3-54m^2+\frac{87}{2}m-\frac{51}{4} < \left(\sum_{k=2m-1}^{\infty}\frac{1}{k^4}\right)^{-1} < 24m^3-54m^2+\frac{87}{2}m-\frac{25}{2},$$

which implies (according to the parity of m):

$$\left[\left(\sum_{k=2m-1}^{\infty}\frac{1}{k^4}\right)^{-1}\right] = 24m^3-54m^2+\left[\frac{174m-51}{4}\right]. \tag{12}$$

Applying (11) and (12) we know that for any integer $n \geqslant 1$, we have the identity

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^4}\right)^{-1}\right] = \begin{cases} 24m^3-18m^2+\left[\frac{3(5m-1)}{2}\right], & \text{if } n=2m;\\ 24m^3-54m^2+\left[\frac{174m-51}{4}\right], & \text{if } n=2m-1. \end{cases}$$

This completes the proof of the theorem. □

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