Journal of Mathematical Inequalities
Volume 11, Number 2 (2017), 551–564 doi:10.7153/jmi-11-45

MAPPING PROPERTIES OF HARDY–TYPE OPERATORS INVOLVING GENERAL FUNCTIONS

PANKAJ JAIN AND BABITA GUPTA

Abstract. Weight characterizations are obtained for the boundedness and compactness of the operator

$$(Sf)(x) = u_1(x)\int_{a(x)}^{b(x)} v_1(t)f(t)dt + u_2(x)\int_{c(x)}^{d(x)} v_2(t)f(t)dt,$$

where $u_i, v_i, i = 1,2$, are certain general measurable functions (not necessarily non negative), between weighted Lebesgue spaces $L^p(I,w_0)$ and $L^q(I,w_1)$, where $1 < p,\ q < \infty$ and $I = (0,\infty)$.

1. Introduction

The problem of studying the boundedness and compactness of the Hardy operator

$$(Hf)(x) = \int_0^x f(t)dt$$

and its conjugate operator

$$(H^*f)(x) = \int_x^\infty f(t)dt$$

between Lebesgue spaces and even in more general setting is well settled now. A complete description of such study can be found in [6, 7, 8] and the references therein.

P. A. Zharov [11] (see also [6]) considered a more general operator

$$(Af)(x) = u_1(x)\int_0^x v_1(t)f(t)dt + u_2(x)\int_x^\infty v_2(t)f(t)dt, \tag{1.1}$$

and characterized its L^p-L^q boundedness, where $u_i, v_i, i = 1,2$, are certain general functions (not necessarily non negative). Through personal communication with some of the mathematicians, we have learnt that perhaps the L^p-L^q compactness of the operator A is also known but we have not been able to find the corresponding literature.

The aim of the present paper is to consider even more general operator than (1.1), i.e, we consider

$$(Sf)(x) = u_1(x)\int_{a(x)}^{b(x)} v_1(t)f(t)dt + u_2(x)\int_{c(x)}^{d(x)} v_2(t)f(t)dt := (S_1f)(x) + (S_2f)(x),$$

Mathematics subject classification (2010): 46E35, 26D10.
Keywords and phrases: Hardy-type operators, boundedness, compactness.

Paper JMI-11-45

where (and throughout the paper) a,b,c,d are strictly increasing differentiable functions on $[0,\infty]$ satisfying

$$\begin{aligned} a(x) < b(x) \leqslant c(x) < d(x), \qquad & x \in (0,\infty) \\ a(0) = b(0) = c(0) = d(0) = 0 & \\ a(\infty) = b(\infty) = c(\infty) = d(\infty) = \infty, & \end{aligned}$$

and $u_i, v_i,\ i = 1,2$ are general measurable functions (not necessarily non-negative) on $(0,\infty)$. This paper deals with the boundedness and compactness of the operator S between weighted Lebesgue spaces.

Various results are available when in the operator S, only one term is involved instead of two and $u \equiv v \equiv 1$, i.e., the operator

$$(Tf)(x) = \int_{a(x)}^{b(x)} f(t)dt.$$

The L^p-L^q boundedness of T was obtained by Heinig and Sinnamon [1] whereas the compactness was characterized by Jain and Gupta [2]. Several variants and generalizations of T have also been studied in [3], [4], [5].

In order to study the compactness of S, the key idea is to reduce its compactness in terms of the compactness of the operators S_1 and S_2. In this direction we first make use of the result from [2] to further reduce the compactness of S_1 to the compactness of the operators

$$(S_{1,L}f)(x) = u_1(b^{-1}(x)) \int_{a_k}^{x} v_1(t)f(t)dt \tag{1.2}$$

and

$$(S_{1,R}f)(x) = u_1(a^{-1}(x)) \int_{x}^{b_k} v_1(t)f(t)dt, \tag{1.3}$$

(a_k and b_k are non-negative real numbers described in Section 3) and then use a result of Stepanov [10] (see also [9]) to obtain the precise weight conditions for the compactness of the operators (1.2) and (1.3). In fact in this paper, Stepanov has studied the L^p-L^q boundedness and compactness of the operator

$$(Kf)(x) = u(x) \int_0^x k(x,y)v(y)f(y)dy,$$

where u,v are general measurable functions and $k(x,y)$ is the so called "Oinarov kernel". On the similar lines, the compactness of S_2 can be studied.

We shall write $I = (0,\infty)$. Let $1 < p < \infty$ and w be a weight function, i.e., measurable, finite and positive $a.e.$ on I. We denote

$$L^p(I,w) = \left\{ f : \|f\|_{L^p(I,w)} := \left(\int_0^\infty |f(x)|^p w(x)dx \right)^{1/p} < \infty \right\}$$

and when $w \equiv 1$, i.e., the non-weighted case, the corresponding space will be denoted by $L^p(I)$.

2. Boundedness of the operator S

The following result was proved by Heinig and Sinnamon [1] :

THEOREM A. *Let* $1 < p \leqslant q < \infty$ *and* w_0, w_1 *be weight functions defined on* I. *Then the inequality*

$$\left(\int_0^\infty \left|\int_{a(x)}^{b(x)} f(t)dt\right|^q w_1(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty |f(x)|^p w_0(x)dx\right)^{1/p} \tag{2.1}$$

holds for all measurable functions f *if and only if* $B < \infty$, *where*

$$B := \sup_{\substack{t<x \\ a(x)<b(t)}} B(x,t) = \sup_{\substack{t<x \\ a(x)<b(t)}} \left(\int_t^x w_1(s)ds\right)^{1/q} \left(\int_{a(x)}^{b(t)} w_0^{1-p'}(s)ds\right)^{1/p'}. \tag{2.2}$$

Now, we prove below the first main result of the paper which gives the L^p-L^q boundedness of the operator S defined by (1.1).

THEOREM 2.1. *Let* $1 < p \leqslant q < \infty$ *and* w_0, w_1 *be weight functions defined on* $(0, \infty)$. *Then the inequality*

$$\left(\int_0^\infty |(Sf)(x)|^q w_1(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty |f(x)|^p w_0(x)dx\right)^{1/p} \tag{2.3}$$

holds for all measurable functions f *if and only if* $\mathscr{B} = \max(\mathscr{B}_1, \mathscr{B}_2) < \infty$, *where*

$$\mathscr{B}_1 := \sup_{\substack{t<x \\ a(x)<b(t)}} \mathscr{B}_1(x,t) = \sup_{\substack{t<x \\ a(x)<b(t)}} \left(\int_t^x w_1(s)\,|u_1(s)|^q\,ds\right)^{1/q} \times$$

$$\times \left(\int_{a(x)}^{b(t)} w_0^{1-p'}(s)\,|v_1(s)|^{p'}\,ds\right)^{1/p'} \tag{2.4}$$

and

$$\mathscr{B}_2 := \sup_{\substack{t<x \\ c(x)<d(t)}} \mathscr{B}_2(x,t) = \sup_{\substack{t<x \\ c(x)<d(t)}} \left(\int_t^x w_1(s)\,|u_2(s)|^q\,ds\right)^{1/q} \times$$

$$\times \left(\int_{c(x)}^{d(t)} w_0^{1-p'}(s)\,|v_2(s)|^{p'}\,ds\right)^{1/p'}. \tag{2.5}$$

Proof. Let us denote

$$(S_1 f)(x) = u_1(x)\int_{a(x)}^{b(x)} v_1(t)f(t)dt \qquad \text{and} \qquad (S_2 f)(x) = u_2(x)\int_{c(x)}^{d(x)} v_2(t)f(t)dt$$

so that $S = S_1 + S_2$ and the triangle inequality gives

$$\|Sf\|_{L^q(I,w_1)} \leqslant \|S_1 f\|_{L^q(I,w_1)} + \|S_2 f\|_{L^q(I,w_1)}.$$

Consequently, if we first assume that $\mathscr{B} < \infty$, then in order to prove the inequality (2.3), it is sufficient to show that for $i = 1,2,$ the inequalities

$$\left(\int_0^\infty |(S_i f)(x)|^q w_1(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty |f(x)|^p w_0(x)dx\right)^{1/p} \tag{2.6}$$

hold. For $i = 1,$ writting $f.v_1 \equiv g,$ (2.6) becomes equivalent to

$$\left(\int_0^\infty \left|\int_{a(x)}^{b(x)} g(t)dt\right|^q w_1(x)\,|u_1(x)|^q\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty |g(x)|^p w_0(x)\,|v_1(x)|^{-p}\,dx\right)^{1/p}$$

which holds in view of Theorem A. Similarly, the case $i = 2$ can be disposed off.

Conversely, assume that the inequality (2.3) holds. Let us first consider the case when $u_i, v_i,\ i = 1,2$ are non-negative functions. Without any loss of generality, we may assume that $f \geqslant 0.$ Then clearly,

$$\|S_i f\|_{L^q(I,w_1)} \leqslant \|Sf\|_{L^q(I,w_1)}, \qquad i = 1,2$$

so that the inequalities (2.6) hold and in view of Theorem A, $\mathscr{B}_i < \infty,\ i = 1,2.$

In case $u_i, v_i,\ i = 1,2$ are not necessarily non-negative, we proceed as follows. For $\varepsilon > 0,$ define a new weight function $w_{0,\varepsilon}$ by

$$w_{0,\varepsilon}(x) = \max\left\{w_0(x), |v_1(x)|^p\,\varepsilon\right\}.$$

Clearly, $w_0(x) \leqslant w_{0,\varepsilon}(x)$ and therefore

$$\|f\|_{L^p(I,w_0)} \leqslant \|f\|_{L^p\left(I,w_{0,\varepsilon}\right)}.$$

Then since (2.3) holds, the inequality

$$\|Sf\|_{L^q(I,w_1)} \leqslant C\,\|f\|_{L^p\left(I,w_{0,\varepsilon}\right)} \tag{2.7}$$

also holds. We claim that $\mathscr{B}_1 < \infty.$

Fix t,x such that $0 < t < x < \infty,\ a(x) < b(t)$ and define a function

$$f_0(y) = |v_1(y)|^{p'-1}\, w_{0,\varepsilon}^{1-p'}(y)\, sgn\, v_1(y)\chi_{(a(x),b(t))}(y).$$

Note that $f_0 \in L^p\left(I, w_{0,\varepsilon}\right)$ since

$$\begin{aligned}
\|f_0\|^p_{L^p\left(I,w_{0,\varepsilon}\right)} &= \int_0^\infty |f_0(y)|^p\, w_{0,\varepsilon}(y)dy \\
&= \int_{a(x)}^{b(t)} |v_1(y)|^{p'}\, w_{0,\varepsilon}^{1-p'}(y)dy \\
&\leqslant \int_{a(x)}^{b(t)} |v_1(y)|^{p'}\left(|v_1(y)|^p\,\varepsilon\right)^{1-p'} dy \\
&= \varepsilon^{1-p'}\left(b(t) - a(x)\right) < \infty.
\end{aligned} \tag{2.8}$$

Next, we have

$$\begin{aligned}
\|Sf_0\|^q_{L^q(I,w_1)} &= \int_0^\infty |(Sf_0)(y)|^q w_1(y)dy \\
&\geqslant \int_t^x \left| u_1(y)\int_{a(y)}^{b(y)} v_1(s)f_0(s)ds + u_2(y)\int_{c(y)}^{d(y)} v_2(s)f_0(s)ds \right|^q w_1(y)dy \\
&= \int_t^x |u_1(y)|^q w_1(y) \left(\int_{a(x)}^{b(t)} |v_1(s)|^{p'} w_{0,\varepsilon}^{1-p'}(s)ds\right)^q dy \\
&= \left(\int_t^x |u_1(y)|^q w_1(y)dy\right)\left(\int_{a(x)}^{b(t)} |v_1(s)|^{p'} w_{0,\varepsilon}^{1-p'}(s)ds\right)^q,
\end{aligned}$$

since for $t<y<x,\ a(t)<a(y)<a(x)<b(t)<b(y)\leqslant c(y)<d(y)$ so that

$$\int_{a(y)}^{b(y)} v_1(s)f_0(s)ds = \int_{a(x)}^{b(t)} v_1(s)f_0(s)ds$$

and

$$\int_{c(y)}^{d(y)} v_2(s)f_0(s)ds = 0.$$

Consequently, by using the function f_0 instead of f in (2.7), we get, in view of (2.8), that

$$\left(\int_t^x |u_1(y)|^q w_1(y)dy\right)^{1/q}\left(\int_{a(x)}^{b(t)} |v_1(s)|^{p'} w_{0,\varepsilon}^{1-p'}(s)ds\right) \leqslant C\left(\int_{a(x)}^{b(t)} |v_1(s)|^{p'} (w_{0,\varepsilon}(s))^{1-p'} ds\right)^{1/p}$$

i.e.,

$$\left(\int_t^x |u_1(y)|^q w_1(y)dy\right)^{1/q}\left(\int_{a(x)}^{b(t)} |v_1(s)|^{p'} w_{0,\varepsilon}^{1-p'}(s)ds\right)^{1/p'} \leqslant C,$$

where the constant C is independent of t,x and ε. Letting $\varepsilon \to 0$, we find that $w_{0,\varepsilon} \to w_0$ and thus $\mathscr{B}_1 < \infty$.

Similarly, by choosing the weight

$$w_{0,\varepsilon}(x) = \max\{w_0(x), |v_2(x)|^p \varepsilon\}$$

and for fixed $0<t<x<\infty$, the function

$$f_0(y) = |v_2(y)|^{p'-1} w_{0,\varepsilon}^{1-p'}(y)\, sgn\, v_2(y)\chi_{(c(x),d(t))}(y),$$

we can show that $\mathscr{B}_2 < \infty$ and the proof is complete.

3. Some auxiliary results

The proofs of the subsequent results require certain well known assertions which we state in the following theorems:

THEOREM B. *Let X and Y be Banach spaces. An operator $A : X \to Y$ is compact if $A^* : Y^* \to X^*$ is weak*-norm sequentially continuous, i.e., for each sequence $\{f_n\}$ in Y^* with $f_n \xrightarrow{w^*} f$ for some f in Y^*, we have $A^*(f_n) \to A^* f$.*

THEOREM C. *Let $1 \leqslant p < \infty$ and $\{f_n\}$ be a sequence in $L^p(\alpha,\beta)$. Then $f_n \xrightarrow{w} f \in L^p(\alpha,\beta)$ if and only if*

(i) $\sup_n \|f_n\|_{L^p(\alpha,\beta)} < \infty$

(ii) $\int_M f_n(t)\,dt \to \int_M f(t)\,dt$ *for every measurable subset $M \subset (\alpha,\beta)$.*

Consider the operator

$$H_L : L^p(I,w_0) \to L^q(I,w_1)$$

defined by

$$(H_L f)(x) = u(x)\int_0^x v(t)\,f(t)\,dt,$$

where u, v are general measurable functions (not necessarily non-negative) on I. The compactness of H_L between non weighted Lebesgue spaces has been given by Stepanov ([10], Theorem 1.1) as follows:

THEOREM D. *Let $1 < p \leqslant q < \infty$. Then the operator $H_L : L^p(I) \to L^q(I)$ is compact if and only if*

$$A_0 := \sup_{t>0} A_0(t) = \sup_{t>0}\left(\int_t^\infty |u(x)|^q\,dx\right)^{\frac{1}{q}}\left(\int_0^t |v(y)|^{p'}\,dy\right)^{\frac{1}{p'}} < \infty$$

and

$$\lim_{t\to 0+} A_0(t) = \lim_{t\to\infty -} A_0(t) = 0.$$

We need to obtain the compactness of H_L between weighted Lebesgue spaces. It can be observed that the compactness of H_L from any set M in $L^q(I,w_1)$ is equivalent to the corresponding compactness from the set $w_1^{1/q}M$ in the non-weighted space $L^q(I)$. Consequently, the following result follows from Theorem D:

THEOREM 3.1. *Let $1 < p \leqslant q < \infty$ and w_0, w_1 be weight functions defined on I. Then the operator $H_L : L^p(I,w_0) \to L^q(I,w_1)$ is compact if and only if*

$$A_1 := \sup_{t>0} A_1(t) = \sup_{t>0}\left(\int_t^\infty |u(x)|^q\,w_1(x)\,dx\right)^{\frac{1}{q}}\left(\int_0^t |v(y)|^{p'}\,w_0^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} < \infty$$

and

$$\lim_{t\to 0+} A_1(t) = \lim_{t\to\infty-} A_1(t) = 0.$$

Likewise we obtain the compactness of

$$H_R : L^p(I,w_0) \to L^q(I,w_1)$$

defined by

$$(H_R f)(x) = u(x)\int_x^\infty v(t)\, f(t)\, dt$$

in the following theorem.

THEOREM 3.2. *Let* $1 < p \leqslant q < \infty$ *and* w_0, w_1 *be weight functions defined on* I*. Then the operator* $H_R : L^p(I,w_0) \to L^q(I,w_1)$ *is compact if and only if*

$$A_2 := \sup_{t>0} A_2(t) = \sup_{t>0}\left(\int_0^t |u(x)|^q\, w_1(x)\, dx\right)^{\frac{1}{q}}\left(\int_t^\infty |v(y)|^{p'}\, w_0^{1-p'}(y)\, dy\right)^{\frac{1}{p'}} < \infty$$

and

$$\lim_{t\to 0+} A_2(t) = \lim_{t\to\infty-} A_2(t) = 0.$$

Recall that the functions a and b are strictly increasing differentiable functions on $[0,\infty]$ satisfying $a(0) = b(0) = 0$, $a(x) < b(x)$ for $x > 0$ and $a(\infty) = b(\infty) = \infty$. Then a^{-1} and b^{-1} exist and are also strictly increasing. Consequently, we may define a sequence $\{m_k\}$, $k \in \mathbb{Z}$ (the set of integers) as follows:

For a fixed $m > 0$,

$$\begin{aligned} m_0 = m, \quad & m_{k+1} = a^{-1}(b(m_k)), && \text{if} \quad k \geqslant 0 \quad \text{and} \\ & m_k = b^{-1}(a(m_{k+1})), && \text{if} \quad k < 0. \end{aligned} \tag{3.1}$$

Clearly, $a(m_{k+1}) = b(m_k)$ for all $k \in \mathbb{Z}$ and by ([1], Lemma 2.1)

$$m_k < m_{k+1} \quad \text{for all} \quad k \in \mathbb{Z}, \quad \lim_{k\to\infty} m_k = \infty \quad \text{and} \quad \lim_{k\to-\infty} m_k = 0.$$

We write $a_k = a(m_k)$ and $b_k = b(m_k)$. We also write $(w_1)_a(y) = w_1(a^{-1}(y))(a^{-1})'(y)$ and $f_a(y) = f(a^{-1}(y)).(a^{-1})'(y)$ so that if $y = a(x)$, we have $(w_1)_a(y)dy = w_1(x)dx$ and $f_a(y)dy = f(x)dx$. We may write $(w_1)_b(y)$, f_b etc. similarly.

Define the operators

$$S_{1,L} \equiv S_{1,L,k} : L^p((a_k,b_k),w_0) \to L^q((a_k,b_k),(w_1)_b)$$

by

$$(S_{1,L}f)(x) = u_1(b^{-1}(x))\int_{a_k}^x v_1(t)f(t)dt \tag{3.2}$$

and

$$S_{1,R} \equiv S_{1,R,k} : L^p((a_k,b_k),w_0) \to L^q((a_k,b_k),(w_1)_a)$$

by

$$(S_{1,R}f)(x) = u_1(a^{-1}(x)) \int_x^{b_k} v_1(t)f(t)dt, \tag{3.3}$$

where $x \in (a_k,b_k)$. Note that $S_{1,L}$ and $S_{1,R}$ are Hardy operators.

To study the compactness conditions of the operator S_1, we require the compactness of S_1 in terms of the compactness of $S_{1,L}$ and $S_{1,R}$. For $u_1 = v_1 = 1$, a similar result was proved in [2]. For general u_1, v_1 also, the proof goes analogously with obvious modifications. Therefore, we only state the result in the form of a lemma.

LEMMA 3.3. *Let $1 < p,q < \infty$ and for some $m > 0$, $\{m_k\}$ be the sequence defined by (3.1). If the operators $S_{1,L}$ and $S_{1,R}$ defined, respectively, by (3.2) and (3.3) for each $k \in Z$ are compact then the operator $S_1 : L^p(I,w_0) \to L^q(I,w_1)$ is also compact.*

Now we give the precise conditions for the compactness of S_1.

THEOREM 3.4. *Let $1 < p \leqslant q < \infty$ and w_0, w_1 be weight functions on I. Further, assume that $\int_0^\infty w_0^{1-p'} < \infty$, $\int_0^\infty w_1 < \infty$. Then the operator $S_1 : L^p(I,w_0) \to L^q(I,w_1)$ defined by*

$$(S_1 f)(x) = u_1(x) \int_{a(x)}^{b(x)} v_1(y) f(y)\, dy$$

is compact if and only if

$$\sup_{\substack{0<t<x<\infty \\ a(x)<b(t)}} \mathscr{B}_1(x,t) = \sup_{\substack{0<t<x<\infty \\ a(x)<b(t)}} \left(\int_t^x |u_1|^q w_1\right)^{\frac{1}{q}} \left(\int_{a(x)}^{b(t)} |v_1|^{p'} w_0^{1-p'}\right)^{\frac{1}{p'}} < \infty,$$

$$\lim_{t\to x-} \mathscr{B}_1(x,t) = \lim_{t\to b^{-1}(a(x))+} \mathscr{B}_1(x,t) = 0, \text{ for every } x > 0$$

and

$$\lim_{x\to t+} \mathscr{B}_1(x,t) = \lim_{x\to a^{-1}(b(t))-} \mathscr{B}_1(x,t) = 0 \text{ , for every } t > 0.$$

Proof. Fixing t and putting $y = a(x)$, we get

$$\begin{aligned}
\mathscr{B}_1(x,t) &= \left(\int_{a(t)}^{y} \left|u_1\left(a^{-1}(s)\right)\right|^q w_1\left(a^{-1}(s)\right)\left(a^{-1}(s)\right)' ds\right)^{\frac{1}{q}} \left(\int_y^{b(t)} |v_1|^{p'} w_0^{1-p'}\right)^{\frac{1}{p'}} \\
&= \left(\int_{a(t)}^{y} \left|u_1\left(a^{-1}(s)\right)\right|^q (w_1)_a(s) ds\right)^{\frac{1}{q}} \left(\int_y^{b(t)} |v_1|^{p'} w_0^{1-p'}\right)^{\frac{1}{p'}} \\
&= \mathscr{A}_2(y).
\end{aligned}$$

Thus for each $t > 0$

$$\sup_{a(t)<y<b(t)} \mathscr{A}_2(y) < \infty,$$

and

$$\lim_{y\to a(t)+} \mathscr{A}_2(y) = \lim_{y\to b(t)-} \mathscr{A}_2(y) = 0,$$

which, in view of Theorem 3.2, are precisely the conditions for the compactness of the operator

$$S_{1,R} : L^p\left(\left(a(t), b(t)\right), w_0\right) \to L^q\left(\left(a(t), b(t)\right), (w_1)_a\right)$$

defined by

$$(S_{1,R}f)(x) = u_1\left(a^{-1}(x)\right) \int_x^{b(t)} v_1(y) f(y)\, dy.$$

Similarly, fixing x, it can be shown that the operator

$$S_{1,L} : L^p\left(\left(a(x), b(x)\right), w_0\right) \to L^q\left(\left(a(x), b(x)\right), (w_1)_b\right)$$

defined by

$$(S_{1,L}f)(t) = u_1\left(b^{-1}(t)\right) \int_{a(x)}^{t} v_1(y) f(y)\, dy$$

is also compact for $x > 0$. Therefore, if $m > 0$ be fixed and $\{m_k\}_{k\in Z}$ be the sequence as defined above, then the operators

$$S_{1,L} : L^p\left((a_k, b_k), w_0\right) \to L^q\left((a_k, b_k), (w_1)_b\right)$$

and

$$S_{1,R} : L^p\left((a_k, b_k), w_0\right) \to L^q\left((a_k, b_k), (w_1)_a\right)$$

are compact for each $k \in Z$ and consequently, we get the assertion by Lemma 3.3.

Conversely, assume that S_1 is compact. Then S_1 is bounded and therefore in view of Theorem 2.1

$$\sup_{\substack{0<t<x<\infty \\ a(x)<b(t)}} \mathscr{B}_1(x,t) < \infty.$$

Now, for t, x such that $0 < t < x < \infty$, $a(x) < b(t)$, we define a function $f_{(x,t)}$ (depending upon x and t) by

$$f_{(x,t)}(y) = \begin{cases} |v_1(y)|^{p'-1} w_0^{1-p'}(y) \operatorname{sgn} v_1(y), & y \in (a(x), b(t)) \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\left\| f_{(x,t)} \right\|_{L^p(I,w_0)} = \left(\int_{a(x)}^{b(t)} |v_1(y)|^{p'} w_0^{1-p'}(y) dy \right)^{1/p}.$$

Thus, if we denote

$$z_{(x,t)} = \frac{f_{(x,t)}}{\|f_{(x,t)}\|_{L^p(I,w_0)}},$$

then, clearly $\|z_{(x,t)}\|_{L^p(I,w_0)} = 1$. Therefore, for every measurable subset $M \subset I$, we get using Holder's inequality

$$\begin{aligned}
\int_M z_{(x,t)}(y)dy &= \frac{\left(\int_{M\cap(a(x),b(t))} |v_1(y)|^{p'-1} w_0^{1-p'}(y)\operatorname{sgn} v_1(y)dy\right)}{\left(\int_{a(x)}^{b(t)} |v_1(y)|^{p'} w_0^{1-p'}(y)dy\right)^{1/p}} \\
&\leqslant \frac{\left(\int_{a(x)}^{b(t)} |v_1(y)|^{p'} w_0^{1-p'}(y)dy\right)^{1/p} \left(\int_{a(x)}^{b(t)} w_0^{1-p'}(y)dy\right)^{1/p'}}{\left(\int_{a(x)}^{b(t)} |v_1(y)|^{p'} w_0^{1-p'}(y)dy\right)^{1/p}} \\
&\leqslant \left(\int_{a(x)}^{b(t)} w_0^{1-p'}(y)dy\right)^{1/p'}
\end{aligned}$$

which tends to 0 as $a(x) \to b(t)-$ for every $t > 0$ or $b(t) \to a(x)+$ for every $x > 0$, since the last integral is finite. Consequently, by Theorem C

$$z_{(x,t)} \xrightarrow{w} 0 \qquad \text{in} \quad L^p(I,w_0)$$

as $a(x) \to b(t)-$ for every $t > 0$ or $b(t) \to a(x)+$ for every $x > 0$ and since S_1 is compact, we further get that

$$\|S_1 z_{(x,t)}\|_{L^q(I,w_1)} \to 0$$

as $a(x) \to b(t)-$ for every $t > 0$ or $b(t) \to a(x)+$ for every $x > 0$. But

$$\begin{aligned}
&\|S_1 z_{(x,t)}\|_{L^q(I,w_1)}^q \\
&\quad = \int_0^\infty |(S_1 z_{(x,t)})(y)|^q w_1(y)dy \\
&\quad \geqslant \int_t^x \left| \frac{u_1(y)\int_{a(y)}^{b(y)} v_1(s) f_{(x,t)}(s)ds}{\|f_{(x,t)}\|_{L^p(I,w_0)}} \right|^q w_1(y)dy \\
&\quad = \int_t^x \left| \frac{u_1(y)\int_{a(x)}^{b(t)} |v_1(s)|^{p'} w_0^{1-p'}(s)ds}{\|f_{(x,t)}\|_{L^p(I,w_0)}} \right|^q w_1(y)dy \\
&\quad = \left(\int_t^x |u_1(y)|^q w_1(y)dy\right) \left(\int_{a(x)}^{b(t)} |v_1(s)|^{p'} w_0^{1-p'}(s)ds\right)^{q-q/p} \\
&\quad = [\mathscr{B}_1(x,t)]^q,
\end{aligned}$$

since for every $t < y < x$, $a(t) < a(y) < a(x) < b(t) < b(y)$. Thus, we have

$$\lim_{x \to a^{-1}(b(t))-} \mathscr{B}_1(x,t) = 0, \qquad \text{for every } t > 0$$

and

$$\lim_{t\to b^{-1}(a(x))+} \mathscr{B}_1(x,t)=0, \qquad \text{for every } x>0.$$

Further, on the similar lines, choosing t,x such that $0<t<x<\infty,\ a(x)<b(t)$ and defining

$$\widetilde{z}_{(x,t)}=\frac{g_{(x,t)}}{\left\|g_{(x,t)}\right\|_{L^{q'}(I,w_1^{1-q'})}},$$

where

$$g_{(x,t)}(y)=\begin{cases} |u_1(y)|^{q-1}w_1(y)\operatorname{sgn}u_1(y), & y\in(t,x)\\ 0, & \text{otherwise}\end{cases}$$

it can be shown that $\widetilde{z}_{(x,t)}\xrightarrow{w}0$ in $L^{q'}(I,w_1^{1-q'})$ as $t\to x-$ for every $x>0$ or $x\to t+$, for every $t>0$.

Consequently, by the compactness of S_1,

$$\left\|S_1^*\widetilde{z}_{(x,t)}\right\|_{L^{p'}(I,w_0^{1-p'})}\to 0$$

as $t\to x-$ for every $x>0$ or $x\to t+$, for every $t>0$. But as before

$$\begin{aligned}
&\left\|S_1^*\widetilde{z}_{(x,t)}\right\|^{p'}_{L^{p'}(I,w_0^{1-p'})}\\
&\quad=\int_0^\infty\left|S_1^*\widetilde{z}_{(x,t)}(y)\right|^{p'}w_0^{1-p'}(y)\,dy\\
&\quad\geqslant\int_{a(x)}^{b(t)}\left|\frac{v_1(y)\int_{b^{-1}(y)}^{a^{-1}(y)}u_1(s)\,g_{(x,t)}(s)\,ds}{\left\|g_{(x,t)}\right\|_{L^{q'}(I,w_1^{1-q'})}}\right|^{p'}w_0^{1-p'}(y)\,dy\\
&\quad=\int_{a(x)}^{b(t)}\left|\frac{v_1(y)\int_t^x|u_1(s)|^q w_1(s)\,ds}{\left\|g_{(x,t)}\right\|_{L^{q'}(I,w_1^{1-q'})}}\right|^{p'}w_0^{1-p'}(y)\,dy\\
&\quad=\left(\int_{a(x)}^{b(t)}|v_1(y)|^{p'}w_0^{1-p'}(y)\,dy\right)\left(\int_t^x|u_1(s)|^q w_1(s)\,ds\right)^{\frac{p'}{q}}\\
&\quad=\left[\mathscr{B}_1(x,t)\right]^{p'}
\end{aligned}$$

since for $a(x)<y<b(t)$, $b^{-1}(y)<t<x<a^{-1}(y)$. Thus, we have

$$\lim_{t\to x-}\mathscr{B}_1(x,t)=0, \qquad \text{for every } x>0$$
$$\lim_{x\to t+}\mathscr{B}_1(x,t)=0, \qquad \text{for every } t>0.$$

4. Compactness of the operator S

THEOREM 4.1. *Let $1 < p \leqslant q < \infty$, w_0, w_1 be weight functions defined on I and $\mathscr{B}_1, \mathscr{B}_2$ be as defined by (2.4) and (2.5). Further assume that $\int_0^\infty w_0^{1-p'}(y)\,dy$, $\int_0^\infty w_1(y)\,dy$ are finite. Then the operator S is compact if and only if S is bounded and*

$$\lim_{t\to x-} \mathscr{B}_1(x,t) = \lim_{t\to b^{-1}(a(x))+} \mathscr{B}_1(x,t) = 0, \qquad \text{for every } x > 0 \tag{4.1}$$

$$\lim_{x\to t+} \mathscr{B}_1(x,t) = \lim_{x\to a^{-1}(b(t))-} \mathscr{B}_1(x,t) = 0, \qquad \text{for every } t > 0 \tag{4.2}$$

$$\lim_{t\to x-} \mathscr{B}_2(x,t) = \lim_{t\to d^{-1}(c(x))+} \mathscr{B}_2(x,t) = 0, \qquad \text{for every } x > 0 \tag{4.3}$$

$$\lim_{x\to t+} \mathscr{B}_2(x,t) = \lim_{x\to c^{-1}(d(t))-} \mathscr{B}_2(x,t) = 0, \qquad \text{for every } t > 0 \tag{4.4}$$

Proof. Assume first that S is bounded and (4.1), (4.2), (4.3), (4.4) hold. Then, in view of Theorem 3.4, we observe that $S_1, S_2 : L^p(I, w_0) \to L^q(I, w_1)$ are compact. To show the compactness of S, let us take a sequence $\{f_n\}$ in $L^{q'}(I, w_1^{1-q'})$ such that $f_n \xrightarrow{w^*} 0$. Then

$$f_n \xrightarrow{w} 0,$$

and by compactness of S_1 and S_2, we get

$$\|S_1^* f_n\|_{L^{p'}(I, w_0^{1-p'})} \to 0, \quad \|S_2^* f_n\|_{L^{p'}(I, w_0^{1-p'})} \to 0\,.$$

Therefore

$$\begin{aligned}\|S^* f_n\|_{L^{p'}(I, w_0^{1-p'})} &\leqslant \|S_1^* f_n\|_{L^{p'}(I, w_0^{1-p'})} + \|S_2^* f_n\|_{L^{p'}(I, w_0^{1-p'})} \\ &\to 0.\end{aligned}$$

Thus S is compact.

Conversely, assume that S is compact. Then S is bounded. Now, for t, x such that $0 < t < x < \infty$, $a(x) < b(t)$, by defining a function $f_{(x,t)}$ (depending upon x and t) by

$$f_{(x,t)}(y) = \begin{cases} |v_1(y)|^{p'-1} w_0^{1-p'}(y)\,\mathrm{sgn}\, v_1(y), & y \in (a(x), b(t)) \\ 0, & \text{otherwise} \end{cases}$$

and

$$z_{(x,t)} = \frac{f_{(x,t)}}{\|f_{(x,t)}\|_{L^p(I, w_0)}},$$

on the lines of Theorem 3.4, we can show that

$$\lim_{x\to a^{-1}(b(t))-} \mathscr{B}_1(x,t) = 0, \qquad \text{for every } t > 0$$

and

$$\lim_{t\to b^{-1}(a(x))+} \mathscr{B}_1(x,t) = 0, \qquad \text{for every } x > 0.$$

Now, $\lim_{t\to d^{-1}(c(x))+} \mathscr{B}_2(x,t) = 0$, for every $x > 0$ and $\lim_{x\to c^{-1}(d(t))-} \mathscr{B}_2(x,t) = 0$, for every $t > 0$ can be shown completely analogously by fixing $0 < t, x < \infty$, $c(x) < d(t)$ and defining

$$f_{(x,t)}(y) = \begin{cases} |v_2(y)|^{p'-1} w_0^{1-p'}(y) \operatorname{sgn} v_2(y), & y \in (c(x), d(t)) \\ 0, & \text{otherwise} \end{cases}$$

and

$$z_{(x,t)} = \frac{f_{(x,t)}}{\|f_{(x,t)}\|_{L^p(I,w_0)}}.$$

Further, on choosing t, x such that $0 < t < x < \infty$, $a(x) < b(t)$ and defining

$$\widetilde{z}_{(x,t)} = \frac{g_{(x,t)}}{\|g_{(x,t)}\|_{L^{q'}(I,w_1^{1-q'})}},$$

where

$$g_{(x,t)}(y) = \begin{cases} |u_1(y)|^{q-1} w_1(y) \operatorname{sgn} u_1(y), & y \in (t, x) \\ 0, & \text{otherwise} \end{cases}$$

it can be shown, by proceeding on the lines of Theorem 3.4, that

$$\lim_{t\to x-} \mathscr{B}_1(x,t) = 0, \qquad \text{for every } x > 0$$
$$\lim_{x\to t+} \mathscr{B}_1(x,t) = 0, \qquad \text{for every } t > 0.$$

Corresponding limits for $\mathscr{B}_2(x,t)$ can be obtained analogously.

REFERENCES

[1] H. P. HEINIG AND G. SINNAMON, *Mapping properties of integral averaging operators*, Studia Math., **129** (1998), 157–177.

[2] P. JAIN AND B. GUPTA, *Compactness of Hardy-Steklov operator*, J. Math. Anal. Appl., **288** (2003), 680–691.

[3] P. JAIN, P. K. JAIN AND B. GUPTA, *On certain double sized integral operators over multidimensional cones*, Proc. A. Razmadze Math. Inst., **131** (2003), 39–60.

[4] P. JAIN, P. K. JAIN AND B. GUPTA, *Compactness of Hardy type operators over star-shaped regions in* $\mathbb{R}^N$, Canad. Math. Bull., **47** (2004), 540–552.

[5] P. JAIN, P. K. JAIN AND B. GUPTA, *On certain weighted integral inequalities with mixed norm*, Italian J. Pure Appl. Math., **17** (2005), 9–20.

[6] A. KUFNER AND L. E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific, 2003.

[7] B. OPIC AND A. KUFNER, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, Longman Scientific & Technical Harlow, 1990.

[8] A. KUFNER, L. MALIGRANDA AND L. E. PERSSON, *The Hardy Inequality, About its History and Some Related Results*, Vydavatelsky Sewis, Pilsen, 2007.

[9] V. D. STEPANOV, *Weighted norm inequalities for integral operators and related topics*, in Proceedings of the Spring School "Nonlinear Analysis, Function Spaces and Applications", 1994, 139–175.

[10] V. D. STEPANOV, *On the boundedness and compactness of a class of integral operators*, Soviet Math. Dokl. **41** (1990), 468–470.

[11] P. A. ZHAROV, *On a two-weight inequality, Generalization of inequalities of Hardy and Poincáre* (in Russian), Trudy Math. Ins. Steklov **194** (1992), 97–110; translation in Proc. Steklov Inst. Math. **194** (4) (1993), 101–114.

Pankaj Jain
Department of Mathematics
South Asian University
Akbar Bhawan, Chanakya Puri
New Delhi -110021, India
e-mail: pankaj.jain@sau.ac.in, pankajkrjain@hotmail.com

Babita Gupta
Department of Mathematics
Shivaji College (University of Delhi)
Raja Garden, Delhi - 110027, India
e-mail: babita.gupta@hotmail.com