Journal of Mathematical Inequalities
Volume 11, Number 4 (2017), 947–964
doi:10.7153/jmi-2017-11-72

OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES RELATED TO POMPEIU'S MEAN VALUE THEOREM WITH COMPLEX EXPONENTIAL WEIGHT

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(*Communicated by A. Aglić Aljinović*)

Abstract. We present some inequalities of Ostrowski and trapezoid type with complex exponential weight, for complex-valued absolutely continuous functions. These inequalities are related to Pompeiu's mean value theorem. Special cases of these inequalities are applied to obtain (i) some approximation results for the finite Fourier and Laplace transforms; (ii) refinements of the Ostrowski and trapezoid inequalities; and (iii) new Ostrowski and trapezoid type inequalities.

1. Introduction

In 1938, Ostrowski [17] proved the following estimate of the integral mean:

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $|f'(t)|\leqslant M<\infty$ for all $t\in(a,b)$. Then, for any $x\in[a,b]$, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leqslant\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]M(b-a). \tag{1}$$

The constant $\frac{1}{4}$ is best possible, in the sense that it cannot be replaced by a smaller quantity.

Inequality (1) is referred to as Ostrowski's inequality. For its generalisations and related results we refer the readers to Dragomir and Rassias [15]. Another estimate of the integral mean is given by the trapezoid rule as follows.

THEOREM 2. (Cerone and Dragomir [7]) *Under the assumptions of Theorem 1, we have*

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leqslant\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]M(b-a), \tag{2}$$

for any $x\in[a,b]$. The constant $\frac{1}{4}$ is best possible.

Mathematics subject classification (2010): Primary: 26D10, 26D15.

Keywords and phrases: Ostrowski inequality, trapezoid inequality, complex exponential weight, Laplace transform, Fourier transform.

Inequality (2) is known as the trapezoid inequality. For its generalisations and related results, we refer the readers to Cerone and Dragomir [7].

It is important to note that the bounds in inequalities (1) and (2) are the same. Cerone [6, Remark 1] stated that there is a strong relationship between the Ostrowski and the trapezoidal functionals which is highlighted by the symmetric transformations amongst their kernels.

In 1946, Pompeiu [19] derived a variant of Lagrange's mean value theorem, known as *Pompeiu's mean value theorem* (cf. Sahoo and Riedel [21, p. 83]), as given below:

THEOREM 3. *For every real-valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{3}$$

Pompeiu's mean value theorem is utilised in order to provide another approximation of the integral mean, as given below.

THEOREM 4. (Dragomir, 2005 [9]) *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x \in [a,b]$, we have the inequality*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leqslant \frac{b-a}{|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \left\| f - \ell f' \right\|_\infty. \tag{4}$$

where $\ell(t) = t,\ t \in [a,b]$. The constant $\frac{1}{4}$ is best possible.

We refer the readers to Popa [20], Pečarić and Ungar [18], Acu and Sofonea [1], and Acu et al. [2] for the generalisations and extensions of Theorem 4. Inequalities of Ostrowski type which are related to the Pompeiu's mean value theorem are given in the papers by Dragomir [10, 11]. Further inequalities of Ostrowski and trapezoid types which are related to the Pompeiu's mean value theorem can be found in Cerone, Dragomir, and Kikianty [8].

Some exponential Pompeiu type inequalities for complex-valued absolutely continuous functions are given in Dragomir [12], with applications to obtain some new Ostrowski type inequalities. We recall the results on the Ostrowski type inequalities, in the next theorem.

THEOREM 5. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha = \beta + i\gamma \in \mathbb{C}$ with $\beta > 0$. Then, for any $x \in [a,b]$ we have*

$$\left| f(x) \frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha} - \exp(\alpha x) \int_a^b f(t)\,dt \right| \tag{5}$$

$$\leqslant \begin{cases} |\beta| B_1(a,b,x,\alpha) \|f' - \alpha f\|_\infty, & f' - \alpha f \in L_\infty[a,b], \\ q^{1/q} |\beta|^{1/q} (b-a)^{1/p} |B_q(a,b,x,\alpha)|^{1/q} \|f' - \alpha f\|_p, & f' - \alpha f \in L_p[a,b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ B_\infty(a,b,x,\alpha) \|f' - \alpha f\|_1. & \end{cases}$$

where

$$B_q(a,b,x,\alpha) = 2\left[e^{xq\beta}\left(x - \frac{a+b}{2}\right) + \frac{1}{q\beta}\left(\frac{e^{bq\beta}+e^{aq\beta}}{2} - e^{xq\beta}\right)\right],$$

for $q \geqslant 1$ *and* $B_\infty(a,b,x,\alpha) := \exp(x\beta)(x-a) + \beta^{-1}\left[\exp(b\beta) - \exp(x\beta)\right]$. *If* $\beta = 0$, *then for any* $x \in [a,b]$ *we have*

$$\left| f(x)\frac{\exp(i\gamma b) - \exp(i\gamma a)}{i\gamma} - \exp(i\gamma x)\int_a^b f(t)\,dt \right| \tag{6}$$

$$\leqslant \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2 \|f' - i\gamma f\|_\infty, & f' - i\gamma f \in L_\infty[a,b], \\ \left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right]\frac{q}{q+1}(b-a)^{\frac{q+1}{q}} \|f' - i\gamma f\|_p, & f' - i\gamma f \in L_p[a,b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ (b-a)\|f' - i\gamma f\|_1. & \end{cases}$$

In this paper, we give refinements of the inequalities in Theorem 5. We also present similar results for trapezoid type inequalities with complex exponential weights.

The paper is organised as follows. We present the main theorems concerning inequalities with complex weights in Section 2. We consider special cases of the main theorems, by choosing $x = (a+b)/2$, in Section 3. The inequalities involving the p-norms (with imaginary weights) where $p \neq 1$ are proven to be sharp.

The Fourier transform has been a principal analytical tool in many fields of research, such as probability theory, quantum physics, and boundary-value problems [3]. The approximations of the finite Fourier transform of different classes of functions have been considered by employing integral inequalities of Ostrowski type. We refer to Barnett and Dragomir [4] for the approximations of the Fourier transform of absolutely continuous functions; to Barnett, Dragomir, and Hanna [5] for functions of bounded variation; and to Dragomir, Cho, and Kim [13] for Lebesgue integrable mappings. Using a pre-Grüss type inequality, Dragomir, Hanna, and Roumeliotis [14] obtained some approximations of the finite Fourier transform for complex-valued functions. We apply the inequalities in Section 3 to obtain some approximation results for the finite Fourier and Laplace transforms.

Finally, we provide some refinements of the Ostrowski and trapezoid inequalities in Section 4 by considering special cases of the main theorems in Section 2. We also obtain some new Ostrowski and trapezoid type inequalities.

2. Main theorems

The main results concerning the Ostrowski and trapezoid type inequalities with complex exponential weights are given in this section.

2.1. Ostrowski type inequalities

We recall the definition of the Gamma and incomplete Gamma functions:

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx, \quad \text{and} \quad \Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt.$$

Throughout the text, for $\alpha = \beta + i\gamma \in \mathbb{C}$ and $1 \leqslant q < \infty$, we use the following notation:

$$\Psi^{+}_{q,\alpha}(s,t) = e^{-s\beta}(q\beta)^{-\frac{q+1}{q}}\left[\Gamma(q+1) - \Gamma(q+1, q\beta(t-s))\right]^{\frac{1}{q}};$$

and

$$\Psi^{-}_{q,\alpha}(s,t) = e^{-t\beta}(-q\beta)^{-\frac{q+1}{q}}\left[\Gamma(q+1) - \Gamma(q+1, -q\beta(t-s))\right]^{\frac{1}{q}}.$$

THEOREM 6. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$, $\alpha = \beta + i\gamma \in \mathbb{C}$ and $1 < p \leqslant \infty$. Let $1 \leqslant q < \infty$ be the Hölder conjugate of p. If $\beta \neq 0$, then*

$$\begin{aligned} &\left|\frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ &\leqslant \Psi^{+}_{q,\alpha}(a,x)\|f' - \alpha f\|_{[a,x],p} + \Psi^{-}_{q,\alpha}(x,b)\|f' - \alpha f\|_{[x,b],p} \\ &\leqslant \left[\Psi^{+}_{q,\alpha}(a,x) + \Psi^{-}_{q,\alpha}(x,b)\right]\|f' - \alpha f\|_{[a,b],p}, \end{aligned} \tag{7}$$

for $x \in [a,b]$. If $\beta = 0$, then

$$\begin{aligned} &\left|\frac{f(x)}{e^{ix\gamma}}(b-a) - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \\ &\leqslant \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f' - i\gamma f\|_{[a,x],p} + \frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f' - i\gamma f\|_{[x,b],p} \\ &\leqslant \frac{(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f' - i\gamma f\|_{[a,b],p}, \end{aligned} \tag{8}$$

for $x \in [a,b]$. The inequalities in (8) are sharp.

Proof. We use the Montgomery identity for the absolutely continuous function $g : [a,b] \to \mathbb{C}$ (cf. Mitrinović, Pečarić, and Fink [16, p. 565]):

$$g(x)(b-a) - \int_a^b g(t)\,dt = \int_a^x (t-a)g'(t)\,dt + \int_x^b (t-b)g'(t)\,dt, \tag{9}$$

where $x \in [a,b]$. If $g(t) = f(t)/e^{\alpha t}$, then $g'(t) = (f'(t) - \alpha f(t))/e^{\alpha t}$; and with this choice of g, (9) becomes:

$$\begin{aligned} &\frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \\ &= \int_a^x (t-a)\frac{f'(t) - \alpha f(t)}{e^{\alpha t}}\,dt + \int_x^b (t-b)\frac{f'(t) - \alpha f(t)}{e^{\alpha t}}\,dt. \end{aligned} \tag{10}$$

Take the modulus of (10) and make use of the Hölder's inequality to obtain the following inequalities for $1 < p \leqslant \infty$ and its Hölder's conjugate q:

$$\begin{aligned}
&\left|\frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leqslant \left(\int_a^x \left((t-a)|e^{-\alpha t}|\right)^q dt\right)^{\frac{1}{q}} \|f'-\alpha f\|_{[a,x],p} + \left(\int_x^b \left((b-t)|e^{\alpha t}|\right)^q dt\right)^{\frac{1}{q}} \|f'-\alpha f\|_{[x,b],p} \\
&= \left(\int_a^x (t-a)^q e^{-tq\beta}\,dt\right)^{\frac{1}{q}} \|f'-\alpha f\|_{[a,x],p} + \left(\int_x^b (b-t)^q e^{-tq\beta}\,dt\right)^{\frac{1}{q}} \|f'-\alpha f\|_{[x,b],p}.
\end{aligned}$$

We evaluate the integral

$$\begin{aligned}
\int_a^x (t-a)^q e^{-tq\beta}\,dt &= e^{-aq\beta}(q\beta)^{-q-1}\int_0^{q\beta(x-a)} z^q e^{-z}\,dz \\
&= e^{-aq\beta}(q\beta)^{-q-1}\left[\Gamma(q+1) - \Gamma(q+1, q\beta(x-a))\right],
\end{aligned}$$

by letting $z = (t-a)q\beta$, and thus

$$\begin{aligned}
&\left(\int_a^x (t-a)^q e^{-tq\beta}\,dt\right)^{\frac{1}{q}} \\
&= e^{-a\beta}(q\beta)^{-\frac{q+1}{q}}\left[\Gamma(q+1) - \Gamma(q+1, q\beta(x-a))\right]^{\frac{1}{q}} = \Psi^{+}_{q,\alpha}(a,x).
\end{aligned}$$

Now, we evaluate the integral

$$\begin{aligned}
\int_x^b (b-t)^q e^{-tq\beta}\,dt &= e^{-bq\beta}(-q\beta)^{-q-1}\int_0^{q(-\beta)(b-x)} z^q e^{-z}\,dz \\
&= e^{-bq\beta}(-q\beta)^{-q-1}\left[\Gamma(q+1) - \Gamma(q+1, -q\beta(b-x))\right],
\end{aligned}$$

by letting $z = -(b-t)q\beta$, and thus

$$\begin{aligned}
&\left(\int_x^b (b-t)^q e^{-tq\beta}\,dt\right)^{\frac{1}{q}} \\
&= e^{-b\beta}(-q\beta)^{-\frac{q+1}{q}}\left[\Gamma(q+1) - \Gamma(q+1, -q\beta(b-x))\right]^{\frac{1}{q}} = \Psi^{-}_{q,\alpha}(x,b),
\end{aligned}$$

and this proves (7). If $\beta = 0$, then

$$\begin{aligned}
&\left|\frac{f(x)}{e^{ix\gamma}}(b-a) - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \\
&\leqslant \left(\int_a^x (t-a)^q\,dt\right)^{\frac{1}{q}} \|f'-i\gamma f\|_{[a,x],p} + \left(\int_x^b (b-t)^q\,dt\right)^{\frac{1}{q}} \|f'-i\gamma f\|_{[x,b],p} \\
&= \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'-i\gamma f\|_{[a,x],p} + \frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'-i\gamma f\|_{[x,b],p} \\
&\leqslant \frac{(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'-i\gamma f\|_{[a,b],p},
\end{aligned}$$

for any $x \in [a,b]$. This completes the proof. The sharpness of the inequalities in (8) is given by Proposition 1. □

In particular, when $p = \infty$ $(q = 1)$ in Theorem 6, inequalities (7) and (8) take simpler forms as follows:

COROLLARY 1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$ and $\alpha = \beta + i\gamma \in \mathbb{C}$. If $\beta \neq 0$, then we have*

$$\begin{aligned} &\left| \frac{f(x)(b-a)}{e^{\alpha x}} - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right| \\ &\leqslant \frac{e^{-a\beta} - [(x-a)\beta + 1]e^{-x\beta}}{\beta^2} \|f' - \alpha f\|_{[a,x],\infty} \\ &\quad + \frac{e^{-b\beta} + [(b-x)\beta - 1]e^{-x\beta}}{\beta^2} \|f' - \alpha f\|_{[x,b],\infty} \\ &\leqslant \frac{1}{\beta^2}\left[e^{-a\beta} + e^{-b\beta} + 2\left(\left(\frac{a+b}{2} - x\right)\beta - 1\right)e^{-x\beta}\right] \|f' - \alpha f\|_{[a,b],\infty}, \end{aligned} \tag{11}$$

for any $x \in [a,b]$. If $\beta = 0$, then we have

$$\begin{aligned} &\left| \frac{f(x)(b-a)}{e^{ix\gamma}} - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt \right| \\ &\leqslant \frac{1}{2}\left[(x-a)^2 \|f' - i\gamma f\|_{[a,x],\infty} + (b-x)^2 \|f' - i\gamma f\|_{[x,b],\infty}\right] \\ &\leqslant \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \|f' - i\gamma f\|_{[a,b],\infty}, \end{aligned} \tag{12}$$

for any $x \in [a,b]$. The constants $\frac{1}{2}$ and $\frac{1}{4}$ in (12) are sharp.

The case for the 1-norm is as follows:

THEOREM 7. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$, $\alpha = \beta + i\gamma \in \mathbb{C}$. If $\beta \neq 0$, then*

$$\left| \frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right| \tag{13}$$

$$\leqslant \begin{cases} \frac{1}{\beta}e^{-(a\beta+1)}\|f' - \alpha f\|_{[a,x],1} + (b-x)e^{-x\beta}\|f' - \alpha f\|_{[x,b],1}, & \beta > 0 \ \&\ a + \frac{1}{\beta} \leqslant x, \\ (x-a)e^{-x\beta}\|f' - \alpha f\|_{[a,x],1} + \frac{1}{-\beta}e^{-(b\beta+1)}\|f' - \alpha f\|_{[x,b],1}, & \beta < 0 \ \&\ b + \frac{1}{\beta} \geqslant x, \\ (x-a)e^{-x\beta}\|f' - \alpha f\|_{[a,x],1} + (b-x)e^{-x\beta}\|f' - \alpha f\|_{[x,b],1}, & \textit{otherwise}, \end{cases}$$

$$\leqslant \begin{cases} \left[\frac{1}{\beta}e^{-(a\beta+1)} + (b-x)e^{-x\beta}\right]\|f' - \alpha f\|_{[a,b],1}, & \beta > 0 \ \&\ a + \frac{1}{\beta} \leqslant x, \\ \left[(x-a)e^{-x\beta} + \frac{1}{-\beta}e^{-(b\beta+1)}\right]\|f' - \alpha f\|_{[a,b],1}, & \beta < 0 \ \&\ b + \frac{1}{\beta} \geqslant x, \\ (b-a)e^{-x\beta}\|f' - \alpha f\|_{[x,b],1}, & \textit{otherwise}, \end{cases}$$

for any $x \in [a,b]$. *If* $\beta = 0$, *then*

$$\left|\frac{f(x)}{e^{ix\gamma}}(b-a) - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \leqslant (x-a)\|f'-i\gamma f\|_{[a,x],1} + (b-x)\|f'-i\gamma f\|_{[x,b],1} \\ \leqslant (b-a)\|f'-i\gamma f\|_{[a,b],1}, \tag{14}$$

for any $x \in [a,b]$.

Proof. Take the modulus of (10) and make use of the Hölder's inequality to obtain:

$$\left|\frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ \leqslant \sup_{t\in[a,x]}(t-a)e^{-t\beta}\|f'-\alpha f\|_{[a,x],1} + \sup_{t\in[x,b]}(b-t)e^{-t\beta}\|f'-\alpha f\|_{[x,b],1}.$$

Define the functions: $A(t) = (t-a)e^{-t\beta}$ for $t \in [a,x]$ and $B(t) = (b-t)e^{-t\beta}$ for $t \in [x,b]$.

Case 1: $\beta > 0$. We have $B'(t) = -e^{-\beta t}(1+\beta(b-t))$. If $\beta > 0$, then $B'(t) \leqslant 0$ for all $t \in [x,b]$, thus, the supremum is attained at $t = x$. We have $A'(t) = e^{-\beta t}(1-\beta(t-a))$. The stationary point of A is $t_a = (a\beta+1)/\beta$.

Case 1a: $t_a \leqslant x$ $(a+\frac{1}{\beta} \leqslant x)$, we have $A''(t_a) = -\beta e^{-(a\beta+1)} \leqslant 0$; thus, the supremum is attained at $t = t_a$.

Case 1b: $t_a > x$ $(a+\frac{1}{\beta} > x)$, we have $1-\beta(t-a) > 1-\beta(x-a) > 0$, which implies that $A'(t) > 0$ for all $t \in [a,x]$. Thus, the supremum is attained at $t = x$.

Case 2: $\beta < 0$. We have $A'(t) \geqslant 0$ for all $t \in [a,x]$, thus, the supremum is attained at $t = x$. The stationary point of B is $t_b = (b\beta+1)/\beta$.

Case 2a: $t_b \geqslant x$ $(b+\frac{1}{\beta} \geqslant x)$, we have $B''(t_a) = \beta e^{-(b\beta+1)} \leqslant 0$; thus, the supremum is attained at $t = t_b$.

Case 2b: $t_b < x$ $(b+\frac{1}{\beta} < x)$, we have $0 < 1+\beta(b-x) < 1+\beta(t-x)$, which implies that $B'(t) < 0$ for all $t \in [x,b]$. Thus, the supremum is attained at $t = x$. This completes the proof of (13). If $\beta = 0$, then

$$\left|\frac{f(x)}{e^{ix\gamma}}(b-a) - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \leqslant \sup_{t\in[a,x]}(t-a)\|f'-i\gamma f\|_{[a,x],1} + \sup_{t\in[x,b]}(b-t)\|f'-i\gamma f\|_{[x,b],1} \\ \leqslant (x-a)\|f'-i\gamma f\|_{[a,x],1} + (b-x)\|f'-i\gamma f\|_{[x,b],1} \\ \leqslant (b-a)\|f'-i\gamma f\|_{[a,b],1}.$$

This completes the proof. □

2.2. Trapezoid type inequalities

The proofs for Theorems 8, 9, and Corollary 2 below are similar to Theorems 6, 7, and Corollary 1. We utilise the trapezoid identity for absolutely continuous function $g : [a,b] \to \mathbb{C}$

$$g(b)(b-x) + g(a)(x-a) - \int_a^b g(t)\,dt = \int_a^b (t-x)g'(t)\,dt, \quad x \in [a,b], \tag{15}$$

instead of the Montgomery identity. We omit the proofs.

THEOREM 8. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$, $\alpha=\beta+i\gamma\in\mathbb{C}$ and $1<p\leqslant\infty$. Let $1\leqslant q<\infty$ be a real number such that $\frac{1}{p}+\frac{1}{q}=1$. If $\beta\neq 0$, then*

$$\begin{aligned}&\left|\frac{f(b)}{e^{b\alpha}}(b-x)+\frac{f(a)}{e^{a\alpha}}(x-a)-\int_a^b\frac{f(t)}{e^{\alpha t}}\,dt\right|\\&\leqslant \Psi^-_{q,\alpha}(a,x)\|f'-\alpha f\|_{[a,x],p}+\Psi^+_{q,\alpha}(x,b)\|f'-\alpha f\|_{[x,b],p}\\&\leqslant\left[\Psi^-_{q,\alpha}(a,x)+\Psi^+_{q,\alpha}(x,b)\right]\|f'-\alpha f\|_{[a,b],p},\end{aligned}\tag{16}$$

for any $x\in[a,b]$. If $\beta=0$, then

$$\begin{aligned}&\left|\frac{f(b)}{e^{ib\gamma}}(b-x)+\frac{f(a)}{e^{ia\gamma}}(x-a)-\int_a^b\frac{f(t)}{e^{it\gamma}}\,dt\right|\\&\leqslant\frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\,\|f'-i\gamma f\|_{[a,x],p}+\frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\,\|f'-i\gamma f\|_{[x,b],p}\\&\leqslant\frac{(x-a)^{\frac{q+1}{q}}+(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\,\|f'-i\gamma f\|_{[a,b],p},\end{aligned}\tag{17}$$

for any $x\in[a,b]$. The inequalities in (17) are sharp.

COROLLARY 2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ and $\alpha=\beta+i\gamma\in\mathbb{C}$. If $\beta\neq0$, then we have*

$$\begin{aligned}&\left|\frac{f(b)}{e^{b\alpha}}(b-x)+\frac{f(a)}{e^{a\alpha}}(x-a)-\int_a^b\frac{f(t)}{e^{\alpha t}}\,dt\right|\\&\leqslant\frac{e^{-x\beta}+[(x-a)\beta-1]e^{-a\beta}}{\beta^2}\|f'-\alpha f\|_{[a,x],\infty}\\&\quad+\frac{e^{-x\beta}-[(b-x)\beta+1]e^{-b\beta}}{\beta^2}\|f'-\alpha f\|_{[x,b],\infty}\\&\leqslant\frac{1}{\beta^2}\left[2e^{-x\beta}+[(x-a)\beta-1]e^{-a\beta}-[(b-x)\beta+1]e^{-b\beta}\right]\|f'-\alpha f\|_{[a,b],\infty},\end{aligned}\tag{18}$$

for any $x\in[a,b]$. If $\beta=0$, then we have

$$\begin{aligned}&\left|\frac{f(b)}{e^{ib\gamma}}(b-x)+\frac{f(a)}{e^{ia\gamma}}(x-a)-\int_a^b\frac{f(t)}{e^{it\gamma}}\,dt\right|\\&\leqslant\frac{1}{2}\left[(x-a)^2\|f'-i\gamma f\|_{[a,x],\infty}+(b-x)^2\|f'-i\gamma f\|_{[x,b],\infty}\right]\\&\leqslant\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'-i\gamma f\|_{[a,b],\infty},\end{aligned}\tag{19}$$

for any $x\in[a,b]$. The constants $\frac{1}{2}$ and $\frac{1}{4}$ in (19) are sharp.

THEOREM 9. *Let* $f:[a,b]\to\mathbb{C}$ *be an absolutely continuous function on* $[a,b]$, $\alpha=\beta+i\gamma\in\mathbb{C}$. *If* $\beta\neq 0$, *then*

$$\left|\frac{f(b)}{e^{b\alpha}}(b-x)+\frac{f(a)}{e^{a\alpha}}(x-a)-\int_a^b\frac{f(t)}{e^{\alpha t}}\,dt\right| \tag{20}$$
$$\leqslant\begin{cases}(x-a)e^{-a\beta}\|f'-\alpha f\|_{[a,x],1}+\frac{1}{\beta}e^{-(x\beta+1)}\|f'-\alpha f\|_{[x,b],1}, & \beta>0\ \&\ x+\frac{1}{\beta}\geqslant a,\\ \frac{1}{-\beta}e^{-(x\beta+1)}\|f'-\alpha f\|_{[a,x],1}+(b-x)e^{-b\beta}\|f'-\alpha f\|_{[x,b],1}, & \beta<0\ \&\ x+\frac{1}{\beta}\leqslant b,\\ (x-a)e^{-a\beta}\|f'-\alpha f\|_{[a,x],1}+(b-x)e^{-b\beta}\|f'-\alpha f\|_{[x,b],1}, & otherwise,\end{cases}$$
$$\leqslant\begin{cases}\left[(x-a)e^{-a\beta}+\frac{1}{\beta}e^{-(x\beta+1)}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta>0\ \&\ x+\frac{1}{\beta}\geqslant a,\\ \left[\frac{1}{-\beta}e^{-(x\beta+1)}+(b-x)e^{-b\beta}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta<0\ \&\ x+\frac{1}{\beta}\leqslant b,\\ \left[(x-a)e^{-a\beta}+(b-x)e^{-b\beta}\right]\|f'-\alpha f\|_{[a,b],1}, & otherwise,\end{cases}$$

for $x\in[a,b]$. *If* $\beta=0$, *then*

$$\begin{aligned}&\left|\frac{f(b)}{e^{ib\gamma}}(b-x)+\frac{f(a)}{e^{ia\gamma}}(x-a)-\int_a^b\frac{f(t)}{e^{it\gamma}}\,dt\right|\\ &\leqslant(x-a)\|f'-i\gamma f\|_{[a,x],1}+(b-x)\|f'-i\gamma f\|_{[x,b],1}\\ &\leqslant(b-a)\|f'-i\gamma f\|_{[a,b],1},\end{aligned} \tag{21}$$

for $x\in[a,b]$.

REMARK 1. Note the similarity of the bounds in Corollaries 1 and 2 [also, Theorems 6 and 8, and similarly, Theorems 7 and 9]. The first set of upper bounds in (11) [(7) and (13), respectively] and (12) [(8) and (14), respectively] can be obtained by letting $a=x$, $x=b$ in the first term, and $x=a$, $b=x$ in the second term in (18) [(16) and (20), respectively] and (19) [(17) and (21), respectively].

3. Special cases of the main theorems and approximations of the Laplace and Fourier transforms

One may obtain simpler inequalities from the main theorems, by choosing $x=(a+b)/2$. These special cases are applied to approximate the Laplace and Fourier transforms. For inequalities involving the p-norms where $p\neq 1$, the choice of $x=(a+b)/2$ proves that the inequalities are sharp. We summarise the results in this section.

COROLLARY 3. *Let* $f:[a,b]\to\mathbb{C}$ *be an absolutely continuous function on* $[a,b]$, $\alpha=\beta+i\gamma\in\mathbb{C}$ *and* $1<p<\infty$. *Let* $q>1$ *be a real number such that* $\frac{1}{p}+\frac{1}{q}=1$. *If* $\beta\neq 0$, *then the following inequalities hold:*

$$\begin{aligned}\left|\frac{f\left(\frac{a+b}{2}\right)(b-a)}{e^{\frac{\alpha(a+b)}{2}}}-\int_a^b\frac{f(t)}{e^{\alpha t}}\,dt\right|&\leqslant\frac{e^{-a\beta}-[\frac{b-a}{2}\beta+1]e^{-\frac{a+b}{2}\beta}}{\beta^2}\|f'-\alpha f\|_{[a,\frac{a+b}{2}],\infty}\\ &\quad+\frac{e^{-b\beta}+[\frac{b-a}{2}\beta-1]e^{-\frac{a+b}{2}\beta}}{\beta^2}\|f'-\alpha f\|_{[\frac{a+b}{2},b],\infty}\\ &\leqslant\frac{1}{\beta^2}\left[e^{-a\beta}+e^{-b\beta}-2e^{-\frac{a+b}{2}\beta}\right]\|f'-\alpha f\|_{[a,b],\infty}.\end{aligned} \tag{22}$$

For the case of the p-norms ($1 < p < \infty$), we have:

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{e^{\frac{\alpha(a+b)}{2}}}(b-a)-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \tag{23}$$
$$\leqslant \Psi^{+}_{q,\alpha}\left(a,\frac{a+b}{2}\right)\|f'-\alpha f\|_{[a,\frac{a+b}{2}],p}+\Psi^{-}_{q,\alpha}\left(\frac{a+b}{2},b\right)\|f'-\alpha f\|_{[\frac{a+b}{2},b],p}$$
$$\leqslant \left[\Psi^{+}_{q,\alpha}\left(a,\frac{a+b}{2}\right)+\Psi^{-}_{q,\alpha}\left(\frac{a+b}{2},b\right)\right]\|f'-\alpha f\|_{[a,b],p}.$$

For the case of the 1-norm, we have:

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{e^{\frac{\alpha(a+b)}{2}}}(b-a)-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \tag{24}$$
$$\leqslant \begin{cases} \frac{1}{\beta}e^{-(a\beta+1)}\|f'-\alpha f\|_{[a,\frac{a+b}{2}],1}+\frac{b-a}{2}e^{-\frac{a+b}{2}\beta}\|f'-\alpha f\|_{[\frac{a+b}{2},b],1}, & \beta>0\ \&\ a+\frac{1}{\beta}\leqslant x,\\ \frac{b-a}{2}e^{-\frac{a+b}{2}\beta}\|f'-\alpha f\|_{[a,\frac{a+b}{2}],1}+\frac{1}{-\beta}e^{-(b\beta+1)}\|f'-\alpha f\|_{[\frac{a+b}{2},b],1}, & \beta<0\ \&\ b+\frac{1}{\beta}\geqslant x,\\ \frac{b-a}{2}e^{-\frac{a+b}{2}\beta}\left[\|f'-\alpha f\|_{[a,\frac{a+b}{2}],1}+\|f'-\alpha f\|_{[\frac{a+b}{2},b],1}\right], & \textit{otherwise}, \end{cases}$$
$$\leqslant \begin{cases} \left[\frac{1}{\beta}e^{-(a\beta+1)}+\frac{b-a}{2}e^{-\frac{a+b}{2}\beta}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta>0\ \&\ a+\frac{1}{\beta}\leqslant x,\\ \left[\frac{b-a}{2}e^{-\frac{a+b}{2}\beta}+\frac{1}{-\beta}e^{-(b\beta+1)}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta<0\ \&\ b+\frac{1}{\beta}\geqslant x,\\ (b-a)e^{-\frac{a+b}{2}\beta}\|f'-\alpha f\|_{[a,b],1}, & \textit{otherwise}, \end{cases}$$

If $\beta = 0$, then the following inequalities hold:

$$\left|\frac{f\left(\frac{a+b}{2}\right)(b-a)}{e^{i\frac{a+b}{2}\gamma}}-\int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \leqslant \frac{1}{8}(b-a)^2\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],\infty}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],\infty}\right]$$
$$\leqslant \frac{1}{4}(b-a)^2\|f'-i\gamma f\|_{[a,b],\infty}. \tag{25}$$

For the case of the p-norms ($1 < p < \infty$), we have:

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{e^{ix\gamma}}(b-a)-\int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \leqslant \frac{(b-a)^{\frac{q+1}{q}}}{2(2q+2)^{\frac{1}{q}}}\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],p}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],p}\right]$$
$$\leqslant \frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}\,\|f'-i\gamma f\|_{[a,b],p}. \tag{26}$$

For the case of the 1-norm, we have:

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{e^{i\frac{a+b}{2}\gamma}}(b-a)-\int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \leqslant \frac{b-a}{2}\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],1}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],1}\right]$$
$$\leqslant (b-a)\|f'-i\gamma f\|_{[a,b],1}. \tag{27}$$

PROPOSITION 1. *Inequalities* (25) *and* (26) *are sharp.*

Proof. We show that the constants $\frac{1}{8}$ and $\frac{1}{4}$ in (25) are sharp. We assume that the inequalities hold for $A,B>0$ instead of $\frac{1}{8}$ and $\frac{1}{4}$, respectively:

$$\left|\frac{f\left(\frac{a+b}{2}\right)(b-a)}{e^{i\frac{a+b}{2}\gamma}}-\int_a^b\frac{f(t)}{e^{it\gamma}}dt\right|\leqslant A(b-a)^2\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],\infty}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],\infty}\right]$$
$$\leqslant B(b-a)^2\|f'-i\gamma f\|_{[a,b],\infty}.$$

Let $\gamma=0$ and choose $f(x)=|x-\frac{a+b}{2}|$ on $[a,b]$. Thus we have $\frac{(b-a)^2}{4}\leqslant 2A(b-a)^2\leqslant B(b-a)^2$; which yields $A\geqslant\frac{1}{8}$ and $B\geqslant\frac{1}{4}$. It implies that the constants $\frac{1}{2}$ and $\frac{1}{4}$ in (12) are sharp.

We show that the inequalities in (26) are sharp. We assume the inequalities hold for $C,D>0$ instead of $\frac{1}{2}$ and 1, respectively:

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{e^{ix\gamma}}(b-a)-\int_a^b\frac{f(t)}{e^{it\gamma}}dt\right|\leqslant C\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],p}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],p}\right]$$
$$\leqslant D\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}\|f'-i\gamma f\|_{[a,b],p}.$$

Let $\gamma=0$ and take $f(x)=|x-\frac{a+b}{2}|$ on $[a,b]$, and we now have

$$\frac{(b-a)^2}{4}\leqslant 2C\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}\leqslant D\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}.$$

Take $q\to 1$, then we have $\frac{(b-a)^2}{4}\leqslant 2C\frac{(b-a)^2}{4}\leqslant D\frac{(b-a)^2}{4}$, which yields $C\geqslant\frac{1}{2}$ and $D\geqslant 1$. It implies that the inequalities in (8) are sharp. □

Let $f:[a,b]\to\mathbb{K}$ ($\mathbb{K}=\mathbb{C},\mathbb{R}$) be a Lebesgue integrable mapping defined on the finite interval $[a,b]$. Let $\mathscr{L}(f)$ and $\mathscr{F}(f)$ be their finite Laplace and Fourier transforms, respectively, defined by

$$\mathscr{L}(f)(\alpha):=\int_a^b f(s)e^{-\alpha s}ds,\quad \alpha\in\mathbb{C},$$
$$\mathscr{F}(f)(t):=\int_a^b f(s)e^{-2\pi its}ds,\quad t\in\mathbb{R}.$$

REMARK 2. (Laplace transform approximations) By rewriting $\int_a^b f(s)e^{-\alpha t}dt=\mathscr{L}(f)(\alpha)$ in (22), (23), and (24), we obtain error bounds in terms of the p-norms ($1\leqslant p\leqslant\infty$), for the approximation of $\mathscr{L}(f)(\alpha)$ by $f(\frac{a+b}{2})(b-a)e^{-\frac{\alpha(a+b)}{2}}$.

REMARK 3. (Fourier transform approximations) Let $u \in \mathbb{R}$. Choose $\gamma = 2\pi u$ in (25), (26), and (27). By rewriting $\int_a^b f(t)e^{-2\pi iut}\,dt = \mathscr{F}(f)(u)$, we obtain error bounds in terms of the p-norms $(1 \leqslant p \leqslant \infty)$, for the approximation of $\mathscr{F}(f)(t)$ by

$$f\left(\frac{a+b}{2}\right)(b-a)e^{-i\pi(a+b)t}.$$

COROLLARY 4. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$, $\alpha=\beta+i\gamma\in\mathbb{C}$ and $1<p<\infty$. Let $q>1$ be a real number such that $\frac{1}{p}+\frac{1}{q}=1$.*

If $\beta\neq 0$, then the following inequalities hold:

$$\begin{aligned}
&\left|\frac{b-a}{2}\left[\frac{f(a)}{e^{a\alpha}}+\frac{f(b)}{e^{b\alpha}}\right]-\int_a^b\frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leqslant \frac{e^{-\frac{a+b}{2}\beta}+[\frac{b-a}{2}\beta-1]e^{-a\beta}}{\beta^2}\|f'-\alpha f\|_{[a,\frac{a+b}{2}],\infty} \\
&\quad+\frac{e^{-\frac{a+b}{2}\beta}-[\frac{b-a}{2}\beta+1]e^{-b\beta}}{\beta^2}\|f'-\alpha f\|_{[\frac{a+b}{2},b],\infty} \\
&\leqslant \frac{1}{\beta^2}\left[2e^{-\frac{a+b}{2}\beta}-e^{-a\beta}-e^{-b\beta}+\frac{b-a}{2}\beta\left(e^{-a\beta}-e^{-b\beta}\right)\right]\|f'-\alpha f\|_{[a,b],\infty}.
\end{aligned} \tag{28}$$

For the case of the p-norms ($1<p<\infty$), we have

$$\begin{aligned}
&\left|\frac{b-a}{2}\left[\frac{f(a)}{e^{a\alpha}}+\frac{f(b)}{e^{b\alpha}}\right]-\int_a^b\frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leqslant \Psi^-_{q,\alpha}\left(a,\frac{a+b}{2}\right)\|f'-\alpha f\|_{[a,\frac{a+b}{2}],p}+\Psi^+_{q,\alpha}\left(\frac{a+b}{2},b\right)\|f'-\alpha f\|_{[\frac{a+b}{2},b],p} \\
&\leqslant \left[\Psi^-_{q,\alpha}\left(a,\frac{a+b}{2}\right)+\Psi^+_{q,\alpha}\left(\frac{a+b}{2},b\right)\right]\|f'-\alpha f\|_{[a,b],p}.
\end{aligned} \tag{29}$$

For the case of the 1-norm, we have

$$\begin{aligned}
&\left|\frac{b-a}{2}\left[\frac{f(a)}{e^{a\alpha}}+\frac{f(b)}{e^{b\alpha}}\right]-\int_a^b\frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leqslant \begin{cases}
\frac{b-a}{2}e^{-a\beta}\|f'-\alpha f\|_{[a,\frac{a+b}{2}],1}+\frac{1}{\beta}e^{-(\frac{a+b}{2}\beta+1)}\|f'-\alpha f\|_{[\frac{a+b}{2},b],1}, & \beta>0\ \&\ x+\frac{1}{\beta}\geqslant a, \\
\frac{1}{-\beta}e^{-(\frac{a+b}{2}\beta+1)}\|f'-\alpha f\|_{[a,\frac{a+b}{2}],1}+\frac{b-a}{2}e^{-b\beta}\|f'-\alpha f\|_{[\frac{a+b}{2},b],1}, & \beta<0\ \&\ x+\frac{1}{\beta}\leqslant b, \\
\frac{b-a}{2}\left[e^{-a\beta}\|f'-\alpha f\|_{[a,\frac{a+b}{2}],1}+e^{-b\beta}\|f'-\alpha f\|_{[\frac{a+b}{2},b],1}\right], & \textit{otherwise,}
\end{cases} \\
&\leqslant \begin{cases}
\left[\frac{b-a}{2}e^{-a\beta}+\frac{1}{\beta}e^{-(\frac{a+b}{2}\beta+1)}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta>0\ \&\ x+\frac{1}{\beta}\geqslant a, \\
\left[\frac{1}{-\beta}e^{-(\frac{a+b}{2}\beta+1)}+\frac{b-a}{2}e^{-b\beta}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta<0\ \&\ x+\frac{1}{\beta}\leqslant b, \\
\frac{b-a}{2}\left[e^{-a\beta}+e^{-b\beta}\right]\|f'-\alpha f\|_{[a,b],1}, & \textit{otherwise,}
\end{cases}
\end{aligned} \tag{30}$$

If $\beta = 0$, then the following inequalities hold:

$$\begin{aligned}
&\left|\frac{b-a}{2}\left[\frac{f(a)}{e^{ia\gamma}}+\frac{f(b)}{e^{ib\gamma}}\right]-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leqslant \frac{1}{8}(b-a)^2\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],\infty}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],\infty}\right] \\
&\leqslant \frac{1}{4}(b-a)^2\|f'-i\gamma f\|_{[a,b],\infty}.
\end{aligned} \tag{31}$$

For the case of the p-norms ($1 < p < \infty$), we have

$$\begin{aligned}
&\left|\frac{b-a}{2}\left[\frac{f(a)}{e^{ia\gamma}}+\frac{f(b)}{e^{ib\gamma}}\right]-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leqslant \frac{(b-a)^{\frac{q+1}{q}}}{2(2q+2)^{\frac{1}{q}}}\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],p}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],p}\right] \\
&\leqslant \frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}\|f'-i\gamma f\|_{[a,b],p}.
\end{aligned} \tag{32}$$

For the case of the 1-norm, we have

$$\begin{aligned}
&\left|\frac{b-a}{2}\left[\frac{f(a)}{e^{ia\gamma}}+\frac{f(b)}{e^{ib\gamma}}\right]-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leqslant \frac{b-a}{2}\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],1}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],1}\right] \\
&\leqslant (b-a)\|f'-i\gamma f\|_{[a,b],1}.
\end{aligned} \tag{33}$$

PROPOSITION 2. *The inequalities* (31) *and* (32) *are sharp.*

The proof follows similarly to that of Proposition 1; and we omit the proof. It follows from Proposition 2 that inequalities (19), and (17) are sharp.

REMARK 4. (Laplace transform approximations) By rewriting $\int_a^b f(s)e^{-\alpha t}\,dt = \mathscr{L}(f)(\alpha)$ in (28), (29), and (30), we obtain error bounds in terms of the p-norms $(1 \leqslant p \leqslant \infty)$, for the approximation of $\mathscr{L}(f)(\alpha)$ by $\frac{b-a}{2}\left[\frac{f(a)}{e^{a\alpha}}+\frac{f(b)}{e^{b\alpha}}\right]$.

REMARK 5. (Fourier transform approximations) Let $u \in \mathbb{R}$. Choose $\gamma = 2\pi u$ in (31), (32) and (33). By rewriting $\int_a^b f(t)e^{-2\pi i u t}\,dt = \mathscr{F}(f)(u)$, we obtain error bounds in terms of the p-norms $(1 \leqslant p \leqslant \infty)$, for the approximation of $\mathscr{F}(f)(t)$ by

$$\frac{b-a}{2}\left[\frac{f(a)}{e^{2a\pi t i}}+\frac{f(b)}{e^{2b\pi t i}}\right].$$

4. Some new and refined Ostrowski and trapezoid type inequalities

Using the results in Section 2, we obtain inequalities of Ostrowski and trapezoid type; and we present the result in the following subsections.

4.1. Refinements of Ostrowski's inequalities

If $\alpha = 0$ in Corollary 1, Theorems 6 and 7, respectively, then we have refinements for the Ostrowski inequality, as follows:

$$\left| f(x)(b-a) - \int_a^b f(t)\,dt \right|$$
$$\leqslant \begin{cases} \frac{1}{2}\left[(x-a)^2\|f'\|_{[a,x],\infty} + (b-x)^2\|f'\|_{[x,b],\infty}\right], \\ \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,x],p} + \frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[x,b],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ (x-a)\|f'\|_{[a,x],1} + (b-x)\|f'\|_{[x,b],1}, \end{cases} \tag{34}$$
$$\leqslant \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty}, \\ \frac{(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ (b-a)\|f'\|_{[a,b],1}, \end{cases} \tag{35}$$

for any $x \in [a,b]$. The constants in cases 1 and 2 in (34) and (35) are sharp (cf. Proposition 1).

4.2. New Ostrowski type inequalities

Let $h_\alpha(t) = e^{\alpha t}$ for $t \in [a,b]$. If $f(t) = g(t)h_\alpha(t) = g(t)e^{\alpha t}$ in Theorem 6, Corollary 1, and Theorem 7, then we have the Ostrowski inequalities: If $\beta \neq 0$, we have

$$\begin{aligned} \left| g(x)(b-a) - \int_a^b g(t)\,dt \right| &\leqslant \Psi^+_{q,\alpha}(a,x)\|g'h_\alpha\|_{[a,x],p} + \Psi^-_{q,\alpha}(x,b)\|g'h_\alpha\|_{[x,b],p} \\ &\leqslant \left[\Psi^+_{q,\alpha}(a,x) + \Psi^-_{q,\alpha}(x,b)\right]\|g'h_\alpha\|_{[a,b],p}, \end{aligned} \tag{36}$$

where $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$,

$$\left| g(x)(b-a) - \int_a^b g(t)\,dt \right| \tag{37}$$
$$\leqslant \frac{e^{-a\beta} - [(x-a)\beta+1]e^{-x\beta}}{\beta^2}\|g'h_\alpha\|_{[a,x],\infty} + \frac{e^{-b\beta} + [(b-x)\beta-1]e^{-x\beta}}{\beta^2}\|g'h_\alpha\|_{[x,b],\infty}$$
$$\leqslant \frac{1}{\beta^2}\left[e^{-a\beta} + e^{-b\beta} + 2\left(\left(\frac{a+b}{2} - x\right)\beta - 1\right)e^{-x\beta}\right]\|g'h_\alpha\|_{[a,b],\infty},$$

and

$$\left|g(x)(b-a)-\int_a^b g(t)\,dt\right| \tag{38}$$

$$\leqslant \begin{cases} \frac{1}{\beta}e^{-(a\beta+1)}\|g'h_\alpha\|_{[a,x],1}+(b-x)e^{-x\beta}\|g'h_\alpha\|_{[x,b],1}, & \beta>0\ \&\ a+\frac{1}{\beta}\leqslant x,\\ (x-a)e^{-x\beta}\|g'h_\alpha\|_{[a,x],1}+\frac{1}{-\beta}e^{-(b\beta+1)}\|g'h_\alpha\|_{[x,b],1}, & \beta<0\ \&\ b+\frac{1}{\beta}\geqslant x,\\ (x-a)e^{-x\beta}\|g'h_\alpha\|_{[a,x],1}+(b-x)e^{-x\beta}\|g'h_\alpha\|_{[x,b],1}, & \text{otherwise}, \end{cases}$$

$$\leqslant \begin{cases} \left[\frac{1}{\beta}e^{-(a\beta+1)}+(b-x)e^{-x\beta}\right]\|g'h_\alpha\|_{[a,b],1}, & \beta>0\ \&\ a+\frac{1}{\beta}\leqslant x,\\ \left[(x-a)e^{-x\beta}+\frac{1}{-\beta}e^{-(b\beta+1)}\right]\|g'h_\alpha\|_{[a,b],1}, & \beta<0\ \&\ b+\frac{1}{\beta}\geqslant x,\\ (b-a)e^{-x\beta}\|g'h_\alpha\|_{[a,b],1}, & \text{otherwise}, \end{cases}$$

for $x\in[a,b]$. If $\beta=0$, then we have

$$\left|g(x)(b-a)-\int_a^b g(t)\,dt\right|$$

$$\leqslant \begin{cases} \frac{1}{2}\left[(x-a)^2\|g'h_\alpha\|_{[a,x],\infty}+(b-x)^2\|g'h_\alpha\|_{[x,b],\infty}\right],\\ \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|g'h_\alpha\|_{[a,x],p}+\frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|g'h_\alpha\|_{[x,b],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ (x-a)\|g'h_\alpha\|_{[a,x],1}+(b-x)\|g'h_\alpha\|_{[x,b],1}, \end{cases} \tag{39}$$

$$\leqslant \begin{cases} \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|g'h_\alpha\|_{[a,b],\infty},\\ \frac{(x-a)^{\frac{q+1}{q}}+(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|g'h_\alpha\|_{[a,b],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ (b-a)\|g'h_\alpha\|_{[a,b],1}, \end{cases} \tag{40}$$

for any $x\in[a,b]$.

4.3. Refinements of the trapezoid inequalities

If $\alpha=0$ in Corollary 2, Theorems 8 and 9, respectively, then we have refinements for the trapezoid inequality as follows:

$$\left|f(a)(x-a)+f(b)(b-x)-\int_a^b f(t)\,dt\right|$$

$$\leqslant \begin{cases} \frac{1}{2}\left[(x-a)^2\|f'\|_{[a,x],\infty}+(b-x)^2\|f'\|_{[x,b],\infty}\right],\\ \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,x],p}+\frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[x,b],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ (x-a)\|f'\|_{[a,x],1}+(b-x)\|f'\|_{[x,b],1}, \end{cases} \tag{41}$$

$$\leqslant \begin{cases} \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty},\\ \frac{(x-a)^{\frac{q+1}{q}}+(b-x)^{\frac{q+1}{q}}}{(+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ (b-a)\|f'\|_{[a,b],1}, \end{cases} \tag{42}$$

for any $x \in [a,b]$. The constants in cases 1 and 2 in (41) and (42) are sharp (cf. Proposition 2).

4.4. New trapezoid type inequalities

Let $h_\alpha(t) = e^{\alpha t}$ for $t \in [a,b]$. If $f(t) = g(t)h_\alpha(t) = g(t)e^{\alpha t}$ in Theorem 8, Corollary 2, and Theorem 9, then we have the trapezoid inequalities: If $\beta \neq 0$, then we have

$$\begin{aligned} &\left| g(a)(x-a) + g(b)(b-x) - \int_a^b g(t)\,dt \right| \\ &\leqslant \Psi^-_{q,\alpha}(a,x)\|g'h_\alpha\|_{[a,x],p} + \Psi^+_{q,\alpha}(x,b)\|g'h_\alpha\|_{[x,b],p} \\ &\leqslant \left[\Psi^-_{q,\alpha}(a,x) + \Psi^+_{q,\alpha}(x,b)\right] \|g'h_\alpha\|_{[a,b],p}, \end{aligned} \tag{43}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} &\left| g(a)(x-a) + g(b)(b-x) - \int_a^b g(t)\,dt \right| \\ &\leqslant \frac{e^{-x\beta} + [(x-a)\beta - 1]e^{-a\beta}}{\beta^2}\|g'h_\alpha|_{[a,x],\infty} + \frac{e^{-x\beta} - [(b-x)\beta + 1]e^{-b\beta}}{\beta^2}\|g'h_\alpha\|_{[x,b],\infty} \\ &\leqslant \frac{1}{\beta^2}\left[2e^{-x\beta} + [(x-a)\beta - 1]e^{-a\beta} - [(b-x)\beta + 1]e^{-b\beta}\right] \|g'h_\alpha\|_{[a,b],\infty}, \end{aligned} \tag{44}$$

and

$$\begin{aligned} &\left| g(a)(x-a) + g(b)(b-x) - \int_a^b g(t)\,dt \right| \\ &\leqslant \begin{cases} (x-a)e^{-a\beta}\|g'h_\alpha\|_{[a,x],1} + \frac{1}{\beta}e^{-(x\beta+1)}\|g'h_\alpha\|_{[x,b],1}, & \beta > 0 \;\&\; x + \frac{1}{\beta} \geqslant a, \\ \frac{1}{-\beta}e^{-(x\beta+1)}\|g'h_\alpha\|_{[a,x],1} + (b-x)e^{-b\beta}\|g'h_\alpha\|_{[x,b],1}, & \beta < 0 \;\&\; x + \frac{1}{\beta} \leqslant b, \\ (x-a)e^{-a\beta}\|g'h_\alpha\|_{[a,x],1} + (b-x)e^{-b\beta}\|g'h_\alpha\|_{[x,b],1}, & \text{otherwise}, \end{cases} \\ &\leqslant \begin{cases} \left[(x-a)e^{-a\beta} + \frac{1}{\beta}e^{-(x\beta+1)}\right]\|g'h_\alpha\|_{[a,b],1}, & \beta > 0 \;\&\; x + \frac{1}{\beta} \geqslant a, \\ \left[\frac{1}{-\beta}e^{-(x\beta+1)} + (b-x)e^{-b\beta}\right]\|g'h_\alpha\|_{[a,b],1}, & \beta < 0 \;\&\; x + \frac{1}{\beta} \leqslant b, \\ \left[(x-a)e^{-a\beta} + (b-x)e^{-b\beta}\right]\|g'h_\alpha\|_{[a,b],1}, & \text{otherwise}, \end{cases} \end{aligned} \tag{45}$$

for $x \in [a,b]$. If $\beta = 0$, then

$$\begin{aligned} &\left| g(a)(x-a) + g(b)(b-x) - \int_a^b g(t)\,dt \right| \\ &\leqslant \begin{cases} \frac{1}{2}\left[(x-a)^2\|g'h_\alpha\|_{[a,x],\infty} + (b-x)^2\|g'h_\alpha\|_{[x,b],\infty}\right], \\ \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|g'h_\alpha\|_{[a,x],p} + \frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|g'h_\alpha|_{[x,b],p}, & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ (x-a)\|g'h_\alpha\|_{[a,x],1} + (b-x)\|g'h_\alpha\|_{[x,b],1}, \end{cases} \end{aligned} \tag{46}$$

$$\leqslant \begin{cases} \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|g'h_\alpha\|_{[a,b],\infty}, \\ \frac{(x-a)^{\frac{q+1}{q}}+(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|g'h_\alpha\|_{[a,b],p}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ (b-a)\|g'h_\alpha\|_{[a,b],1}, \end{cases} \tag{47}$$

for $x \in [a,b]$.

Acknowledgement. The authors would like to thank the anonymous referees for valuable suggestions that have been incorporated in the final version of the manuscript.

REFERENCES

[1] A. M. ACU AND F. D. SOFONEA, *On an inequality of Ostrowski type*, J. Sci. Arts, no. **3** (16) (2011), 281–287.

[2] A. M. ACU, A. BABOŞ AND F. D. SOFONEA, *The mean value theorems and inequalities of Ostrowski type*, Sci. Stud. Res. Ser. Math. Inform. **21** (1) (2011), 5–16.

[3] E. O. BRIGHAM, *The Fast Fourier Transform and its Applications*, Englewood Cliffs, NJ: Prentice-Hall Inc., 1988.

[4] N. S. BARNETT AND S. S. DRAGOMIR, *An approximation for the Fourier transform of absolutely continuous mappings*, Proc. 4th Int. Conf. on Modelling and Simulation. Victoria University (2002), 351–355. RGMIA Res. Rep. Coll. **5** (Supplement) (2002), Article 33.

[5] N. S. BARNETT, S. S. DRAGOMIR AND G. HANNA, *Error estimates for approximating the Fourier transform of functions of bounded variation*, RGMIA Res. Rep. Coll. **7** (1) (2004), Article 11.

[6] P. CERONE, *On relationships between Ostrowski, trapezoidal and Chebychev identities and inequalities*, Soochow J. Math. **28** (3) (2002), 311–328.

[7] P. CERONE AND S. S. DRAGOMIR, *Trapezoidal-type rules from an inequalities point of view, Handbook of analytic-computational methods in applied mathematics*, Chapman & Hall/CRC, Boca Raton, FL, 2000, 65–134.

[8] P. CERONE, S. S. DRAGOMIR AND E. KIKIANTY, *Ostrowski and Trapezoid Type Inequalities Related to Pompeiu's Mean Value Theorem*, J. Math. Inequal. **9** (3) (2015), 739–762.

[9] S. S. DRAGOMIR, *An inequality of Ostrowski type via Pompeiu's mean value theorem*, J. Inequal. Pure Appl. Math. **6** (3) (2005), Article 83.

[10] S. S. DRAGOMIR, *Another Ostrowski type inequality via Pompeiu's mean value theorem*, General Mathematics **21** (3) (2013), 3–15

[11] S. S. DRAGOMIR, *Inequalities of Pompeiu's type for absolutely continuous functions with applications to Ostrowski's inequality*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **30** (1) (2014), 43–52.

[12] S. S. DRAGOMIR, *Exponential Pompeiu's type inequalities with applications to Ostrowski's inequality*, Acta Math. Univ. Comenian. (N.S.) **84** (1) (2015), 39–50.

[13] S. S. DRAGOMIR, Y. J. CHO AND S. S. KIM, *An approximation for the Fourier transform of Lebesgue integrable mappings*, Fixed point Theory and Applications **4**. Y. J. Cho, J. K. Kim, and S. M. Kong (Eds.), Nova Science Publishers Inc., 2003, 67–76.

[14] S. S. DRAGOMIR, G. HANNA AND J. ROUMELIOTIS, *A reverse of the Cauchy-Bunyakovsky-Schwarz integral inequality for complex-valued functions and applications for Fourier transform*, Bull. Korean Math. Soc., **42** (4) (2005), 725–738.

[15] S. S. DRAGOMIR AND T. M. RASSIAS, *Generalisations of the Ostrowski inequality and applications*, Ostrowski type inequalities and applications in numerical integration. Kluwer Acad. Publ., Dordrecht, 2002, 1–63.

[16] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Inequalities involving functions and their integrals and derivatives*, Mathematics and its Applications (East European Series) **53**. Kluwer Academic Publishers Group, Dordrecht, 1991.

[17] A. OSTROWSKI, *Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Helv. **10** (1938), 226–227.

[18] J. PEČARIĆ AND Š. UNGAR, *On an inequality of Ostrowski type*, J. Ineq. Pure Appl. Math.**7** (4) (2006), Article 151.

[19] D. POMPEIU, *Sur une proposition analogue au théorème des accroissements finis*, Mathematica, Timişoara **22** (1946), 143–146.

[20] E. C. POPA, *An inequality of Ostrowski type via a mean value theorem*, Gen. Math. **15** (1) (2007), 93–100.

[21] P. K. SAHOO AND T. RIEDEL, *Mean Value Theorems and Functional Equations*, World Scientific, Singapore, New Jersey, London, Hong Kong, 2000.

(Received January 7, 2015)

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