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THE JENSEN AND HERMITE–HADAMARD INEQUALITY ON THE TRIANGLE

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Abstract. We study the functional forms of the most important inequalities concerning convex functions on the triangle. Our intension is to construct the functional form which implies the integral and discrete form of the Jensen inequality, the Fejér, and so the Hermite-Hadamard inequality. To reach this goal, we combine features of positive linear functionals and convex functions.

1. Introduction

1.1. Objectives

The first objective of the article is to promote the inequality for convex functions on the triangle that contains the Jensen, Fejér and Hermite-Hadamard inequality. The second objective is to refine the resulting inequality.

As is known, one proven way to achieve the first objective is to include positive linear functionals. The significant results concerning the application of positive functionals to convex analysis were obtained by Jessen (see [7] and [8]) for one variable convex functions, and McShane (see [14]) for multivariate convex functions.

The article also has a slightly wider scope, because the methods applied to the triangle can be transferred to higher dimensions.

1.2. Convex functions on the triangle

The basic structure which is used in the article is a real linear space $\mathbb{S}$. Let us remind the initial notions of convexity upgrading the signification of the space $\mathbb{S}$.

A set $C \subseteq \mathbb{S}$ is said to be convex if the inclusion

$$\alpha x + \beta y \in C \tag{1}$$

holds for all points $x, y \in C$ and all coefficients $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta = 1$. The sum $\alpha x + \beta y$ with the above coefficients is called convex combination.

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A function $f: C \to \mathbb{R}$ is said to be convex if the inequality

$$f(\alpha x+\beta y) \leqslant \alpha f(x)+\beta f(y) \tag{2}$$

holds for all convex combinations $\alpha x+\beta y$ of points $x,y \in C$.

The triangle with vertices $a,b,c \in \mathbb{R}^2$ that do not belong to the same line is used permanently throughout the article. It will be marked by $\triangle$ or abc, and its interior will be marked by $\triangle^o$. Each point $x \in \triangle$ can be represented by the unique trinomial convex combination

$$x=\alpha_x a+\beta_x b+\gamma_x c. \tag{3}$$

The coefficients can be expressed by the ratio of areas

$$\alpha_x=\frac{\operatorname{ar}(xbc)}{\operatorname{ar}(abc)}, \qquad \beta_x=\frac{\operatorname{ar}(xac)}{\operatorname{ar}(abc)}, \qquad \gamma_x=\frac{\operatorname{ar}(xab)}{\operatorname{ar}(abc)}, \tag{4}$$

and thus by the ratio of determinants

$$\alpha_x=\frac{\begin{vmatrix} x_1 & x_2 & 1\\ b_1 & b_2 & 1\\ c_1 & c_2 & 1\end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & 1\\ b_1 & b_2 & 1\\ c_1 & c_2 & 1\end{vmatrix}}, \qquad \beta_x=-\frac{\begin{vmatrix} x_1 & x_2 & 1\\ a_1 & a_2 & 1\\ c_1 & c_2 & 1\end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & 1\\ b_1 & b_2 & 1\\ c_1 & c_2 & 1\end{vmatrix}}, \qquad \gamma_x=\frac{\begin{vmatrix} x_1 & x_2 & 1\\ a_1 & a_2 & 1\\ b_1 & b_2 & 1\end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & 1\\ b_1 & b_2 & 1\\ c_1 & c_2 & 1\end{vmatrix}}. \tag{5}$$

Let $f: \triangle \to \mathbb{R}$ be a convex function. The secant plane of f passes through the respective graph points of a, b and c, and its equation is

$$f_{abc}^{\mathrm{sec}}(x)=\frac{\operatorname{ar}(xbc)}{\operatorname{ar}(abc)}f(a)+\frac{\operatorname{ar}(xac)}{\operatorname{ar}(abc)}f(b)+\frac{\operatorname{ar}(xab)}{\operatorname{ar}(abc)}f(c). \tag{6}$$

Let $d \in \triangle^o$ be a point of the triangle interior. The support planes of f at d pass through the respective graph point of d, and their equations depend on the pairs of slope coefficients $\kappa_1 \in [f_1'(d-),f_1'(d+)]$ and $\kappa_2 \in [f_2'(d-),f_2'(d+)]$. For the specified pair of coefficients κ_1 and κ_2, the corresponding equation is

$$f_d^{\mathrm{sup}}(x_1,x_2)=\kappa_1(x_1-d_1)+\kappa_2(x_2-d_2)+f(d_1,d_2). \tag{7}$$

The support-secant plane inequality

$$f_d^{\mathrm{sup}}(x) \leqslant f(x) \leqslant f_{abc}^{\mathrm{sec}}(x) \tag{8}$$

holds for every $x \in \triangle$.

1.3. Positive functionals on the space of functions

Let S be a nonempty set, and let $\mathbb{F}=\mathbb{F}(S)$ be a subspace of the linear space of all real functions on the domain S. We assume that the space $\mathbb{F}$ contains the unit function

u defined by $u(s)=1$ for every $s\in S$. Then the space $\mathbb{F}$ contains every real constant κ within the meaning of $\kappa=\kappa u$, and $\mathbb{F}$ contains every composite function $f(g_1,g_2)$ of an affine function $f:\mathbb{R}^2\to\mathbb{R}$ and a pair of functions $g_1,g_2\in\mathbb{F}$. Specifically, using the equation $f(x_1,x_2)=\kappa_1x_1+\kappa_2x_2+\kappa_3$, we have the composition

$$f(g_1,g_2)=\kappa_1g_1+\kappa_2g_2+\kappa_3u \tag{9}$$

which belongs to the space $\mathbb{F}$.

We will use linear functionals on the space of real functions. Let $\mathbb{L}=\mathbb{L}(\mathbb{F}(S))$ be the space of all linear functionals on the space $\mathbb{F}(S)$. A functional $L\in\mathbb{L}$ is said to be positive (nonnegative) if the inequality $L(g)\geqslant 0$ holds for every nonnegative function $g\in\mathbb{F}$. If $g_1,g_2\in\mathbb{F}$ are functions such that $g_1(s)\leqslant g_2(s)$ for every $s\in S$, then a positive functional L satisfies the inequality

$$L(g_1)\leqslant L(g_2). \tag{10}$$

A functional $L\in\mathbb{L}$ is said to be unital (normalized) if $L(u)=1$. Such functional has the property $L(\kappa u)=\kappa$ for every real constant κ.

For more details on positive linear functionals and related topics, we recommend an interesting book of functional analysis in [1].

2. Main results

We start with two initial lemmas as a basis for our research.

The first lemma provides a basic inclusion relating to the image of a pair of functions $g_1,g_2\in\mathbb{F}$ and a positive unital functional $L\in\mathbb{L}$. The proof of lemma includes a convex analytics through the application of convex combinations.

LEMMA 1. *Let $g_1,g_2\in\mathbb{F}$ be functions such that $(g_1(s),g_2(s))\in\triangle$ for every point $s\in S$.*

Then each positive unital functional $L\in\mathbb{L}$ satisfies the inclusion

$$(L(g_1),L(g_2))\in\triangle. \tag{11}$$

Proof. Taking a point s in S, we get the plane point $(g_1(s),g_2(s))$ in $\triangle$, and its unique convex combination

$$(g_1(s),g_2(s))=\alpha(s)(a_1,a_2)+\beta(s)(b_1,b_2)+\gamma(s)(c_1,c_2). \tag{12}$$

Using equations in (5), we can determine functions α, β and γ showing that they belong to $\mathbb{F}$. For example, we have $\alpha(s)=\alpha_1g_1(s)+\alpha_2g_2(s)+\alpha_3$. Since the functional L is positive, the numbers $L(\alpha)$, $L(\beta)$ and $L(\gamma)$ are nonnegative, and since $\alpha(s)+\beta(s)+\gamma(s)=u(s)$, it follows that $L(\alpha)+L(\beta)+L(\gamma)=1$.

Acting with the functional L to each coordinate of equation in (12), we obtain the convex combination

$$(L(g_1),L(g_2))=L(\alpha)(a_1,a_2)+L(\beta)(b_1,b_2)+L(\gamma)(c_1,c_2) \tag{13}$$

ensuring that the point $(L(g_1), L(g_2))$ belongs to the triangle $\triangle$. $\square$

Using hyperplanes, McShane proved that the inclusion in (11) generally applies to closed convex sets in $\mathbb{R}^n$. That part of his work (see [14]) he called the geometric formulation of Jensen's inequality.

The second lemma provides a basic equality relating to the composition of an affine function and a unital functional.

LEMMA 2. *Let $g_1, g_2 \in \mathbb{F}$ be functions, and let $L \in \mathbb{L}$ be a unital functional. Then each affine function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the equality*

$$f\big(L(g_1), L(g_2)\big) = L\big(f(g_1, g_2)\big). \tag{14}$$

Proof. Using the affine equation $f(x_1, x_2) = \kappa_1 x_1 + \kappa_2 x_2 + \kappa_3$, and applying the unital property of L, we obtain

$$\begin{aligned} f\big(L(g_1), L(g_2)\big) &= \kappa_1 L(g_1) + \kappa_2 L(g_2) + \kappa_3 \\ &= L(\kappa_1 g_1 + \kappa_2 g_2 + \kappa_3 u) \\ &= L\big(f(g_1, g_2)\big) \end{aligned} \tag{15}$$

proving the equality in formula (14). $\square$

In further consideration, we include continuous convex functions.

THEOREM 1. *Let $g_1, g_2 \in \mathbb{F}$ be functions such that $(g_1(s), g_2(s)) \in \triangle$ for every $s \in S$. Let $L \in \mathbb{L}$ be a positive unital functional, and let $l = (L(g_1), L(g_2))$.*

Then each continuous convex function $f : \triangle \to \mathbb{R}$ such that $f(g_1, g_2) \in \mathbb{F}$ satisfies the double inequality

$$f(l) \leqslant L\big(f(g_1, g_2)\big) \leqslant f_{\triangle}^{sec}(l). \tag{16}$$

Proof. The point l belongs to the triangle $\triangle$ by Lemma 1. We sketch the proof in two steps depending on the position of l.

If l belongs to the interior $\triangle^o$, we take any support plane of f at l. Acting with the positive functional L to the support-secant inequality in formula (8) with $x = (g_1(s), g_2(s))$, we obtain

$$L\big(f_l^{\mathrm{sup}}(g_1, g_2)\big) \leqslant L\big(f(g_1, g_2)\big) \leqslant L\big(f_{\triangle}^{\mathrm{sec}}(g_1, g_2)\big).$$

By applying Lemma 2 to affine functions f_l^{sup} and $f_{\triangle}^{\mathrm{sec}}$, and writing the point $(L(g_1), L(g_2))$ as l, the above inequality takes the form

$$f_l^{\mathrm{sup}}(l) \leqslant L\big(f(g_1, g_2)\big) \leqslant f_{\triangle}^{\mathrm{sec}}(l), \tag{17}$$

where the first term

$$f_l^{\mathrm{sup}}(l) = f(l).$$

If l belongs to the boundary $\partial\triangle$, we rely on the continuity of f using a support plane at a point of the interior $\triangle^0$ that is close enough to l. Given $\varepsilon > 0$, we can find $d \in \triangle^0$ so that

$$f(l) - \varepsilon < f_d^{\sup}(l).$$

Combining the above inequality, and the inequality in formula (17) with the support plane at d, we obtain

$$f(l) - \varepsilon < L\big(f(g_1, g_2)\big) \leqslant f_{\triangle}^{\sec}(l).$$

The inequality in formula (16) follows by sending ε to zero. □

Formula (16) can be expressed in the form that includes the convex combination of the triangle vertices a, b and c. The respective form of Theorem 1 is as follows.

COROLLARY 1. *Let $g_1, g_2 \in \mathbb{F}$ be functions such that $(g_1(s), g_2(s)) \in \triangle$ for every $s \in S$. Let $L \in \mathbb{L}$ be a positive unital functional, and let*

$$l = (L(g_1), L(g_2)) = \alpha_l a + \beta_l b + \gamma_l c. \tag{18}$$

Then each continuous convex function $f : \triangle \to \mathbb{R}$ such that $f(g_1, g_2) \in \mathbb{F}$ satisfies the double inequality

$$f(\alpha_l a + \beta_l b + \gamma_l c) \leqslant L\big(f(g_1, g_2)\big) \leqslant \alpha_l f(a) + \beta_l f(b) + \gamma_l f(c). \tag{19}$$

Proof. As regards the last terms of formulae (16) and (19), the equality

$$f_{\triangle}^{\sec}(l) = \alpha_l f_{\triangle}^{\sec}(a) + \beta_l f_{\triangle}^{\sec}(b) + \gamma_l f_{\triangle}^{\sec}(c) = \alpha_l f(a) + \beta_l f(b) + \gamma_l f(c)$$

holds because of the affinity of $f_{\triangle}^{\sec}$, and its coincidence with f at vertices. □

We want to refine the inequality in formula (16). In addition to the secant plane relating to $\triangle$, we will use three more secant planes. These planes will be specified by a point d belonging to the interior $\triangle^o$.

LEMMA 3. *Let $d \in \triangle^o$ be an interior point, and let $\triangle_1 = dbc$, $\triangle_2 = dac$ and $\triangle_3 = dab$ be subtriangles.*

Then each convex function $f : \triangle \to \mathbb{R}$ satisfies the secant planes inequality

$$\min\left\{f_{\triangle_1}^{sec}(x), f_{\triangle_2}^{sec}(x), f_{\triangle_3}^{sec}(x)\right\} \leqslant f(x) \leqslant \max\left\{f_{\triangle_1}^{sec}(x), f_{\triangle_2}^{sec}(x), f_{\triangle_3}^{sec}(x)\right\} \tag{20}$$

for every $x \in \triangle$.

Proof. The cases $x \in \triangle_1$, $x \in \triangle_2$ and $x \in \triangle_3$ should be considered. □

Let $\triangle_i$ be triangles as in the previous lemma, and let $g_1, g_2 \in \mathbb{F}$ be functions such that $(g_1(s_i), g_2(s_i)) \in \triangle_i$ for some triplet of points $s_1, s_2, s_3 \in S$. Using these assumptions, we can find a triplet of positive unital functionals $L_1, L_2, L_3 \in \mathbb{L}$ meeting the inclusions

$$(L_i(g_1), L_i(g_2)) \in \triangle_i. \tag{21}$$

For example, each of functionals L_i may be taken as the evaluation at the given point s_i defined by $L_i(g) = g(s_i)$ for every $g \in \mathbb{F}$.

The following refining theorem uses functionals L_i that satisfy the inclusions in formula (21).

THEOREM 2. *Let $d \in \triangle^o$, and let $\triangle_1 = dbc$, $\triangle_2 = dac$, $\triangle_3 = dab$. Let $g_1, g_2 \in \mathbb{F}$ be functions such that $(g_1(s), g_2(s)) \in \triangle$ for every $s \in S$, and let $L_i \in \mathbb{L}$ be positive unital functionals such that $l_i = (L_i(g_1), L_i(g_2)) \in \triangle_i$. Let $L = \sum_{i=1}^3 \lambda_i L_i$ be a convex combination of functionals L_i, and let $l = (L(g_1), L(g_2))$.*

Then each continuous convex function $f : \triangle \to \mathbb{R}$ such that $f(g_1, g_2) \in \mathbb{F}$ satisfies the series of inequalities

$$\begin{aligned} f(l) &\leqslant \sum_{i=1}^{3} \lambda_i f(l_i) \leqslant L\big(f(g_1, g_2)\big) \\ &\leqslant \sum_{i=1}^{3} \lambda_i f_{\triangle_i}^{sec}(l_i) \leqslant f_{\triangle}^{sec}(l). \end{aligned} \tag{22}$$

Proof. Applying the discrete form of Jensen's inequality (see [5]) to the convex combination $l = \sum_{i=1}^3 \lambda_i l_i$, we get

$$f(l) \leqslant \sum_{i=1}^{3} \lambda_i f(l_i). \tag{23}$$

Applying the left-hand side of formula (16) to each L_i, we obtain

$$\sum_{i=1}^{3} \lambda_i f(l_i) \leqslant \sum_{i=1}^{3} \lambda_i L_i\big(f(g_1, g_2)\big) = L\big(f(g_1, g_2)\big). \tag{24}$$

Acting with L_i to the right-hand side of the secant planes inequality in formula (20) with $x = (g_1(s), g_2(s))$, and using the assumption that l_i belongs to $\triangle_i$, we find

$$L_i\big(f(g_1, g_2)\big) \leqslant L_i\big(f_{\triangle_i}^{\mathrm{sec}}(g_1, g_2)\big) = f_{\triangle_i}^{\mathrm{sec}}(l_i).$$

Multiplications by λ_i and summation yield

$$\sum_{i=1}^{3} \lambda_i L_i\big(f(g_1, g_2)\big) \leqslant \sum_{i=1}^{3} \lambda_i f_{\triangle_i}^{\mathrm{sec}}(l_i) \leqslant f_{\triangle}^{\mathrm{sec}}(l) \tag{25}$$

because

$$\sum_{i=1}^{3} \lambda_i f_{\triangle_i}^{\mathrm{sec}}(l_i) \leqslant \sum_{i=1}^{3} \lambda_i f_{\triangle}^{\mathrm{sec}}(l_i) = f_{\triangle}^{\mathrm{sec}}(l).$$

Putting together the inequalities in formulae (23), (24) and (25) into a series, we produce the inequality in formula (22). □

The geometric presentation of the series of inequalities in formula (22) can be seen in Figure 1. The inequality terms are represented by five black dots above the point l. If the function f is strictly convex, and if the functional L is strictly positive, then all black dots are different.

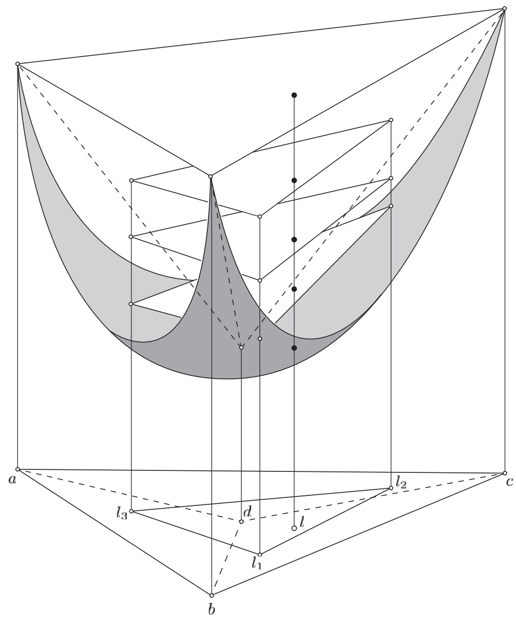

Figure 1: *Geometric presentation of the inequality in formula* (22).

In order to adapt Theorem 2 for applications, we need to emphasize the points a, b, c and d in formula (22). In the representation of the point l, we have already used the convex combination of vertices of the triangle $\triangle$. We can similarly represent the points l_i by the convex combinations of vertices of the triangles $\triangle_i$. Due to the affinity of the secant planes $f_{\triangle_i}^{\mathrm{sec}}$, the representations of l_i may be employed to obtain the convex combination

$$\sum_{i=1}^{3}\lambda_i f_{\triangle_i}^{\mathrm{sec}}(l_i)=\alpha f(a)+\beta f(b)+\gamma f(c)+\delta f(d).$$

COROLLARY 2. *Let $d\in\triangle^o$, and let $\triangle_1=dbc$, $\triangle_2=dac$, $\triangle_3=dab$. Let $g_1,g_2\in\mathbb{F}$ be functions such that $(g_1(s),g_2(s))\in\triangle$ for every $s\in S$, and let $L_i\in\mathbb{L}$ be positive unital functionals such that $l_i=(L_i(g_1),L_i(g_2))\in\triangle_i$. Let $L=\sum_{i=1}^{3}\lambda_iL_i$ be a convex combination of L_i, and let $l=(L(g_1),L(g_2))=\alpha_la+\beta_lb+\gamma_lc$.*

Then each continuous convex function $f:\triangle\to\mathbb{R}$ *such that* $f(g_1,g_2)\in\mathbb{F}$ *satisfies the series of inequalities*

$$\begin{aligned} f(\alpha_l a+\beta_l b+\gamma_l c) &\leqslant \sum_{i=1}^{3}\lambda_i f(l_i)\leqslant L\big(f(g_1,g_2)\big)\\ &\leqslant \alpha f(a)+\beta f(b)+\gamma f(c)+\delta f(d)\\ &\leqslant \alpha_l f(a)+\beta_l f(b)+\gamma_l f(c) \end{aligned} \tag{26}$$

where

$$\alpha=\lambda_2\frac{\operatorname{ar}(dcl_2)}{\operatorname{ar}(\triangle_2)}+\lambda_3\frac{\operatorname{ar}(dbl_3)}{\operatorname{ar}(\triangle_3)} \tag{27}$$

$$\beta=\lambda_1\frac{\operatorname{ar}(dcl_1)}{\operatorname{ar}(\triangle_1)}+\lambda_3\frac{\operatorname{ar}(dal_3)}{\operatorname{ar}(\triangle_3)} \tag{28}$$

$$\gamma=\lambda_1\frac{\operatorname{ar}(dbl_1)}{\operatorname{ar}(\triangle_1)}+\lambda_2\frac{\operatorname{ar}(dal_2)}{\operatorname{ar}(\triangle_2)} \tag{29}$$

$$\delta=\lambda_1\frac{\operatorname{ar}(bcl_1)}{\operatorname{ar}(\triangle_1)}+\lambda_2\frac{\operatorname{ar}(acl_2)}{\operatorname{ar}(\triangle_2)}+\lambda_3\frac{\operatorname{ar}(abl_3)}{\operatorname{ar}(\triangle_3)} \tag{30}$$

Proof. By including the convex combinations

$$l_1=\beta_1 b+\gamma_1 c+\delta_1 d$$

$$l_2=\alpha_2 a+\gamma_2 c+\delta_2 d$$

$$l_3=\alpha_3 a+\beta_3 b+\delta_3 d$$

it follows that

$$\begin{aligned}\sum_{i=1}^{3}\lambda_i f_{\triangle_i}^{\sec}(l_i) &= (\lambda_2\alpha_2+\lambda_3\alpha_3)f(a)+(\lambda_1\beta_1+\lambda_3\beta_3)f(b)\\ &\quad+(\lambda_1\gamma_1+\lambda_2\gamma_2)f(c)+(\lambda_1\delta_1+\lambda_2\delta_2+\lambda_3\delta_3)f(d).\end{aligned}$$

The above representation shows the required coefficients. Using formula (4), we can determine the coefficient

$$\alpha=\lambda_2\alpha_2+\lambda_3\alpha_3=\lambda_2\frac{\operatorname{ar}(dcl_2)}{\operatorname{ar}(\triangle_2)}+\lambda_3\frac{\operatorname{ar}(dbl_3)}{\operatorname{ar}(\triangle_3)},$$

and similarly the coefficients β, γ and δ. $\square$

Generalizations of important inequalities for convex functions on the triangle without using functionals were discussed in [11]. Certain functional inequalities for functions that are not necessarily convex were considered in [10].

3. Applications

We utilize Theorem 1 to derive the generalizations of important integral and discrete inequalities for convex functions on the triangle.

COROLLARY 3. *Let $g_1, g_2 : \triangle \to \mathbb{R}$ be integrable functions such that $(g_1(x), g_2(x)) \in \triangle$ for every $x \in \triangle$, and let $h : \triangle \to \mathbb{R}$ be a positive integrable function. Let*

$$l = \left(\frac{\int_\triangle g_1 h\,dx}{\int_\triangle h\,dx}, \frac{\int_\triangle g_2 h\,dx}{\int_\triangle h\,dx} \right). \tag{31}$$

Then each convex function $f : \triangle \to \mathbb{R}$ satisfies the double inequality

$$\begin{aligned} f\left(\frac{\int_\triangle g_1 h\,dx}{\int_\triangle h\,dx}, \frac{\int_\triangle g_2 h\,dx}{\int_\triangle h\,dx} \right) &\leqslant \frac{\int_\triangle f(g_1, g_2) h\,dx}{\int_\triangle h\,dx} \\ &\leqslant \frac{\operatorname{ar}(lbc)}{\operatorname{ar}(\triangle)} f(a) + \frac{\operatorname{ar}(lac)}{\operatorname{ar}(\triangle)} f(b) + \frac{\operatorname{ar}(lab)}{\operatorname{ar}(\triangle)} f(c). \end{aligned} \tag{32}$$

Proof. Let $\mathbb{F}$ be the space of all integrable functions over the domain $S = \triangle$. The composition $f(g_1, g_2)$ is integrable over $\triangle$ because it is bounded, and continuous almost everywhere in $\triangle$ (the Lebesgue theorem on the Riemann integral, see [9]).

We define the integrating linear functional L for every function $g \in \mathbb{F}$ by the formula

$$L(g) = L(g;h) = \frac{\int_\triangle gh\,dx}{\int_\triangle h\,dx}.$$

The functional L is positive and unital. Using this functional in formula (16), we obtain formula (32) if the function f is continuous.

Let us verify that the inequality in formula (32) applies to a convex function which is not continuous on the boundary $\partial\triangle$. We use convex combinations associated with points $x \in \triangle$,

$$(g_1(x), g_2(x)) = \alpha(x)a + \beta(x)b + \gamma(x)c. \tag{33}$$

Multiplying the above equation by $h(x)$, integrating each coordinate over $\triangle$, and dividing by $\int_\triangle h\,dx$, we obtain the convex combination with integral coefficients,

$$l = \frac{\int_\triangle \alpha h\,dx}{\int_\triangle h\,dx} a + \frac{\int_\triangle \beta h\,dx}{\int_\triangle h\,dx} b + \frac{\int_\triangle \gamma h\,dx}{\int_\triangle h\,dx} c. \tag{34}$$

If l belongs to the interior $\triangle^o$, then the continuous extension $\widetilde{f}$ of $f/\triangle^o$ to $\triangle$ may be utilized in formula (32). The first two terms are the same as we use f, and the last terms satisfy the inequality

$$\alpha_l \widetilde{f}(a) + \beta_l \widetilde{f}(b) + \gamma_l \widetilde{f}(c) < \alpha_l f(a) + \beta_l f(b) + \gamma_l f(c).$$

If l belongs to the relative interior I^o of the side $I = \operatorname{conv}\{a, b\}$, then $\gamma(x) = 0$ for almost every $x \in \triangle$ by equation (34), and so $(g_1(x), g_2(x)) \in I$ for almost every $x \in \triangle$

by equation (33). Any continuous convex extension $\widehat{f}$ of f/I^o to $\triangle$ may be utilized in formula (32). The first two terms are the same as we use f, and the last terms satisfy the inequality

$$\alpha_l\widehat{f}(a)+\beta_l\widehat{f}(b)\leqslant\alpha_l f(a)+\beta_l f(b).$$

If l is equal to the vertex a, then $\alpha(x)-1=\beta(x)=\gamma(x)=0$ and $(g_1(x),g_2(x))=a$ for almost every $x\in\triangle$. Consequently, $f(g_1(x),g_2(x))=f(a)$ for almost every $x\in\triangle$, and the inequality $f(a)\leqslant f(a)\leqslant f(a)$ represents formula (32).

Respecting all considerations, we may conclude that the inequality in formula (32) applies to any convex function f. □

To demonstrate the generality of the inequality in formula (32), we will say a few words about barycenters. Let μ be a positive measure on the plane $\mathbb{R}^2$. The μ-barycenter of the measurable set $S\subseteq\mathbb{R}^2$ such that $\mu(S)>0$ is defined by

$$\left(\frac{\int_S x_1\,d\mu}{\mu(S)},\frac{\int_S x_2\,d\mu}{\mu(S)}\right), \tag{35}$$

and the μ-barycenter of the nonnegative μ-integrable function $h:S\to\mathbb{R}$ such that $\int_S h\,d\mu>0$ can be defined by

$$\left(\frac{\int_S x_1h\,d\mu}{\int_S h\,d\mu},\frac{\int_S x_2h\,d\mu}{\int_S h\,d\mu}\right). \tag{36}$$

The inequality in formula (32) contains the extended integral form of Jensen's inequality (see [6]), the Fejér inequality (see [2]), and consequently the Hermite-Hadamard inequality (see [4] and [3]). Let us demonstrate the simplifications of this inequality relating to the unit function, projections and symmetric functions.

Using the unit function $h(x_1,x_2)=1$, in which case

$$l=\left(\frac{\int_\triangle g_1\,dx}{\mathrm{ar}(\triangle)},\frac{\int_\triangle g_2\,dx}{\mathrm{ar}(\triangle)}\right), \tag{37}$$

we get the extended integral form of Jensen's inequality for convex functions on the triangle,

$$\begin{aligned} f\left(\frac{\int_\triangle g_1\,dx}{\mathrm{ar}(\triangle)},\frac{\int_\triangle g_2\,dx}{\mathrm{ar}(\triangle)}\right)&\leqslant\frac{\int_\triangle f(g_1,g_2)\,dx}{\mathrm{ar}(\triangle)}\\ \leqslant\frac{\mathrm{ar}(lbc)}{\mathrm{ar}(\triangle)}f(a)+\frac{\mathrm{ar}(lac)}{\mathrm{ar}(\triangle)}f(b)&+\frac{\mathrm{ar}(lab)}{\mathrm{ar}(\triangle)}f(c). \end{aligned} \tag{38}$$

Now we use the projections $g_1(x_1,x_2)=x_1$ and $g_2(x_1,x_2)=x_2$, and a positive integrable function $h(x_1,x_2)$ whose barycenter falls into $(a+b+c)/3$. Thus

$$l=\left(\frac{\int_\triangle x_1h\,dx}{\int_\triangle h\,dx},\frac{\int_\triangle x_2h\,dx}{\int_\triangle h\,dx}\right)=\frac{a+b+c}{3} \tag{39}$$

indicating that $\alpha_l = \beta_l = \gamma_l = 1/3$, and we have the extension of the Fejér inequality to convex functions on the triangle,

$$f\left(\frac{a+b+c}{3}\right) \leqslant \frac{\int_\triangle fh\,dx}{\int_\triangle h\,dx} \leqslant \frac{f(a)+f(b)+f(c)}{3}. \tag{40}$$

Putting the unit function $h(x_1,x_2)=1$ in Fejér's inequality in formula (40), we obtain the Hermite-Hadamard inequality for convex functions on the triangle,

$$f\left(\frac{a+b+c}{3}\right) \leqslant \frac{\int_\triangle f\,dx}{\mathrm{ar}(\triangle)} \leqslant \frac{f(a)+f(b)+f(c)}{3}. \tag{41}$$

In a context of the measure theory, the aspect of the Fejér inequality determines the μ-barycenter of the function h on the triangle $\triangle$. If

$$\left(\frac{\int_\triangle x_1 h\,d\mu}{\int_\triangle h\,d\mu}, \frac{\int_\triangle x_2 h\,d\mu}{\int_\triangle h\,d\mu}\right) = \alpha a + \beta b + \gamma c \tag{42}$$

respecting the measure μ, then

$$f(\alpha a + \beta b + \gamma c) \leqslant \frac{\int_\triangle fh\,d\mu}{\int_\triangle h\,d\mu} \leqslant \alpha f(a) + \beta f(b) + \gamma f(c). \tag{43}$$

Equally, the aspect of the Hermite-Hadamard inequality determines the μ-barycenter of the triangle $\triangle$.

At the end of the section, let us present the discrete version of Corollary 1. In the next corollary, we will use functions $u = g_1$ and $v = g_2$. The evaluations at points $x_i \in \triangle$ will be marked by $u(x_i) = u_i$, $v(x_i) = v_i$ and $h(x_i) = h_i$.

COROLLARY 4. *Let $u,v : \triangle \to \mathbb{R}$ be functions such that $(u(x),v(x)) \in \triangle$ for every $x \in \triangle$, and let $h : \triangle \to \mathbb{R}$ be a positive function. Let $x_1,\dots,x_n \in \triangle$ be points, and let*

$$l = \left(\frac{\sum_{i=1}^n u_i h_i}{\sum_{i=1}^n h_i}, \frac{\sum_{i=1}^n v_i h_i}{\sum_{i=1}^n h_i}\right). \tag{44}$$

Then each convex function $f : \triangle \to \mathbb{R}$ satisfies the double inequality

$$\begin{aligned} f\left(\frac{\sum_{i=1}^n u_i h_i}{\sum_{i=1}^n h_i}, \frac{\sum_{i=1}^n v_i h_i}{\sum_{i=1}^n h_i}\right) &\leqslant \frac{\sum_{i=1}^n f(u_i,v_i) h_i}{\sum_{i=1}^n h_i} \\ &\leqslant \frac{\mathrm{ar}(lbc)}{\mathrm{ar}(\triangle)} f(a) + \frac{\mathrm{ar}(lac)}{\mathrm{ar}(\triangle)} f(b) + \frac{\mathrm{ar}(lab)}{\mathrm{ar}(\triangle)} f(c). \end{aligned} \tag{45}$$

Proof. Having $\mathbb{F}$ as the space of all real functions on the domain $S = \triangle$, and taking the summarizing linear functional defined by

$$L(g) = L(g;h) = \frac{\sum_{i=1}^n g_i h_i}{\sum_{i=1}^n h_i} \tag{46}$$

for every $g \in \mathbb{F}$, we can follow the proof of Corollary 3. $\square$

Note that the right-hand side of equation (46) is the convex combination of points g_i with coefficients $\lambda_i = h_i / \sum_{i=1}^{n} h_i$.

Jensen type inequalities, and generalizations of the Hermite-Hadamard inequality for convex functions of several variables were considered in [12]. The Hermite-Hadamard-Fejér type inequalities were considered in [13].

4. Refinements

We exploit Corollary 2 to obtain the series of inequalities which refines the double inequality in formula (32).

COROLLARY 5. *Let $d \in \triangle^o$, and let $\triangle_1 = dbc$, $\triangle_2 = dac$, $\triangle_3 = dab$. Let $g_1, g_2 : \triangle \to \mathbb{R}$ be integrable functions such that $(g_1(x), g_2(x)) \in \triangle_i$ for every $x \in \triangle_i$, and let $h : \triangle \to \mathbb{R}$ be a positive integrable function.*

Then each convex function $f : \triangle \to \mathbb{R}$ satisfies the series of inequalities

$$
\begin{aligned}
f\left(\frac{\int_{\triangle} g_1 h\,dx}{\int_{\triangle} h\,dx}, \frac{\int_{\triangle} g_2 h\,dx}{\int_{\triangle} h\,dx}\right) &\leqslant \sum_{i=1}^{3} \frac{\int_{\triangle_i} h\,dx}{\int_{\triangle} h\,dx} f\left(\frac{\int_{\triangle_i} g_1 h\,dx}{\int_{\triangle_i} h\,dx}, \frac{\int_{\triangle_i} g_2 h\,dx}{\int_{\triangle_i} h\,dx}\right) \\
&\leqslant \frac{\int_{\triangle} f(g_1, g_2) h\,dx}{\int_{\triangle} h\,dx} \\
&\leqslant \alpha f(a) + \beta f(b) + \gamma f(c) + \delta f(d) \\
&\leqslant \frac{\operatorname{ar}(lbc)}{\operatorname{ar}(\triangle)} f(a) + \frac{\operatorname{ar}(lac)}{\operatorname{ar}(\triangle)} f(b) + \frac{\operatorname{ar}(lab)}{\operatorname{ar}(\triangle)} f(c),
\end{aligned}
\tag{47}
$$

where coefficients α, β, γ and δ are as in Corollary 2 with respect to

$$
l_i = \left(\frac{\int_{\triangle_i} g_1 h\,dx}{\int_{\triangle_i} h\,dx}, \frac{\int_{\triangle_i} g_2 h\,dx}{\int_{\triangle_i} h\,dx}\right), \qquad \lambda_i = \frac{\int_{\triangle_i} h\,dx}{\int_{\triangle} h\,dx}, \qquad l = \left(\frac{\int_{\triangle} g_1 h\,dx}{\int_{\triangle} h\,dx}, \frac{\int_{\triangle} g_2 h\,dx}{\int_{\triangle} h\,dx}\right).
$$

Proof. As in Corollary 3, let $\mathbb{F}$ be the space of all integrable functions over the triangle $S = \triangle$. We take the integrating linear functionals L_i defined by the formulae

$$
L_i(g) = L_i(g; h) = \frac{\int_{\triangle_i} g h\,dx}{\int_{\triangle_i} h\,dx}
$$

for every $g \in \mathbb{F}$. The functionals L_i are positive and unital. The point

$$
(L_i(g_1), L_i(g_2)) = \left(\frac{\int_{\triangle_i} g_1 h\,dx}{\int_{\triangle_i} h\,dx}, \frac{\int_{\triangle_i} g_2 h\,dx}{\int_{\triangle_i} h\,dx}\right) = l_i \tag{48}
$$

falls into the triangle $\triangle_i$ by Lemma 1. Applying the functional convex combination $L=\sum_{i=1}^{3}\lambda_i L_i$, we get

$$(L(g_1),L(g_2))=\left(\frac{\int_\triangle g_1 h\,dx}{\int_\triangle h\,dx},\frac{\int_\triangle g_2 h\,dx}{\int_\triangle h\,dx}\right)=l \tag{49}$$

and

$$L\big(f(g_1,g_2)\big)=\frac{\int_\triangle f(g_1,g_2)h\,dx}{\int_\triangle h\,dx}. \tag{50}$$

Inserting the integrals of formulae (48), (49) and (50) into formula (26), we obtain the series of inequalities in formula (47). $\square$

We will just apply the series of inequalities in formula (47) to obtain the simplest refinement of the Hermite-Hadamard inequality in formula (41). Formula (47) with $g_1(x_1,x_2)=x_1$, $g_2(x_1,x_2)=x_2$, $h(x_1,x_2)=1$ and

$$d=\left(\frac{\int_\triangle x_1\,dx}{\mathrm{ar}(\triangle)},\frac{\int_\triangle x_2\,dx}{\mathrm{ar}(\triangle)}\right)=\frac{a+b+c}{3}$$

gives the inequality including a plurality of convex combinations,

$$\begin{aligned}
f\left(\frac{a+b+c}{3}\right)&\leqslant\frac{1}{3}f\left(\frac{a+4b+4c}{9}\right)+\frac{1}{3}f\left(\frac{b+4a+4c}{9}\right)+\frac{1}{3}f\left(\frac{c+4a+4b}{9}\right)\\
&\leqslant\frac{\int_\triangle f\,dx}{\mathrm{ar}(\triangle)}\\
&\leqslant\frac{2}{9}f(a)+\frac{2}{9}f(b)+\frac{2}{9}f(c)+\frac{1}{3}f\left(\frac{a+b+c}{3}\right)\\
&\leqslant\frac{f(a)+f(b)+f(c)}{3}.
\end{aligned} \tag{51}$$

We will briefly explain the above inequality. The triangles $\triangle_i$ have the same area equal to $\mathrm{ar}(\triangle)/3$, and therefore

$$\lambda_i=\frac{\int_{\triangle_i}dx}{\int_\triangle dx}=\frac{1}{3}.$$

The point

$$l_i=\left(\frac{\int_{\triangle_i}x_1\,dx}{\mathrm{ar}(\triangle_i)},\frac{\int_{\triangle_i}x_2\,dx}{\mathrm{ar}(\triangle_i)}\right)$$

is the barycenter of the triangle $\triangle_i$, which consequently yields the representations

$$l_1=\frac{b+c+d}{3}=\frac{a+4b+4c}{9},$$

$$l_2=\frac{a+c+d}{3}=\frac{b+4a+4c}{9},$$

$$l_3 = \frac{a+b+d}{3} = \frac{c+4a+4b}{9},$$

and $l = d$. All three subtriangles of $\triangle_i$ have the same area equal to $\mathrm{ar}(\triangle_i)/3$. By applying formulae (27)-(30), it follows that

$$\alpha = \beta = \gamma = \frac{2}{9}, \quad \delta = \frac{1}{3}.$$

The series of inequalities in formula (51) is also obtained in [11, Theorem 3.1] by using the triangle barycenter and its convex combinations.

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