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NEW INEQUALITIES FOR SOME GENERALIZED MATHIEU TYPE SERIES AND THE RIEMANN ZETA FUNCTION

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Abstract. Our aim in this paper is to derive several new inequalities for the Mathieu type series and the Riemann zeta function. In particular, we prove Turán type inequalities and some monotonicity and log-convexity results for these special functions. New Laplace type integral representations for the Mathieu type series and the Riemann zeta function are also presented.

1. Introduction

The following infinite series:

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2}\qquad (r\in\mathbb{R}^+) \tag{1.1}$$

is popularly known as the Mathieu series. It was introduced and studied by Émile Leonard Mathieu (1835–1890) in his 1890 book [10] devoted to the elasticity of solid bodies. Bounds for this series are needed for the solution of boundary value problems for the bi-harmonic equations in a two-dimensional rectangular domain (see, for example, [17, p. 258, Eq. (54)]). A remarkably useful integral representation for $S(r)$ is given by Emersleben [8] in the following form:

$$S(r)=\frac{1}{r}\int_0^{\infty}\frac{x\sin(rx)}{e^x-1}\,dx. \tag{1.2}$$

The so-called generalized Mathieu series with a *fractional* power reads as follows (see [3]):

$$S_\mu(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^{\mu+1}}\qquad (\mu>0;\ r>0) \tag{1.3}$$

Such series as $S_\mu(r)$ given by (1.3) have been widely considered in the mathematical literature (see, for details, [3], [19] and [22]). In particular, Cerone and Lenard [3] derived also the following integral expression for the generalized Mathieu series $S_\mu(r)$:

$$S_\mu(r)=C_\mu(r)\int_0^{\infty}\frac{x^{\mu+\frac{1}{2}}}{e^x-1}\,J_{\mu-\frac{1}{2}}(rx)\,dx\qquad (\mu>0), \tag{1.4}$$

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where

$$C_\mu(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu+1)}$$

and $J_\nu(z)$ denotes the ordinary Bessel function of order ν (see, for details, [23]).

The study of Mathieu's series $S(r)$ and its associated inequalities has a rich literature. We choose to recall here an interesting result of Alzer *et al.* [1], who showed that the best constants κ_1 and κ_2 in the following two-sided inequality:

$$\frac{1}{\kappa_1 + r^2} < S(r) < \frac{1}{\kappa_2 + r^2} \qquad (r \neq 0) \tag{1.5}$$

are given by

$$\kappa_1 = \frac{1}{2\zeta(3)} \qquad \text{and} \qquad \kappa_2 = \frac{1}{6},$$

where $\zeta(s)$ denotes the Riemann zeta function defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\[2ex] \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0;\ s \neq 1\big) \end{cases} \tag{1.6}$$

and by its meromorphic continuation to the whole complex s-plane except for a simple pole at $s = 1$ with the residue 1 (see, for details, [18]). Many interesting refinements and
extensions of the Mathieu type series and their associated inequalities can be found in (for example) [19], [21] and [22] (see also several other investigations and developments on the subject of the Mathieu type series, which are reported in [4], [7], [15], [16] and [20]).

The present sequel to some of the aforementioned investigations is organized as follows. In Section 2, we state some useful lemmas which will be needed in the proofs of our results. In Section 3, we prove several (presumably new) inequalities for the Mathieu and related series. In particular, we present a Turán type inequality for this special function. Moreover, we derive some monotonicity and convexity results for the function $\mu \mapsto S_\mu(r)$. As a consequence, we establish a number of functional inequalities. At the end of Section 3, we derive the Laplace integral representation of such types of Mathieu series. Finally, in Section 4, we apply some of our main results of Section 3 with a view to deriving several new inequalities for the Riemann zeta function $\zeta(s)$ defined by (1.6).

Each of the following definitions will be used in our investigation.

DEFINITION 1. A function $f : (0,\infty) \to \mathbb{R}$ is said to be completely monotonic if f has derivatives of all orders and satisfies the following inequalities:

$$(-1)^n f^{(n)}(x) \geqq 0 \qquad (\forall\, x \geqq 0 \quad \text{and} \quad \forall\, n \in \mathbb{N} = \{1, 2.3.\cdots\}).$$

DEFINITION 2. A function $g : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ is said to be log-convex if its natural logarithm $\log g$ is convex, that is, if

$$g\big(\alpha x + (1-\alpha)y\big) \leqq [g(x)]^{\alpha}\, [g(y)]^{1-\alpha} \qquad (\forall\, x,y \in [a,b] \quad \text{and} \quad \forall\, \alpha \in [0,1]).$$

2. Preliminary lemmas

In this section, we recall each of the following lemmas, which are needed in the proofs of our results.

LEMMA 1. (see [9, p. 313, Eq. (10)]) *Let* $\mu > 2$. *Then the following integral formula:*

$$\int_0^{\infty} \frac{t^{\mu-1}}{(e^t-1)^2}\, dt = \Gamma(\mu)\,[\zeta(\mu-1) - \zeta(\mu)] \tag{2.1}$$

holds true.

LEMMA 2. (see [11]) *Let the function* $f : (0,\infty) \longrightarrow (0,1)$ *be continuous. If* f *is completely monotonic, then*

$$f(x)f(y) \leqq f(x+y) \qquad (\forall\, x,y \geqq 0). \tag{2.2}$$

LEMMA 3. (see [2]) *Let* $\mu > 0$ *and* $r > 0$. *Then the following integral formula:*

$$\int_0^{\infty} S_\mu(r)\, dr = \frac{\sqrt{\pi}\,\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(\mu+1)}\,\zeta(2\mu) \qquad (\mu > 0;\ r > 0) \tag{2.3}$$

holds true for the generalized Mathieu series $S_\mu(r)$.

3. Inequalities and integral representations for the Mathieu type series

Our first main result is asserted by the following theorem.

THEOREM 1. *Let* $\mu > \frac{3}{2}$. *Then the following inequality:*

$$S_\mu(r) \geqq \left(\frac{2\mu-1}{2\mu r^3}\right) S_{\mu-1}(r) - \left(\frac{(2\mu-1)\sqrt{\pi}}{2^{2(\mu-1)}\, r^3}\right) \frac{\Gamma(2\mu)\zeta(2\mu-1)}{\Gamma(\mu+1)\Gamma\left(\mu+\frac{1}{2}\right)} \tag{3.1}$$

is valid for all $r > 0$.

Proof. Let us consider the function $\mathscr{J}_\mu(x) : \mathbb{R} \longrightarrow (-\infty, 1]$ defined by

$$\mathscr{J}_\mu(x) = \left(\frac{2}{x}\right)^{\mu} \Gamma(\mu+1) J_\mu(x) \qquad (\mu > -1).$$

Thus, by applying (1.4), we can write the generalized Mathieu series $S_\mu(r)$ in the following form:

$$S_\mu(r) = c_{\mu,1} \int_0^\infty \frac{x^{2\mu}}{e^x - 1} \mathscr{J}_{\mu-\frac{1}{2}}(rx)\, dx \qquad (\mu \geqq 1), \tag{3.2}$$

where

$$c_{\mu,1} = \frac{\sqrt{\pi}}{2^{2\mu-1}\,\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\mu+1)}.$$

Now, upon using the following derivative formula [23, p. 18]:

$$\mathscr{J}_\mu'(x) = -\frac{x}{2(\mu+1)} \mathscr{J}_{\mu+1}(x),$$

if we evaluate the right-hand side of (3.2) by integration by parts, we get

$$\begin{aligned}
S_\mu(r) &= -\frac{(2\mu-1)c_{\mu,1}}{r^2} \int_0^\infty \frac{x^{2\mu-1}}{e^x-1} \mathscr{J}_{\mu-\frac{3}{2}}'(rx)\, dx \\
&= \frac{(2\mu-1)c_{\mu,1}}{r^2} \left(\int_0^\infty \frac{(2\mu-1)x^{2\mu-2}}{r(e^x-1)} \mathscr{J}_{\mu-\frac{3}{2}}(rx)\, dx - \int_0^\infty \frac{x^{2\mu-1}e^x}{r(e^x-1)^2} \mathscr{J}_{\mu-\frac{3}{2}}(rx)\, dx \right) \\
&= \frac{(2\mu-1)^2\, c_{\mu,1}}{c_{\mu-1,1}\, r^3} S_{\mu-1}(r) \\
&\quad - \frac{(2\mu-1)c_{\mu,1}}{r^3} \left(\int_0^\infty \frac{x^{2\mu-1}}{e^x-1} \mathscr{J}_{\mu-\frac{3}{2}}(rx)\, dx + \int_0^\infty \frac{x^{2\mu-1}}{(e^x-1)^2} \mathscr{J}_{\mu-\frac{3}{2}}(rx)\, dx \right) \\
&\geqq \frac{(2\mu-1)^2\, c_{\mu,1}}{c_{\mu-1,1}\, r^3} S_{\mu-1}(r) - \frac{(2\mu-1)c_{\mu,1}}{r^3} \left(\int_0^\infty \frac{x^{2\mu-1}}{e^x-1}\, dx + \int_0^\infty \frac{x^{2\mu-1}}{(e^x-1)^2}\, dx \right) \\
&= \frac{(2\mu-1)^2 c_{\mu,1}}{c_{\mu-1,1} r^3} S_{\mu-1}(r) - \frac{(2\mu-1)c_{\mu,1}}{r^3} \left[\Gamma(2\mu)\zeta(2\mu) + \Gamma(2\mu)[\zeta(2\mu-1) - \zeta(2\mu)] \right] \\
&= \frac{(2\mu-1)^2\, c_{\mu,1}}{c_{\mu-1,1}\, r^3} S_{\mu-1}(r) - \frac{(2\mu-1)c_{\mu,1}}{r^3} \Gamma(2\mu)\zeta(2\mu-1).
\end{aligned} \tag{3.3}$$

In this last equation (3.3), we use the following bound given by by Minakshisundaram and Szász [14]:

$$|\mathscr{J}_\mu(x)| \leqq 1 \qquad \left(\mu > -\frac{1}{2};\ x \in \mathbb{R}\right)$$

together with Lemma 1. The desired inequality (3.1) is thus established. □

In Theorem 2 below, we establish a Turán type inequality for the generalized Mathieu serie $S_\mu(r)$.

THEOREM 2. *Let* $\mu > 0$. *Then the following Turán type inequality:*

$$S_{\mu+2}(r)S_\mu(r) - [S_{\mu+1}(r)]^2 \geqq 0 \tag{3.4}$$

holds true for all $r \in (0,\infty)$.

Proof. By applying the Cauchy product, we find that

$$
\begin{aligned}
&S_{\mu+2}(r)S_{\mu}(r)-[S_{\mu+1}(r)]^2\\
&\quad=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{4k(n-k)}{(k^2+r^2)^{\mu+1}[(n-k)^2+r^2]^{\mu+3}}-\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{4k(n-k)}{(k^2+r^2)^{\mu+2}[(n-k)^2+r^2]^{\mu+2}}\\
&\quad=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{4nk(n-k)(2k-n)}{(k^2+r^2)^{\mu+2}[(n-k)^2+r^2]^{\mu+3}}\\
&\quad=\sum_{n=1}^{\infty}\sum_{k=0}^{n}4n(2k-n)T_{n,k},
\end{aligned}
\tag{3.5}
$$

where

$$
T_{n,k}=\frac{k(n-k)}{[(n-k)^2+r^2]^{\mu+3}(k^2+r^2)^{\mu+2}}.
$$

Case 1. Let n be an even positive integer. Then

$$
\begin{aligned}
\sum_{k=0}^{n}T_{n,k}(2k-n)&=\sum_{k=0}^{\frac{n}{2}-1}T_{n,k}(2k-n)+\sum_{k=\frac{n}{2}+1}^{n}T_{n,k}(2k-n)\\
&=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(T_{n,k}-T_{n,n-k})(2k-n),
\end{aligned}
\tag{3.6}
$$

where, as usual, $[\kappa]$ denotes the greatest integer part of $\kappa\in\mathbb{R}$.

Case 2. Let n be an odd positive integer. Then, just as in Case 1, we get

$$
\sum_{k=0}^{n}T_{n,k}(2k-n)=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(T_{n,k}-T_{n,n-k})(2k-n).
$$

Thus, by combining Case 1 and Case 2, we have

$$
S_{\mu+2}(r)S_{\mu}(r)-[S_{\mu+1}(r)]^2=\sum_{n=1}^{\infty}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(T_{n,k}-T_{n,n-k})(2k-n),
\tag{3.7}
$$

which, upon simplifying, yields

$$
T_{n,k}-T_{n,n-k}=\frac{n^2k\,(n-k)(2k-n)}{[(n-k)^2+r^2]^{\mu+3}(k^2+r^2)^{\mu+3}}.
\tag{3.8}
$$

Finally, in view of (3.7) and (3.8), we deduce the Turán type inequality (3.4). $\square$

THEOREM 3. *Each of the following assertions holds true:*

1. *The function* $\mu\mapsto S_{\mu}(r)$ *is completely monotonic and log-convex on* $(0,\infty)$ *for each* $r>0$.

2. *The function* $\mu\mapsto\dfrac{S_{\mu+1}(r)}{S_{\mu}(r)}$ *is increasing on* $(0,\infty)$.

3. *For all* $r>0$, *the following inequalities are valid:*

$$S_{\mu+\nu}(r)S(r) \geqq S_{\mu}(r)S_{\nu}(r) \qquad (\mu,\nu>0), \tag{3.9}$$

$$\left(\frac{S_{\nu}(r)}{2\zeta(2\nu+1)}\right)^{\frac{1}{\nu+1}} \geqq \left(\frac{S_{\mu}(r)}{2\zeta(2\mu+1)}\right)^{\frac{1}{\mu+1}} \qquad (\mu \geqq \nu > 0) \tag{3.10}$$

and

$$\left(\frac{S_{\mu}(r)}{2\zeta(2\mu+1)}\right)^{\frac{1}{\mu+1}} + \frac{\zeta(2\mu+3)S_{\mu}(r)}{\zeta(2\mu+1)S_{\mu+1}(r)} \geqq 2 \qquad (\mu>0). \tag{3.11}$$

Proof. 1. For all $m\in\mathbb{N}$ and $\mu>0$, we have

$$(-1)^m \frac{\partial^m S_{\mu}(r)}{\partial \mu^m} = \sum_{n=1}^{\infty} \frac{2n[\log(n^2+r^2)]^m}{(n^2+r^2)^{\mu+1}} \geqq 0.$$

Thus, clearly, the function $\mu \mapsto S_{\mu}(r)$ is completely monotonic and log-convex on $(0,\infty)$, since every completely monotonic function is log-convex (see [24, p. 167]).

2. In view of Part 1 of Theorem 3, the function $\mu \mapsto \log S_{\mu}(r)$ is convex. Hence it follows that the function:

$$\mu \mapsto \log S_{\mu+1}(r) - \log S_{\mu}(r)$$

is increasing on $(0,\infty)$.

3. Since the function $\mu \mapsto \dfrac{S_{\mu}(r)}{S(r)}$ is completely monotonic on $(0,\infty)$ and maps $(0,\infty)$ to $(0,1)$, according to Lemma 2, we conclude the asserted inequality (3.9).

We next prove the inequality (3.10). Suppose that $\mu \geqq \nu > 0$ and define the function $H:(0,\infty) \longrightarrow \mathbb{R}$ with the following relation:

$$H(r) = \left(\frac{\nu+1}{\mu+1}\right)\log S_{\mu}(r) - \log S_{\nu}(r).$$

On the other hand, by using the fact that

$$S'_{\mu}(r) = -2r(\mu+1)S_{\mu+1}(r),$$

we have

$$H'(r) = 2(\nu+1)r\left(\frac{S_{\nu+1}(r)}{S_{\nu}(r)} - \frac{S_{\mu+1}(r)}{S_{\mu}(r)}\right) \leqq 0.$$

So, by Part 2 of Theorem 3, we conclude that the function $H(r)$ is decreasing on $(0,\infty)$. Consequently, we have

$$H(r) \leqq H(0) \qquad (r>0).$$

Now, upon replacing μ by $\mu+1$ and ν by μ in the inequality (3.11), if we use the inequality (3.10) and the *Arithmetic-Geometric Mean Inequality*, we find that

$$\begin{aligned}\frac{1}{2}\left[\left(\frac{S_\mu(r)}{\zeta(2\mu+1)}\right)^{\frac{1}{\mu+1}}+\frac{\zeta(2\mu+3)S_\mu(r)}{\zeta(2\mu+1)S_{\mu+1}(r)}\right]&\\ &\geqq\sqrt{\left(\frac{S_\mu(r)}{\zeta(2\mu+1)}\right)^{\frac{1}{\mu+1}}\cdot\frac{\zeta(2\mu+3)S_\mu(r)}{\zeta(2\mu+1)S_{\mu+1}(r)}}\\ &\geqq 1,\end{aligned}$$

which evidently completes the proof of Theorem 3. □

REMARK 1. There are other proofs of the Turán type inequality (3.4). For example, since the function $\nu\mapsto S_\mu(r)$ is log-convex on $(0,\infty)$ for $r>0$, it follows, for all $\mu_1,\mu_2\geqq 1,\ \alpha\in[0,1]$ and $r>0$, that

$$S_{\alpha\mu_1+(1-\alpha)\mu_2}(r)\leqq\left[S_{\mu_1}(r)\right]^{\alpha}\left[S_{\mu_2}(r)\right]^{1-\alpha}.$$

Upon setting

$$\mu_1=\mu,\quad \mu_2=\mu+2\qquad\text{and}\qquad \alpha=\frac{1}{2},$$

the above inequality reduces to the Turán inequality (3.4).

REMARK 2. Yet another proof of the Turán inequality (3.4) can be given as follows. By using the fact that the function $\mu\mapsto\dfrac{S_{\mu+1}(r)}{S_\mu(r)}$ is increasing on $(0,\infty)$, we have

$$\frac{S_{\mu+2}(r)}{S_{\mu+1}(r)}\geqq\frac{S_{\mu+1}(r)}{S_\mu(r)},$$

which leads us readily to the required result.

THEOREM 4. *For $\mu>\frac{3}{2}$, let c_μ be given by*

$$c_\mu=\frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}}\,\Gamma(\mu+1)}.$$

Also let

$$\mathscr{K}_\mu(t)=t^{\mu+\frac{1}{2}}\,g_\mu(t)$$

in terms of the Schlömilch series $g_\mu(t)$ defined by [5]

$$g_\mu(t)=\sum_{n=1}^{\infty}\frac{J_{\mu+\frac{1}{2}}(nt)}{n^{\mu-\frac{1}{2}}}$$

for the Bessel function $J_\nu(z)$*. Then the Mathieu series* $S_\mu(r)$ *has the following integral representation:*

$$S_\mu(r) = c_\mu \int_0^\infty e^{-rt}\ \mathscr{K}_\mu(t)\, dt. \tag{3.12}$$

Furthermore, it is asserted that

$$\zeta(2\mu+1) = \frac{c_\mu}{2} \int_0^\infty \mathscr{K}_\mu(t)\, dt \tag{3.13}$$

and

$$S(r) = \int_0^\infty e^{-rt}\ \mathscr{K}(t)\, dt, \tag{3.14}$$

where

$$\mathscr{K}(t) = h(t) - t\, h'(t)$$

and the Schlömilch series $h(t)$ *is defined by* [5]

$$h(t) = \sum_{n=1}^\infty \frac{\sin nt}{n^2}$$

in terms of the Bessel function $J_\nu(z)$.

Proof. By using the following known result [6, p. 397, Eq. (42)]:

$$\frac{1}{(a^2+s^2)^{\mu+\frac{1}{2}}} = \frac{\sqrt{\pi}}{(2a)^\mu\, \Gamma\left(\mu+\frac{1}{2}\right)} \int_0^\infty t^\mu\, e^{-st}\, J_\mu(at)\, dt \qquad \left(\mu > -\frac{1}{2}\right), \tag{3.15}$$

we get

$$\frac{2n}{(n^2+r^2)^{\mu+1}} = \frac{\sqrt{\pi}}{(2n)^{\mu-\frac{1}{2}}\, \Gamma(\mu+1)} \int_0^\infty t^{\mu+\frac{1}{2}}\, e^{-rt}\, J_{\mu+\frac{1}{2}}(nt)\, dt, \tag{3.16}$$

which, upon interchanging the order of integration and summation, yields (3.12). Next, if we let $r \to 0$ in (3.12), we obtain (3.13). Finally, if we set $\mu = 1$ in (3.12) and use the fact that

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin x}{x^{\frac{3}{2}}} - \frac{\cos x}{x^{\frac{1}{2}}}\right),$$

we obtain

$$\begin{aligned} S(r) &= \int_0^\infty e^{-rt} \left(\sum_{n=1}^\infty \frac{\sin nt}{n^2} - t\sum_{n=1}^\infty \frac{\cos nt}{n}\right) dt \\ &= \int_0^\infty e^{-rt}\, [h(t) - t\, h'(t)]\, dt \\ &= \int_0^\infty e^{-rt}\ \mathscr{K}(t)\, dt. \end{aligned}$$

The proof of Theorem 4 is thus completed. □

EXAMPLE. From the integral representation (3.14), we can write $S(r)$ in the following form:

$$S(r) = \int_0^\infty e^{-rt} \left[\mathrm{Cl}_2(t) - t\,\mathrm{Cl}_2'(t)\right] dt, \tag{3.17}$$

where $\mathrm{Cl}_2(x)$ is the Clausen function defined by (see, for example, [5])

$$\mathrm{Cl}_2(x) = \sum_{n=1}^\infty \frac{\sin kx}{k^2} = -\int_0^x \log\left[2\sin\left(\frac{1}{2}\eta\right)\right] d\eta \qquad (x \in \mathbb{R}). \tag{3.18}$$

Upon letting $r \to 0+$ in (3.17), we get the following integral formula for the Apéry constant $\zeta(3)$:

$$\zeta(3) = \frac{1}{2}\int_0^\infty \mathscr{K}(t)\,dt = \frac{1}{2}\int_0^\infty \left[\mathrm{Cl}_2(t) - t\,\mathrm{Cl}_2'(t)\right] dt. \tag{3.19}$$

PROPOSITION. *Let $\mu > \frac{7}{6}$. Then the following inequality holds true:*

$$S_\mu(r) \leqq \left(\frac{c_L\sqrt{\pi}}{2^{\mu-\frac{1}{2}}\, r^{\mu+\frac{7}{6}}}\right) \frac{\Gamma\left(\mu+\frac{7}{6}\right)\zeta\left(\mu-\frac{1}{6}\right)}{\Gamma(\mu+1)} \qquad (r>0), \tag{3.20}$$

where

$$c_L = \sup_{x>0}\{x^{\frac{1}{3}} J_0(x)\} = 0.78574687\cdots. \tag{3.21}$$

Proof. We recall the Landau estimate (see [12]):

$$|J_\mu(x)| \leqq c_L x^{-\frac{1}{3}},$$

which holds true uniformly in μ and where c_L is given by the equation (3.21). If we now make use of the integral representation (3.12), we are led easily to the inequality (3.20). □

4. New inequalities for the Riemann zeta function

In this section, we first present a new Turán type inequality for the Riemann zeta function $\zeta(s)$ defined by (1.6).

THEOREM 5. *Let $\mu > 1$. Then the following Turán type inequality:*

$$\zeta(\mu)\zeta(\mu+2) - [\zeta(\mu+1)]^2 \geqq 0. \tag{4.1}$$

holds true.

Proof. By letting $r \to 0$ in (3.4), we get

$$\zeta(2\mu+1)\zeta(2\mu+3) - [\zeta(2\mu+2)]^2 \geqq 0,$$

which, upon replacing μ by $\frac{1}{2}(\mu-1)$, yields the inequality (4.1) asserted by Theorem 5. □

REMARK 3. By using a generalization of the familiar Schwarz inequality, Laforgia and Natalini [13] proved the following Turán type inequality for the Riemann zeta function:

$$\zeta(\mu)\zeta(\mu+2) \geqq \frac{\mu}{\mu+1}[\zeta(\mu+1)]^2 \qquad (\mu > 1). \tag{4.2}$$

We note that the inequality (4.1), which is asserted by Theorem 5, is sharper than the inequality (4.2).

In Theorem 6 below, we establish a simple upper bound for the Riemann zeta function. Our main tool will be the formula (2.3) in Lemma 3.

THEOREM 6. *Let* $\mu \geqq 1$. *Then each of the following inequalities holds true:*

$$\zeta(2\mu) \leqq \sqrt{\frac{3\pi}{2}}\,\frac{\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{1}{2}\right)} \tag{4.3}$$

and

$$\frac{[\zeta(2\mu+1)]^{\frac{3}{2}}}{[\zeta(2\mu)]^2\,\zeta(2\mu+3)} \leqq (\mu+1)\left(\frac{\left[\Gamma\left(\mu+\frac{1}{2}\right)\right]}{[\Gamma(\mu+1)]}\right)^2. \tag{4.4}$$

Proof. Alzer *et al.* [1] proved that

$$S(r) < \frac{1}{r^2+\frac{1}{6}} \qquad (r > 0), \tag{4.5}$$

which is the second part of the inequality in (1.5). Since the function $\mu \mapsto S_\mu(r)$ is decreasing on $[1,\infty)$, we get

$$S_\mu(r) < \frac{1}{r^2+\frac{1}{6}} \qquad (r > 0;\ \mu \geqq 1). \tag{4.6}$$

Integrating both sides of this last equation (4.6) over the interval $(0,\infty)$, we obtain

$$\int_0^\infty S_\mu(r)\,dr \leqq \pi\sqrt{\frac{3}{2}}.$$

Thus, clearly, Lemma 3 completes the proof of the first inequality (4.3) asserted by Theorem 6.

We next prove the second inequality (4.4) asserted by Theorem 6. We recall that Tomovski and Mehrez [22] derived the following inequality:

$$2\zeta(2\mu+1)\exp\left(-(\mu+1)\,\frac{\zeta(2\mu+3)}{\zeta(2\mu+1)}\,r^2\right) \leqq S_\mu(r) \qquad (r > 0). \tag{4.7}$$

Therefore, upon integrating both sides of (4.7) over the interval $(0,\infty)$, if we apply Lemma 3, we see that the inequality (4.4) holds true. This evidently completes the proof of Theorem 6. $\square$

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