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PROPERTIES AND REFINEMENTS OF ACZÉL–TYPE INEQUALITIES

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Abstract. In this paper, we present some new properties of Aczél-type inequalities, and then we obtain some new refinements of Aczél-type inequalities.

1. Introduction

The famous Aczél's inequality, which is of wide application in the theory of functional equations in non-Euclidean geometry, was given by Aczél [1] as follows.

THEOREM A. *Let $n \in \mathbb{N}^+$, $n \geqslant 2$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be real numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ and $b_1^2 - \sum_{i=2}^n b_i^2 > 0$. Then*

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \leqslant \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2. \tag{1}$$

In 1959, Popoviciu [3] first established an exponential extension of the inequality (1) in the following theorem.

THEOREM B. *Let $n \in \mathbb{N}^+$, $n \geqslant 2$, let $\lambda_1 > 1$, $\lambda_2 > 1$, $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be nonnegative real numbers such that $a_1^{\lambda_1} - \sum_{i=2}^n a_i^{\lambda_1} > 0$ and $b_1^{\lambda_2} - \sum_{i=2}^n b_i^{\lambda_2} > 0$. Then*

$$\left(a_1^{\lambda_1} - \sum_{i=2}^n a_i^{\lambda_1}\right)^{\frac{1}{\lambda_1}}\left(b_1^{\lambda_2} - \sum_{i=2}^n b_i^{\lambda_2}\right)^{\frac{1}{\lambda_2}} \leqslant a_1 b_1 - \sum_{i=2}^n a_i b_i, \tag{2}$$

which is called as Aczél-Popoviciu inequality.

In 1982, Vasić and Pečarić [9] presented the following reversed version of Aczél-Popoviciu inequality (2).

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THEOREM C. *Let $n \in \mathbb{N}^+$, $n \geqslant 2$, let $\lambda_1 < 1$ $(\lambda_1 \neq 0)$, $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive real numbers such that $a_1^{\lambda_1} - \sum_{i=2}^n a_i^{\lambda_1} > 0$ and $b_1^{\lambda_2} - \sum_{i=2}^n b_i^{\lambda_2} > 0$. Then*

$$\left(a_1^{\lambda_1} - \sum_{i=2}^n a_i^{\lambda_1}\right)^{\frac{1}{\lambda_1}} \left(b_1^{\lambda_2} - \sum_{i=2}^n b_i^{\lambda_2}\right)^{\frac{1}{\lambda_2}} \geqslant a_1 b_1 - \sum_{i=2}^n a_i b_i, \tag{3}$$

which is called as Aczél-Vasić-Pečarić inequality.

In another paper, Vasić and Pečarić [10] obtained a further extension of the Aczél inequality (1) as follows.

THEOREM D. *Let $n, m \in \mathbb{N}^+$, $n \geqslant 2$, let $\lambda_j > 0$, $\sum_{j=1}^m \frac{1}{\lambda_j} \geqslant 1$, and let a_{rj} $(r = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$ be positive real numbers such that $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$ $(j = 1, 2, \cdots, m)$. Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leqslant \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \tag{4}$$

In 2012, Tian [5] gave the reversed version of inequality (4) as follows.

THEOREM E. *Let $n, m \in \mathbb{N}^+$, $n \geqslant 2$, let $\lambda_1 \neq 0$, $\lambda_j < 0$ $(j = 2, 3, \cdots, m)$, $\sum_{j=1}^m \frac{1}{\lambda_j} \leqslant 1$, and let a_{rj} $(r = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$ be positive real numbers such that $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$ $(j = 1, 2, \cdots, m)$. Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geqslant \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \tag{5}$$

Moreover, Bjelica in [2] obtained a new interesting Aczél-type inequality as follows.

THEOREM F. *Let $n \in \mathbb{N}^+$, $n \geqslant 2$, let $0 < \lambda \leqslant 2$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be nonnegative real numbers such that $a_1^{\lambda} - \sum_{i=2}^n a_i^{\lambda} > 0$ and $b_1^{\lambda} - \sum_{i=2}^n b_i^{\lambda} > 0$. Then*

$$\left(a_1^{\lambda} - \sum_{i=2}^n a_i^{\lambda}\right)^{\frac{1}{\lambda}} \left(b_1^{\lambda} - \sum_{i=2}^n b_i^{\lambda}\right)^{\frac{1}{\lambda}} \leqslant a_1 b_1 - \sum_{i=2}^n a_i b_i, \tag{6}$$

which is called as Aczél-Bjelica inequality.

Recently, Tian and Zhou [8] presented the reversed version of Aczél-Bjelica inequality (6) as follows.

THEOREM G. *Let $n \in \mathbb{N}^+$, $n \geqslant 2$, let $\lambda < 0$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive real numbers such that $a_1^{\lambda} - \sum_{i=2}^n a_i^{\lambda} > 0$ and $b_1^{\lambda} - \sum_{i=2}^n b_i^{\lambda} > 0$. Then*

$$\left(a_1^{\lambda} - \sum_{i=2}^n a_i^{\lambda}\right)^{\frac{1}{\lambda}} \left(b_1^{\lambda} - \sum_{i=2}^n b_i^{\lambda}\right)^{\frac{1}{\lambda}} \geqslant a_1 b_1 - \sum_{i=2}^n a_i b_i. \tag{7}$$

An important research subject in analyzing inequality is to convert an univariate into the monotonicity of functions [4, 6, 7]. In this paper, we give some new monotonicity properties of the above Aczél-type inequalities (2)–(7), and then we obtain some new refinements of Aczél-type inequalities (2)–(7).

2. Main results

LEMMA 2.1. [10] *Let* $a_{ij} > 0 \ (i = 1,2,\cdots,n,\ j = 1,2,\cdots,m)$.
(a) If $\lambda_j > 0$, *and if* $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1$, *then*

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \leqslant \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}. \tag{8}$$

(b) If $\lambda_j < 0 \ (j = 1,2,\cdots,m)$, *then*

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \geqslant \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}. \tag{9}$$

(c) If $\lambda_1 > 0,\ \lambda_j < 0 \ (j = 2,3,\cdots,m)$, *and if* $\sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$, *then*

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \geqslant \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}. \tag{10}$$

The above inequalities (8), (9) and (10) are called as generalized Hölder's inequalities.

THEOREM 2.2. *Let* $n,m \in \mathbb{N}^+$, $n \geqslant 2$, *let* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m > 0$ *with* $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1$, *and let* $a_{rj} \ (r = 1,2,\cdots,n;\ j = 1,2,\cdots,m)$ *be positive real numbers such that* $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0 \ (j = 1,2,\cdots,m)$. *If we denote*

$$\widetilde{V}(n) = \prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{2}{\lambda_j}} - \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right)^2,$$

then

$$\widetilde{V}(n+1) \leqslant \widetilde{V}(n) \leqslant 0. \tag{11}$$

Proof. Put

$$\Phi(n) = \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right)^2,$$

$$\Psi(n) = \prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{2}{\lambda_j}}.$$

1° Let $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$, and m be even. After simple calculation we get

$$\begin{aligned}
\Psi(n) = &\left[\left(a_{12}^{\lambda_2} - \sum_{i=2}^{n} a_{i2}^{\lambda_2}\right)\left(a_{12}^{\lambda_2} - \sum_{j=2}^{n} a_{j2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}} \\
&\times\left[\left(a_{11}^{\lambda_1} - \sum_{i=2}^{n} a_{i1}^{\lambda_1}\right)\left(a_{12}^{\lambda_2} - \sum_{j=2}^{n} a_{j2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_1}} \\
&\times\left[\left(a_{12}^{\lambda_2} - \sum_{i=2}^{n} a_{i2}^{\lambda_2}\right)\left(a_{11}^{\lambda_1} - \sum_{j=2}^{n} a_{j1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_1}} \\
&\times\left[\left(a_{14}^{\lambda_4} - \sum_{i=2}^{n} a_{i4}^{\lambda_4}\right)\left(a_{14}^{\lambda_4} - \sum_{j=2}^{n} a_{j4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}} \\
&\times\left[\left(a_{13}^{\lambda_3} - \sum_{i=2}^{n} a_{i3}^{\lambda_3}\right)\left(a_{14}^{\lambda_4} - \sum_{j=2}^{n} a_{j4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_3}} \\
&\times\left[\left(a_{14}^{\lambda_4} - \sum_{i=2}^{n} a_{i4}^{\lambda_4}\right)\left(a_{13}^{\lambda_3} - \sum_{j=2}^{n} a_{j3}^{\lambda_3}\right)\right]^{\frac{1}{\lambda_3}} \\
&\times\ldots \\
&\times\left[\left(a_{1m}^{\lambda_m} - \sum_{i=2}^{n} a_{im}^{\lambda_m}\right)\left(a_{1m}^{\lambda_m} - \sum_{j=2}^{n} a_{jm}^{\lambda_m}\right)\right]^{\frac{1}{\lambda_m}-\frac{1}{\lambda_{m-1}}} \\
&\times\left[\left(a_{1(m-1)}^{\lambda_{m-1}} - \sum_{i=2}^{n} a_{i(m-1)}^{\lambda_{m-1}}\right)\left(a_{1m}^{\lambda_m} - \sum_{j=2}^{n} a_{jm}^{\lambda_m}\right)\right]^{\frac{1}{\lambda_{m-1}}} \\
&\times\left[\left(a_{1m}^{\lambda_m} - \sum_{i=2}^{n} a_{im}^{\lambda_m}\right)\left(a_{1(m-1)}^{\lambda_{m-1}} - \sum_{j=1}^{n} a_{j(m-1)}^{\lambda_{m-1}}\right)\right]^{\frac{1}{\lambda_{m-1}}}. \qquad (12)
\end{aligned}$$

By (4) and performing some simple computations, we have

$$\begin{aligned}
&\Phi(n+1) - \Phi(n) \\
&= \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1}\prod_{j=1}^{m} a_{rj}\right)^2 - \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}\right)^2 \\
&= \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1}\prod_{j=1}^{m} a_{rj} - \prod_{j=1}^{m} a_{1j} + \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}\right) \\
&\quad\times\left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1}\prod_{j=1}^{m} a_{rj} + \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}\right) \\
&= -\left(\prod_{j=1}^{k} a_{(n+1)j}\right)\left[\left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}\right) + \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1}\prod_{j=1}^{m} a_{rj}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant -\left(\prod_{j=1}^{m} a_{(n+1)j}\right)\left[\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n+1} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]\\
&= -\left[\left(\prod_{j=1}^{m} a_{(n+1)j}\right)\left(a_{11}^{\lambda_1}-\sum_{i=2}^{n} a_{i1}^{\lambda_1}\right)^{\frac{1}{\lambda_1}}\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n} a_{i2}^{\lambda_2}\right)^{\frac{1}{\lambda_2}}\cdot\ldots\cdot\left(a_{1m}^{\lambda_m}-\sum_{i=2}^{n} a_{im}^{\lambda_m}\right)^{\frac{1}{\lambda_m}}\right.\\
&\quad\left.+\left(\prod_{j=1}^{m} a_{(n+1)j}\right)\left(a_{11}^{\lambda_1}-\sum_{i=2}^{n+1} a_{i1}^{\lambda_1}\right)^{\frac{1}{\lambda_1}}\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n+1} a_{i2}^{\lambda_2}\right)^{\frac{1}{\lambda_2}}\cdot\ldots\cdot\left(a_{1m}^{\lambda_m}-\sum_{i=2}^{n+1} a_{im}^{\lambda_m}\right)^{\frac{1}{\lambda_m}}\right]\\
&= -\left\{\left[a_{(n+1)2}^{\lambda_2}\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n} a_{i2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}\left[a_{(n+1)1}^{\lambda_1}\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n} a_{i2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_1}}\right.\\
&\quad\times\left[a_{(n+1)2}^{\lambda_2}\left(a_{11}^{\lambda_1}-\sum_{i=2}^{n} a_{i1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_1}}\left[a_{(n+1)4}^{\lambda_4}\left(a_{14}^{\lambda_4}-\sum_{i=2}^{n} a_{i4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}}\\
&\quad\times\left[a_{(n+1)3}^{\lambda_3}\left(a_{14}^{\lambda_4}-\sum_{i=2}^{n} a_{i4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_3}}\left[a_{(n+1)4}^{\lambda_4}\left(a_{13}^{\lambda_3}-\sum_{i=2}^{n} a_{i3}^{\lambda_3}\right)\right]^{\frac{1}{\lambda_3}}\\
&\quad\times\ldots\\
&\quad\times\left[a_{(n+1)m}^{\lambda_m}\left(a_{1m}^{\lambda_m}-\sum_{i=2}^{n} a_{im}^{\lambda_m}\right)\right]^{\frac{1}{\lambda_m}-\frac{1}{\lambda_{m-1}}}\left[a_{(n+1)(m-1)}^{\lambda_{m-1}}\left(a_{1m}^{\lambda_m}-\sum_{i=2}^{n} a_{im}^{\lambda_m}\right)\right]^{\frac{1}{\lambda_{m-1}}}\\
&\quad\times\left[a_{(n+1)m}^{\lambda_m}\left(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n} a_{i(m-1)}^{\lambda_{m-1}}\right)\right]^{\frac{1}{\lambda_{m-1}}}+\left[a_{(n+1)2}^{\lambda_2}\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n+1} a_{i2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}\\
&\quad\times\left[a_{(n+1)2}^{\lambda_2}\left(a_{11}^{\lambda_1}-\sum_{i=2}^{n+1} a_{i1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_1}}\left[a_{(n+1)1}^{\lambda_1}\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n+1} a_{i2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_1}}\\
&\quad\times\left[a_{(n+1)4}^{\lambda_4}\left(a_{14}^{\lambda_4}-\sum_{i=2}^{n+1} a_{i4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}}\left[a_{(n+1)4}^{\lambda_4}\left(a_{13}^{\lambda_3}-\sum_{i=2}^{n+1} a_{i3}^{\lambda_3}\right)\right]^{\frac{1}{\lambda_3}}\\
&\quad\times\left[a_{(n+1)3}^{\lambda_3}\left(a_{14}^{\lambda_4}-\sum_{i=2}^{n+1} a_{i4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_3}}\\
&\quad\times\ldots\\
&\quad\times\left[a_{(n+1)m}^{\lambda_m}\left(a_{1m}^{\lambda_m}-\sum_{i=2}^{n+1} a_{im}^{\lambda_m}\right)\right]^{\frac{1}{\lambda_m}-\frac{1}{\lambda_{m-1}}}\left[a_{(n+1)m}^{\lambda_m}\left(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n+1} a_{i(m-1)}^{\lambda_{m-1}}\right)\right]^{\frac{1}{\lambda_{m-1}}}\\
&\quad\left.\times\left[a_{(n+1)(m-1)}^{\lambda_{m-1}}\left(a_{1m}^{\lambda_m}-\sum_{i=2}^{n+1} a_{im}^{\lambda_m}\right)\right]^{\frac{1}{\lambda_{m-1}}}\right\}. \qquad (13)
\end{aligned}
$$

Hence, by (13), (12) and (8), we get

$$\begin{aligned}
&\Phi(n+1)-\Phi(n)-\Psi(n+1)\\
\geqslant &-\Bigg\{\Bigg[a_{(n+1)2}^{\lambda_2}\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n}a_{i2}^{\lambda_2}\right)+a_{(n+1)2}^{\lambda_2}\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n+1}a_{i2}^{\lambda_2}\right)\\
&+\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n+1}a_{i2}^{\lambda_2}\right)\left(a_{12}^{\lambda_2}-\sum_{j=2}^{n+1}a_{j2}^{\lambda_2}\right)\Bigg]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}\\
&\times\Bigg[a_{(n+1)1}^{\lambda_1}\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n}a_{i2}^{\lambda_2}\right)+a_{(n+1)2}^{\lambda_2}\left(a_{11}^{\lambda_1}-\sum_{i=2}^{n+1}a_{i1}^{\lambda_1}\right)\\
&+\left(a_{11}^{\lambda_1}-\sum_{i=2}^{n+1}a_{i1}^{\lambda_1}\right)\left(a_{12}^{\lambda_2}-\sum_{j=2}^{n+1}a_{j2}^{\lambda_2}\right)\Bigg]^{\frac{1}{\lambda_1}}\\
&\times\Bigg[a_{(n+1)2}^{\lambda_2}\left(a_{11}^{\lambda_1}-\sum_{i=2}^{n}a_{i1}^{\lambda_1}\right)+a_{(n+1)1}^{\lambda_1}\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n+1}a_{i2}^{\lambda_2}\right)\\
&+\left(a_{12}^{\lambda_2}-\sum_{i=2}^{n+1}a_{i2}^{\lambda_2}\right)\left(a_{11}^{\lambda_1}-\sum_{j=2}^{n+1}a_{j1}^{\lambda_1}\right)\Bigg]^{\frac{1}{\lambda_1}}\\
&\times\Bigg[a_{(n+1)4}^{\lambda_4}\left(a_{14}^{\lambda_4}-\sum_{i=2}^{n}a_{i4}^{\lambda_4}\right)+a_{(n+1)4}^{\lambda_4}\left(a_{14}^{\lambda_4}-\sum_{i=2}^{n+1}a_{i4}^{\lambda_4}\right)\\
&+\left(a_{14}^{\lambda_4}-\sum_{i=2}^{n+1}a_{i4}^{\lambda_4}\right)\left(a_{14}^{\lambda_4}-\sum_{j=2}^{n+1}a_{j4}^{\lambda_4}\right)\Bigg]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}}\\
&\times\Bigg[a_{(n+1)3}^{\lambda_3}\left(a_{14}^{\lambda_4}-\sum_{i=2}^{n}a_{i4}^{\lambda_4}\right)+a_{(n+1)4}^{\lambda_4}\left(a_{13}^{\lambda_3}-\sum_{i=2}^{n+1}a_{i3}^{\lambda_3}\right)\\
&+\left(a_{13}^{\lambda_3}-\sum_{i=2}^{n+1}a_{i3}^{\lambda_3}\right)\left(a_{14}^{\lambda_4}-\sum_{j=2}^{n+1}a_{j4}^{\lambda_4}\right)\Bigg]^{\frac{1}{\lambda_3}}\\
&\times\Bigg[a_{(n+1)4}^{\lambda_4}\left(a_{13}^{\lambda_3}-\sum_{i=2}^{n}a_{i3}^{\lambda_3}\right)+a_{(n+1)3}^{\lambda_3}\left(a_{14}^{\lambda_4}-\sum_{i=2}^{n+1}a_{i4}^{\lambda_4}\right)\\
&+\left(a_{14}^{\lambda_4}-\sum_{i=2}^{n+1}a_{i4}^{\lambda_4}\right)\left(a_{13}^{\lambda_3}-\sum_{j=2}^{n+1}a_{j3}^{\lambda_3}\right)\Bigg]^{\frac{1}{\lambda_3}}\\
&\times\dots\\
&\times\Bigg[a_{(n+1)m}^{\lambda_m}\left(a_{1m}^{\lambda_m}-\sum_{i=2}^{n}a_{im}^{\lambda_m}\right)+a_{(n+1)m}^{\lambda_m}\left(a_{1m}^{\lambda_m}-\sum_{i=2}^{n+1}a_{im}^{\lambda_m}\right)\\
&+\left(a_{1m}^{\lambda_m}-\sum_{i=2}^{n+1}a_{im}^{\lambda_m}\right)\left(a_{1m}^{\lambda_m}-\sum_{j=2}^{n+1}a_{jm}^{\lambda_m}\right)\Bigg]^{\frac{1}{\lambda_m}-\frac{1}{\lambda_{m-1}}}
\end{aligned}$$

$$
\begin{aligned}
&\times\Bigg[a_{(n+1)(m-1)}^{\lambda_{m-1}}\Bigg(a_{1m}^{\lambda_m}-\sum_{i=2}^{n}a_{im}^{\lambda_m}\Bigg)+a_{(n+1)m}^{\lambda_m}\Bigg(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n+1}a_{i(m-1)}^{\lambda_{m-1}}\Bigg)\\
&+\Bigg(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n+1}a_{i(m-1)}^{\lambda_{m-1}}\Bigg)\Bigg(a_{1m}^{\lambda_m}-\sum_{j=2}^{n+1}a_{jm}^{\lambda_m}\Bigg)\Bigg]^{\frac{1}{\lambda_{m-1}}}\\
&\times\Bigg[a_{(n+1)m}^{\lambda_m}\Bigg(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n}a_{i(m-1)}^{\lambda_{m-1}}\Bigg)+a_{(n+1)(m-1)}^{\lambda_{m-1}}\Bigg(a_{1m}^{\lambda_m}-\sum_{i=2}^{n+1}a_{im}^{\lambda_m}\Bigg)\\
&+\Bigg(a_{1m}^{\lambda_m}-\sum_{i=2}^{n+1}a_{im}^{\lambda_m}\Bigg)\Bigg(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{j=2}^{n+1}a_{j(m-1)}^{\lambda_{m-1}}\Bigg)\Bigg]^{\frac{1}{\lambda_{m-1}}}\Bigg\}\\
=&-\Bigg\{\Bigg[a_{(n+1)2}^{\lambda_2}\Bigg(2\Bigg(a_{12}^{\lambda_2}-\sum_{i=2}^{n+1}a_{i2}^{\lambda_2}\Bigg)+a_{(n+1)2}^{\lambda_2}\Bigg)+\Bigg(a_{12}^{\lambda_2}-\sum_{i=2}^{n+1}a_{i2}^{\lambda_2}\Bigg)^2\Bigg]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}\\
&\times\Bigg[a_{(n+1)1}^{\lambda_1}\Bigg(a_{12}^{\lambda_2}-\sum_{i=2}^{n}a_{i2}^{\lambda_2}\Bigg)+\Bigg(a_{11}^{\lambda_1}-\sum_{i=2}^{n+1}a_{i1}^{\lambda_1}\Bigg)\Bigg(a_{12}^{\lambda_2}-\sum_{j=2}^{n}a_{j2}^{\lambda_2}\Bigg)\Bigg]^{\frac{1}{\lambda_1}}\\
&\times\Bigg[a_{(n+1)2}^{\lambda_2}\Bigg(a_{11}^{\lambda_1}-\sum_{i=2}^{n}a_{i1}^{\lambda_1}\Bigg)+\Bigg(a_{12}^{\lambda_2}-\sum_{i=2}^{n+1}a_{i2}^{\lambda_2}\Bigg)\Bigg(a_{11}^{\lambda_1}-\sum_{j=2}^{n}a_{j1}^{\lambda_1}\Bigg)\Bigg]^{\frac{1}{\lambda_1}}\\
&\times\Bigg[a_{(n+1)4}^{\lambda_4}\Bigg(2\Bigg(a_{14}^{\lambda_4}-\sum_{i=2}^{n+1}a_{i4}^{\lambda_4}\Bigg)+a_{(n+1)4}^{\lambda_4}\Bigg)+\Bigg(a_{14}^{\lambda_4}-\sum_{i=2}^{n+1}a_{i4}^{\lambda_4}\Bigg)^2\Bigg]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}}\\
&\times\Bigg[a_{(n+1)3}^{\lambda_3}\Bigg(a_{14}^{\lambda_4}-\sum_{i=2}^{n}a_{i4}^{\lambda_4}\Bigg)+\Bigg(a_{13}^{\lambda_3}-\sum_{i=2}^{n+1}a_{i3}^{\lambda_3}\Bigg)\Bigg(a_{14}^{\lambda_4}-\sum_{j=2}^{n}a_{j4}^{\lambda_4}\Bigg)\Bigg]^{\frac{1}{\lambda_3}}\\
&\times\Bigg[a_{(n+1)4}^{\lambda_4}\Bigg(a_{13}^{\lambda_3}-\sum_{i=2}^{n}a_{i3}^{\lambda_3}\Bigg)+\Bigg(a_{14}^{\lambda_4}-\sum_{i=2}^{n+1}a_{i4}^{\lambda_4}\Bigg)\Bigg(a_{13}^{\lambda_3}-\sum_{j=2}^{n}a_{j3}^{\lambda_3}\Bigg)\Bigg]^{\frac{1}{\lambda_3}}\\
&\times\dots\\
&\times\Bigg[a_{(n+1)m}^{\lambda_m}\Bigg(2\Bigg(a_{1m}^{\lambda_m}-\sum_{i=2}^{n+1}a_{im}^{\lambda_m}\Bigg)+a_{(n+1)m}^{\lambda_m}\Bigg)+\Bigg(a_{1m}^{\lambda_m}-\sum_{i=2}^{n+1}a_{im}^{\lambda_m}\Bigg)^2\Bigg]^{\frac{1}{\lambda_m}-\frac{1}{\lambda_{m-1}}}\\
&\times\Bigg[a_{(n+1)(m-1)}^{\lambda_{m-1}}\Bigg(a_{1m}^{\lambda_m}-\sum_{i=2}^{n}a_{im}^{\lambda_m}\Bigg)+\Bigg(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n+1}a_{i(m-1)}^{\lambda_{m-1}}\Bigg)\Bigg(a_{1m}^{\lambda_m}-\sum_{j=2}^{n}a_{jm}^{\lambda_m}\Bigg)\Bigg]^{\frac{1}{\lambda_{m-1}}}\\
&\times\Bigg[a_{(n+1)m}^{\lambda_m}\Bigg(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n}a_{i(m-1)}^{\lambda_{m-1}}\Bigg)\\
&+\Bigg(a_{1m}^{\lambda_m}-\sum_{i=2}^{n+1}a_{im}^{\lambda_m}\Bigg)\Bigg(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{j=2}^{n}a_{j(m-1)}^{\lambda_{m-1}}\Bigg)\Bigg]^{\frac{1}{\lambda_{m-1}}}\Bigg\}
\end{aligned}
$$

$$= -\Bigg\{\Bigg[\left(a_{(n+1)2}^{\lambda_2}\right)^2 + 2a_{(n+1)2}^{\lambda_2}\left(a_{12}^{\lambda_2} - \sum_{i=2}^{n+1} a_{i2}^{\lambda_2}\right) + \left(a_{12}^{\lambda_2} - \sum_{i=2}^{n+1} a_{i2}^{\lambda_2}\right)^2\Bigg]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}$$

$$\times\left[\left(a_{12}^{\lambda_2} - \sum_{i=2}^{n} a_{i2}^{\lambda_2}\right)\left(a_{11}^{\lambda_1} - \sum_{i=2}^{n} a_{i1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_1}}\left[\left(a_{11}^{\lambda_1} - \sum_{i=2}^{n} a_{i1}^{\lambda_1}\right)\left(a_{12}^{\lambda_2} - \sum_{i=2}^{n} a_{i2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_1}}$$

$$\times\Bigg[\left(a_{(n+1)4}^{\lambda_4}\right)^2 + 2a_{(n+1)4}^{\lambda_4}\left(a_{14}^{\lambda_4} - \sum_{i=2}^{n+1} a_{i4}^{\lambda_4}\right) + \left(a_{14}^{\lambda_4} - \sum_{i=2}^{n+1} a_{i4}^{\lambda_4}\right)^2\Bigg]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}}$$

$$\times\left[\left(a_{14}^{\lambda_4} - \sum_{i=2}^{n} a_{i4}^{\lambda_4}\right)\left(a_{13}^{\lambda_3} - \sum_{i=2}^{n} a_{i3}^{\lambda_3}\right)\right]^{\frac{1}{\lambda_3}}\left[\left(a_{13}^{\lambda_3} - \sum_{i=2}^{n} a_{i3}^{\lambda_3}\right)\left(a_{14}^{\lambda_4} - \sum_{i=2}^{n} a_{i4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_3}}$$

$$\times \dots$$

$$\times\Bigg[\left(a_{(n+1)m}^{\lambda_m}\right)^2 + 2a_{(n+1)m}^{\lambda_m}\left(a_{1m}^{\lambda_m} - \sum_{i=2}^{n+1} a_{im}^{\lambda_m}\right) + \left(a_{1m}^{\lambda_m} - \sum_{i=2}^{n+1} a_{im}^{\lambda_m}\right)^2\Bigg]^{\frac{1}{\lambda_m}-\frac{1}{\lambda_{m-1}}}$$

$$\times\left[\left(a_{1m}^{\lambda_m} - \sum_{i=2}^{n} a_{im}^{\lambda_m}\right)\left(a_{1(m-1)}^{\lambda_{m-1}} - \sum_{i=2}^{n} a_{i(m-1)}^{\lambda_{m-1}}\right)\right]^{\frac{1}{\lambda_{m-1}}}$$

$$\times\left[\left(a_{1(m-1)}^{\lambda_{m-1}} - \sum_{i=2}^{n} a_{i(m-1)}^{\lambda_{m-1}}\right)\left(a_{1m}^{\lambda_m} - \sum_{i=2}^{n} a_{im}^{\lambda_m}\right)\right]^{\frac{1}{\lambda_{m-1}}}\Bigg\}$$

$$= -\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}} = -\Psi(n), \tag{14}$$

which implies

$$\Phi(n+1) - \Psi(n+1) \geqslant \Phi(n) - \Psi(n).$$

So

$$\Psi(n+1) - \Phi(n+1) \leqslant \Psi(n) - \Phi(n),$$

and then, we have

$$\widetilde{V}(n+1) \leqslant \widetilde{V}(n). \tag{15}$$

Thus, by inequalities (4) and (15), we obtain immediately the desired inequality (11).

2° Let $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ and k be odd. After simple calculation we get

$$\Psi(n) = \left[\left(a_{12}^{\lambda_2} - \sum_{i=2}^{n} a_{i2}^{\lambda_2}\right)\left(a_{12}^{\lambda_2} - \sum_{j=2}^{n} a_{j2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}\left[\left(a_{11}^{\lambda_1} - \sum_{i=2}^{n} a_{i1}^{\lambda_1}\right)\left(a_{12}^{\lambda_2} - \sum_{j=2}^{n} a_{j2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_1}}$$

$$\times\left[\left(a_{12}^{\lambda_2} - \sum_{i=2}^{n} a_{i2}^{\lambda_2}\right)\left(a_{11}^{\lambda_1} - \sum_{j=2}^{n} a_{j1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_1}}\left[\left(a_{14}^{\lambda_4} - \sum_{i=2}^{n} a_{i4}^{\lambda_4}\right)\left(a_{14}^{\lambda_4} - \sum_{j=2}^{n} a_{j4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}}$$

$$\times\left[\left(a_{13}^{\lambda_3} - \sum_{i=2}^{n} a_{i3}^{\lambda_3}\right)\left(a_{14}^{\lambda_4} - \sum_{j=2}^{n} a_{j4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_3}}\left[\left(a_{14}^{\lambda_4} - \sum_{i=2}^{n} a_{i4}^{\lambda_4}\right)\left(a_{13}^{\lambda_3} - \sum_{j=2}^{n} a_{j3}^{\lambda_3}\right)\right]^{\frac{1}{\lambda_3}}$$

$$
\begin{aligned}
&\times \dots \\
&\times \left[\left(a_{1(m-1)}^{\lambda_{m-1}} - \sum_{i=2}^{n} a_{i(m-1)}^{\lambda_{m-1}} \right) \left(a_{1(m-1)}^{\lambda_{m-1}} - \sum_{j=2}^{n} a_{j(m-1)}^{\lambda_{m-1}} \right) \right]^{\frac{1}{\lambda_{m-1}} - \frac{1}{\lambda_{m-2}}} \\
&\times \left[\left(a_{1(m-2)}^{\lambda_{m-2}} - \sum_{i=2}^{n} a_{i(m-2)}^{\lambda_{m-2}} \right) \left(a_{1(m-1)}^{\lambda_{m-1}} - \sum_{j=2}^{n} a_{j(m-1)}^{\lambda_{m-1}} \right) \right]^{\frac{1}{\lambda_{m-2}}} \\
&\times \left[\left(a_{1(m-1)}^{\lambda_{m-1}} - \sum_{i=2}^{n} a_{i(m-1)}^{\lambda_{m-1}} \right) \left(a_{1(m-2)}^{\lambda_{m-2}} - \sum_{j=1}^{n} a_{j(m-2)}^{\lambda_{m-2}} \right) \right]^{\frac{1}{\lambda_{m-2}}} \left(a_{1m}^{\lambda_m} - \sum_{i=2}^{n} a_{im}^{\lambda_m} \right)^{\frac{2}{\lambda_m}}.
\end{aligned}
\tag{16}
$$

From inequality (5) and performing some simple computations, we get

$$
\begin{aligned}
&\Phi(n+1) - \Phi(n) \\
&= \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1} \prod_{j=1}^{m} a_{rj} \right)^2 - \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right)^2 \\
&= \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1} \prod_{j=1}^{m} a_{rj} - \prod_{j=1}^{m} a_{1j} + \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right) \\
&\quad \times \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1} \prod_{j=1}^{m} a_{rj} + \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right) \\
&= -\left(\prod_{j=1}^{k} a_{(n+1)j} \right) \left[\left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right) + \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1} \prod_{j=1}^{m} a_{rj} \right) \right] \\
&\leqslant -\left(\prod_{j=1}^{m} a_{(n+1)j} \right) \left[\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} + \prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n+1} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \right] \\
&= -\left[\left(\prod_{j=1}^{m} a_{(n+1)j} \right) \left(a_{11}^{\lambda_1} - \sum_{i=2}^{n} a_{i1}^{\lambda_1} \right)^{\frac{1}{\lambda_1}} \left(a_{12}^{\lambda_2} - \sum_{i=2}^{n} a_{i2}^{\lambda_2} \right)^{\frac{1}{\lambda_2}} \cdot \ldots \cdot \left(a_{1m}^{\lambda_m} - \sum_{i=2}^{n} a_{im}^{\lambda_m} \right)^{\frac{1}{\lambda_m}} \right. \\
&\quad \left. + \left(\prod_{j=1}^{m} a_{(n+1)j} \right) \left(a_{11}^{\lambda_1} - \sum_{i=2}^{n+1} a_{i1}^{\lambda_1} \right)^{\frac{1}{\lambda_1}} \left(a_{12}^{\lambda_2} - \sum_{i=2}^{n+1} a_{i2}^{\lambda_2} \right)^{\frac{1}{\lambda_2}} \cdot \ldots \cdot \left(a_{1m}^{\lambda_m} - \sum_{i=2}^{n+1} a_{im}^{\lambda_m} \right)^{\frac{1}{\lambda_m}} \right] \\
&= -\left\{ \left[a_{(n+1)2}^{\lambda_2} \left(a_{12}^{\lambda_2} - \sum_{i=2}^{n} a_{i2}^{\lambda_2} \right) \right]^{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}} \left[a_{(n+1)1}^{\lambda_1} \left(a_{12}^{\lambda_2} - \sum_{i=2}^{n} a_{i2}^{\lambda_2} \right) \right]^{\frac{1}{\lambda_1}} \right. \\
&\quad \times \left[a_{(n+1)2}^{\lambda_2} \left(a_{11}^{\lambda_1} - \sum_{i=2}^{n} a_{i1}^{\lambda_1} \right) \right]^{\frac{1}{\lambda_1}} \left[a_{(n+1)4}^{\lambda_4} \left(a_{14}^{\lambda_4} - \sum_{i=2}^{n} a_{i4}^{\lambda_4} \right) \right]^{\frac{1}{\lambda_4} - \frac{1}{\lambda_3}} \\
&\quad \times \left[a_{(n+1)3}^{\lambda_3} \left(a_{14}^{\lambda_4} - \sum_{i=2}^{n} a_{i4}^{\lambda_4} \right) \right]^{\frac{1}{\lambda_3}} \left[a_{(n+1)4}^{\lambda_4} \left(a_{13}^{\lambda_3} - \sum_{i=2}^{n} a_{i3}^{\lambda_3} \right) \right]^{\frac{1}{\lambda_3}} \\
&\quad \times \dots
\end{aligned}
$$

$$\times\left[a_{(n+1)(m-1)}^{\lambda_{m-1}}\left(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n}a_{i(m-1)}^{\lambda_{m-1}}\right)\right]^{\frac{1}{\lambda_{m-1}}-\frac{1}{\lambda_{m-2}}}$$
$$\times\left[a_{(n+1)(m-2)}^{\lambda_{m-2}}\left(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n}a_{i(m-1)}^{\lambda_{m-1}}\right)\right]^{\frac{1}{\lambda_{m-2}}}$$
$$\times\left[a_{(n+1)(m-1)}^{\lambda_{m-1}}\left(a_{1(m-2)}^{\lambda_{m-2}}-\sum_{i=2}^{n}a_{i(m-2)}^{\lambda_{m-2}}\right)\right]^{\frac{1}{\lambda_{m-2}}}\left[a_{(n+1)m}^{\lambda_{m}}\left(a_{1m}^{\lambda_{m}}-\sum_{i=2}^{n}a_{im}^{\lambda_{m}}\right)\right]^{\frac{1}{\lambda_{m}}}$$
$$+\left[a_{(n+1)2}^{\lambda_{2}}\left(a_{12}^{\lambda_{2}}-\sum_{i=2}^{n+1}a_{i2}^{\lambda_{2}}\right)\right]^{\frac{1}{\lambda_{2}}-\frac{1}{\lambda_{1}}}\left[a_{(n+1)2}^{\lambda_{2}}\left(a_{11}^{\lambda_{1}}-\sum_{i=2}^{n+1}a_{i1}^{\lambda_{1}}\right)\right]^{\frac{1}{\lambda_{1}}}$$
$$\times\left[a_{(n+1)1}^{\lambda_{1}}\left(a_{12}^{\lambda_{2}}-\sum_{i=2}^{n+1}a_{i2}^{\lambda_{2}}\right)\right]^{\frac{1}{\lambda_{1}}}\left[a_{(n+1)4}^{\lambda_{4}}\left(a_{14}^{\lambda_{4}}-\sum_{i=2}^{n+1}a_{i4}^{\lambda_{4}}\right)\right]^{\frac{1}{\lambda_{4}}-\frac{1}{\lambda_{3}}}$$
$$\times\left[a_{(n+1)4}^{\lambda_{4}}\left(a_{13}^{\lambda_{3}}-\sum_{i=2}^{n+1}a_{i3}^{\lambda_{3}}\right)\right]^{\frac{1}{\lambda_{3}}}\left[a_{(n+1)3}^{\lambda_{3}}\left(a_{14}^{\lambda_{4}}-\sum_{i=2}^{n+1}a_{i4}^{\lambda_{4}}\right)\right]^{\frac{1}{\lambda_{3}}}$$
$$\times\ldots$$
$$\times\left[a_{(n+1)(m-1)}^{\lambda_{m-1}}\left(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n+1}a_{i(m-1)}^{\lambda_{m-1}}\right)\right]^{\frac{1}{\lambda_{m-1}}-\frac{1}{\lambda_{m-2}}}$$
$$\times\left[a_{(n+1)(m-1)}^{\lambda_{m-1}}\left(a_{1(m-2)}^{\lambda_{m-2}}-\sum_{i=2}^{n+1}a_{i(m-2)}^{\lambda_{m-2}}\right)\right]^{\frac{1}{\lambda_{m-2}}}$$
$$\times\left[a_{(n+1)(m-2)}^{\lambda_{m-2}}\left(a_{1(m-1)}^{\lambda_{m-1}}-\sum_{i=2}^{n+1}a_{i(m-1)}^{\lambda_{m-1}}\right)\right]^{\frac{1}{\lambda_{m-2}}}\Bigg\}\left[a_{(n+1)m}^{\lambda_{m}}\left(a_{1m}^{\lambda_{m}}-\sum_{i=2}^{n+1}a_{im}^{\lambda_{m}}\right)\right]^{\frac{1}{\lambda_{m}}}\Bigg\}. \tag{17}$$

Therefore, from (17), (16) and (8), and by the same methods as in Case 1°, we can obtain the desired inequality (11).

3° Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m > 0$ and at least one of "$=$" be valid, and let k be even. By the same method as in 1°, we can obtain the inequality (11).

4° Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m > 0$ and at least one of "$=$" be valid, and let k be odd. By the same method as in 2°, we can obtain the inequality (11).

The proof of Theorem 2.2 is completed. □

By the same method as in Theorem 2.2, but using Theorem E in place of Theorem D, we can obtain the following Theorems.

THEOREM 2.3. *Let $n,m \in \mathbb{N}^+$, $n \geqslant 2$, let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m < 0$, and let a_{rj} $(r = 1,2,\cdots,n; j = 1,2,\cdots,m)$ be positive real numbers such that $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$*

$(j=1,2,\cdots,m)$. *If we denote*

$$\widetilde{V}(n)=\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}}-\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)^2,$$

then

$$\widetilde{V}(n+1)\geqslant\widetilde{V}(n)\geqslant 0. \tag{18}$$

THEOREM 2.4. *Let $n,m\in\mathbb{N}^+$, $n\geqslant 2$, let $\lambda_1>0$, $\lambda_2\leqslant\ldots\leqslant\lambda_m<0$ with $\sum_{j=1}^{m}\frac{1}{\lambda_j}\leqslant 1$, and let a_{rj} $(r=1,2,\cdots,n;j=1,2,\cdots,m)$ be positive real numbers such that $a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}>0$ $(j=1,2,\cdots,m)$. If we denote*

$$\widetilde{V}(n)=\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}}-\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)^2,$$

then

$$\widetilde{V}(n+1)\geqslant\widetilde{V}(n)\geqslant 0. \tag{19}$$

Setting $m=2$, $a_{r1}=a_r$, $a_{r2}=b_r$ $(r=1,2,\cdots,n)$ in Theorem 2.3 and Theorem 2.4, we get the following result.

COROLLARY 2.5. *Let $n\in\mathbb{N}^+$, $n\geqslant 2$, let $\lambda_1\neq 0$, $\lambda_2<0$, $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\leqslant 1$, and let a_i, b_i $(i=1,2,\cdots,n)$ be positive real numbers such that $a_1^{\lambda_1}-\sum_{i=2}^{n}a_i^{\lambda_1}>0$ and $b_1^{\lambda_2}-\sum_{i=2}^{n}b_i^{\lambda_2}>0$. If we denote*

$$V^*(n)=\left(a_1^{\lambda_1}-\sum_{r=2}^{n}a_r^{\lambda_1}\right)^{\frac{2}{\lambda_1}}\left(b_1^{\lambda_2}-\sum_{r=2}^{n}b_r^{\lambda_2}\right)^{\frac{2}{\lambda_2}}-\left(a_1b_1-\sum_{r=2}^{n}a_rb_r\right)^2,$$

then

$$V^*(n+1)\geqslant V^*(n)\geqslant 0. \tag{20}$$

Similarly, setting $m=2$, $a_{r1}=a_r$, $a_{r2}=b_r$ $(r=1,2,\cdots,n)$ in Theorem 2.2, we have the following Corollary 2.6.

COROLLARY 2.6. *Let $n\in\mathbb{N}^+$, $n\geqslant 2$, let $\lambda_1\geqslant\lambda_2>0$, $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\geqslant 1$, and let a_i, b_i $(i=1,2,\cdots,n)$ be positive real numbers such that $a_1^{\lambda_1}-\sum_{i=2}^{n}a_i^{\lambda_1}>0$ and $b_1^{\lambda_2}-\sum_{i=2}^{n}b_i^{\lambda_2}>0$. If we denote*

$$V^*(n)=\left(a_1^{\lambda_1}-\sum_{r=2}^{n}a_r^{\lambda_1}\right)^{\frac{2}{\lambda_1}}\left(b_1^{\lambda_2}-\sum_{r=2}^{n}b_r^{\lambda_2}\right)^{\frac{2}{\lambda_2}}-\left(a_1b_1-\sum_{r=2}^{n}a_rb_r\right)^2,$$

then

$$V^*(n+1)\leqslant V^*(n)\leqslant 0. \tag{21}$$

More particularly, if we set $\lambda_1 = \lambda_2 = \lambda < 0$ in Corollary 2.5, then we have the following property of reversed Aczél-Bjelica inequality (7).

COROLLARY 2.7. *Let $n \in \mathbb{N}^+$, $n \geqslant 2$, let $\lambda < 0$, and let $a_i,\ b_i\ (i = 1,2,\cdots,n)$ be positive real numbers such that $a_1^\lambda - \sum_{i=2}^n a_i^\lambda > 0$ and $b_1^\lambda - \sum_{i=2}^n b_i^\lambda > 0$. If we denote*

$$\widetilde{V^*}(n) = \left(a_1^\lambda - \sum_{r=2}^n a_r^\lambda\right)^{\frac{2}{\lambda}} \left(b_1^\lambda - \sum_{r=2}^n b_r^\lambda\right)^{\frac{2}{\lambda}} - \left(a_1 b_1 - \sum_{r=2}^n a_r b_r\right)^2,$$

then we have

$$\widetilde{V^*}(n+1) \geqslant \widetilde{V^*}(n) \geqslant 0. \tag{22}$$

Similarly, if we set $\lambda_1 = \lambda_2 = \lambda$ in Corollary 2.6, then we have the following property of Aczél-Bjelica inequality (6).

COROLLARY 2.8. *Let $n \in \mathbb{N}^+$, $n \geqslant 2$, let $0 < \lambda \leqslant 2$, and let $a_i,\ b_i\ (i = 1,2,\cdots,n)$ be positive real numbers such that $a_1^\lambda - \sum_{i=2}^n a_i^\lambda > 0$ and $b_1^\lambda - \sum_{i=2}^n b_i^\lambda > 0$. If we denote*

$$\widetilde{V^*}(n) = \left(a_1^\lambda - \sum_{r=2}^n a_r^\lambda\right)^{\frac{2}{\lambda}} \left(b_1^\lambda - \sum_{r=2}^n b_r^\lambda\right)^{\frac{2}{\lambda}} - \left(a_1 b_1 - \sum_{r=2}^n a_r b_r\right)^2,$$

then we have

$$\widetilde{V^*}(n+1) \leqslant \widetilde{V^*}(n) \leqslant 0. \tag{23}$$

From Theorem 2.3, we obtain a new refinement of generalized Aczél inequality (5) as follows.

COROLLARY 2.9. *Let $n,m \in \mathbb{N}^+$, $n \geqslant 2$, let $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_m < 0$, and let $a_{rj}\ (r = 1,2,\cdots,n; j = 1,2,\cdots,m)$ be positive real numbers such that $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0\ (j = 1,2,\cdots,m)$. Then*

$$\begin{aligned}\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} &\geqslant \left|\prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}\right| \left[1 + \frac{\widetilde{V}(2)}{(\prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj})^2}\right]^{\frac{1}{2}} \\ &\geqslant \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj},\end{aligned} \tag{24}$$

where $\widetilde{V}(2) = \prod_{j=1}^m \left(a_{1j}^{\lambda_j} - a_{2j}^{\lambda_j}\right)^{\frac{2}{\lambda_j}} - \left(\prod_{j=1}^m a_{1j} - \prod_{j=1}^m a_{2j}\right)^2$.

Proof. From Theorem 2.3, we find

$$\widetilde{V}(n) \geqslant \widetilde{V}(2) \geqslant 0, \tag{25}$$

and then, we have

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geqslant \left[\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)^2+\widetilde{V}(2)\right]^{\frac{1}{2}}$$
$$\geqslant \left|\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right|\left[1+\frac{\widetilde{V}(2)}{(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj})^2}\right]^{\frac{1}{2}}. \tag{26}$$

The proof of Corollary 2.9 is completed. □

Making similar technique as in the proof of Corollary 2.9 and by using Theorem 2.4 and Theorem 2.2, we get the following refinements of generalized Aczél inequalities (5) and (4).

COROLLARY 2.10. *Let* $n,m\in\mathbb{N}^+$, $n\geqslant 2$, *let* $\lambda_1>0$, $\lambda_2\leqslant\lambda_3\leqslant\ldots\leqslant\lambda_m<0$ *with* $\sum_{j=1}^{m}\frac{1}{\lambda_j}\leqslant 1$, *and let* a_{rj} $(r=1,2,\cdots,n;\ j=1,2,\cdots,m)$ *be positive real numbers such that* $a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}>0$ $(j=1,2,\cdots,m)$. *Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geqslant \left|\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right|\left[1+\frac{\widetilde{V}(2)}{(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj})^2}\right]^{\frac{1}{2}}$$
$$\geqslant \prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}, \tag{27}$$

where $\widetilde{V}(2)=\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-a_{2j}^{\lambda_j}\right)^{\frac{2}{\lambda_j}}-\left(\prod_{j=1}^{m}a_{1j}-\prod_{j=1}^{m}a_{2j}\right)^2$.

COROLLARY 2.11. *Let* $n,m\in\mathbb{N}^+$, $n\geqslant 2$, *let* $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_m>0$ *with* $\sum_{j=1}^{m}\frac{1}{\lambda_j}\geqslant 1$, *and let* a_{rj} $(r=1,2,\cdots,n;\ j=1,2,\cdots,m)$ *be positive real numbers such that* $a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}>0$ $(j=1,2,\cdots,m)$. *Then, we have*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leqslant \left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)\left[1+\frac{\widetilde{V}(2)}{(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj})^2}\right]^{\frac{1}{2}}$$
$$\leqslant \prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}, \tag{28}$$

where $\widetilde{V}(2)=\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-a_{2j}^{\lambda_j}\right)^{\frac{2}{\lambda_j}}-\left(\prod_{j=1}^{m}a_{1j}-\prod_{j=1}^{m}a_{2j}\right)^2$.

Similarly, we have the following refinements of Aczél-type inequality.

COROLLARY 2.12. *Let* $n \in \mathbb{N}^+$, $n \geqslant 2$, *let* $\lambda_1 \neq 0$, $\lambda_2 < 0$, $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \leqslant 1$, *and let* a_i, b_i $(i = 1, 2, \cdots, n)$ *be positive real numbers such that* $a_1^{\lambda_1} - \sum_{i=2}^n a_i^{\lambda_1} > 0$ *and* $b_1^{\lambda_2} - \sum_{i=2}^n b_i^{\lambda_2} > 0$. *Then*

$$\left(a_1^{\lambda_1} - \sum_{r=2}^n a_r^{\lambda_1}\right)^{\frac{1}{\lambda_1}} \left(b_1^{\lambda_2} - \sum_{r=2}^n b_r^{\lambda_2}\right)^{\frac{1}{\lambda_2}} \geqslant \left|a_1 b_1 - \sum_{r=2}^n a_r b_r\right| \left[1 + \frac{V^*(2)}{(a_1 b_1 - \sum_{r=2}^n a_r b_r)^2}\right]^{\frac{1}{2}} \geqslant a_1 b_1 - \sum_{r=2}^n a_r b_r, \tag{29}$$

where $V^*(2) = \left(a_1^{\lambda_1} - a_2^{\lambda_1}\right)^{\frac{2}{\lambda_1}} \left(b_1^{\lambda_2} - b_2^{\lambda_2}\right)^{\frac{2}{\lambda_2}} - \left(a_1 b_1 - a_2 b_2\right)^2 \geqslant 0$.

COROLLARY 2.13. *Let* $n \in \mathbb{N}^+$, $n \geqslant 2$, *let* $\lambda_1 > 0$, $\lambda_2 > 0$, $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \geqslant 1$, *and let* a_i, b_i $(i = 1, 2, \cdots, n)$ *be positive real numbers such that* $a_1^{\lambda_1} - \sum_{i=2}^n a_i^{\lambda_1} > 0$ *and* $b_1^{\lambda_2} - \sum_{i=2}^n b_i^{\lambda_2} > 0$. *Then*

$$\left(a_1^{\lambda_1} - \sum_{r=2}^n a_r^{\lambda_1}\right)^{\frac{1}{\lambda_1}} \left(b_1^{\lambda_2} - \sum_{r=2}^n b_r^{\lambda_2}\right)^{\frac{1}{\lambda_2}} \leqslant \left(a_1 b_1 - \sum_{r=2}^n a_r b_r\right) \left[1 + \frac{V^*(2)}{(a_1 b_1 - \sum_{r=2}^n a_r b_r)^2}\right]^{\frac{1}{2}} \leqslant a_1 b_1 - \sum_{r=2}^n a_r b_r, \tag{30}$$

where $V^*(2) = \left(a_1^{\lambda_1} - a_2^{\lambda_1}\right)^{\frac{2}{\lambda_1}} \left(b_1^{\lambda_2} - b_2^{\lambda_2}\right)^{\frac{2}{\lambda_2}} - \left(a_1 b_1 - a_2 b_2\right)^2 \leqslant 0$.

COROLLARY 2.14. *Let* $n \in \mathbb{N}^+$, $n \geqslant 2$, *let* $\lambda < 0$, *and let* a_i, b_i $(i = 1, 2, \cdots, n)$ *be positive real numbers such that* $a_1^{\lambda} - \sum_{i=2}^n a_i^{\lambda} > 0$ *and* $b_1^{\lambda} - \sum_{i=2}^n b_i^{\lambda} > 0$. *Then*

$$\left(a_1^{\lambda} - \sum_{r=2}^n a_r^{\lambda}\right)^{\frac{1}{\lambda}} \left(b_1^{\lambda} - \sum_{r=2}^n b_r^{\lambda}\right)^{\frac{1}{\lambda}} \geqslant \left|a_1 b_1 - \sum_{r=2}^n a_r b_r\right| \left[1 + \frac{\widetilde{V^*}(2)}{(a_1 b_1 - \sum_{r=2}^n a_r b_r)^2}\right]^{\frac{1}{2}} \geqslant a_1 b_1 - \sum_{r=2}^n a_r b_r, \tag{31}$$

where $\widetilde{V^*}(2) = \left(a_1^{\lambda} - a_2^{\lambda}\right)^{\frac{2}{\lambda}} \left(b_1^{\lambda} - b_2^{\lambda}\right)^{\frac{2}{\lambda}} - \left(a_1 b_1 - a_2 b_2\right)^2 \geqslant 0$.

COROLLARY 2.15. *Let* $n \in \mathbb{N}^+$, $n \geqslant 2$, *let* $0 < \lambda \leqslant 2$, *and let* a_i, b_i $(i = 1, 2, \cdots, n)$ *be positive real numbers such that* $a_1^{\lambda} - \sum_{i=2}^n a_i^{\lambda} > 0$ *and* $b_1^{\lambda} - \sum_{i=2}^n b_i^{\lambda} > 0$. *Then*

$$\left(a_1^{\lambda} - \sum_{r=2}^n a_r^{\lambda}\right)^{\frac{1}{\lambda}} \left(b_1^{\lambda} - \sum_{r=2}^n b_r^{\lambda}\right)^{\frac{1}{\lambda}} \leqslant \left(a_1 b_1 - \sum_{r=2}^n a_r b_r\right) \left[1 + \frac{\widetilde{V^*}(2)}{(a_1 b_1 - \sum_{r=2}^n a_r b_r)^2}\right]^{\frac{1}{2}} \leqslant a_1 b_1 - \sum_{r=2}^n a_r b_r, \tag{32}$$

where $\widetilde{V^*}(2) = \left(a_1^\lambda - a_2^\lambda\right)^{\frac{2}{\lambda}}\left(b_1^\lambda - b_2^\lambda\right)^{\frac{2}{\lambda}} - \left(a_1b_1 - a_2b_2\right)^2 \leqslant 0.$

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