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OPTIMAL BOUNDS FOR THE FIRST SEIFFERT MEAN IN TERMS OF THE CONVEX COMBINATION OF THE LOGARITHMIC AND NEUMAN–SÁNDOR MEAN

JIAN-JUN LEI, JING-JING CHEN AND BO-YONG LONG

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Abstract. In this paper, we find the least value α and the greatest value β such that the double inequality

$$\alpha L(a,b)+(1-\alpha)M(a,b)<P(a,b)<\beta L(a,b)+(1-\beta)M(a,b)$$

holds for all $a,b>0$ with $a\neq b$, where $L(a,b),M(a,b)$ and $P(a,b)$ are the logarithmic, the Neuman-Sándor, and the first Seiffert means of two positive numbers a and b, respectively.

1. Introduction

For $a,b>0$ with $a\neq b$, the Neuman-Sándor mean $M(a,b)$, the first Seiffert mean $P(a,b)$, and the logarithmic mean $L(a,b)$ are defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}((a-b)/(a+b))}, \tag{1.1}$$

$$P(a,b)=\frac{a-b}{4\tan^{-1}(\sqrt{a/b})-\pi},$$

$$L(a,b)=\frac{b-a}{\log b-\log a},$$

respectively. It can be observed that the first Seiffert mean $P(a,b)$ and the logarithmic mean can be rewritten as (see as [12])

$$P(a,b)=\frac{a-b}{2\sin^{-1}((a-b)/(a+b))}, \tag{1.2}$$

$$L(a,b)=\frac{a-b}{2\tanh^{-1}((a-b)/(a+b))}, \tag{1.3}$$

where $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$, $\tanh^{-1}(x)=\frac{1}{2}\log[(1-x)/(1+x)]$ and $\sin^{-1}(x)=\arcsin x$ are the inverse hyperbolic sine, inverse hyperbolic tangent and inverse sine functions, respectively.

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Recently, the means L, M, and P and other means have been the subject of intensive research. Many remarkable inequalities for means can be found in the literature [2, 4, 6, 10, 15-18].

Let $H(a,b)=2ab/(a+b)$, $G(a,b)=\sqrt{ab}$, $I(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b)=(a+b)/2$, $S(a,b)=\sqrt{(a^2+b^2)/2}$, $T(a,b)=(a-b)/[2\tan^{-1}((a-b)/(a+b))]$ and

$$M_p(a,b)=\begin{cases}(\frac{a^p+b^p}{2})^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0\end{cases}$$

denote the harmonic, geometric, identric, arithmetic, root-square, second Seiffert and the p-th power means of two positive numbers a and b with $a\neq b$ respectively. Then it is well-known that the inequalities

$$H(a,b)<G(a,b)<L(a,b)<P(a,b)<I(a,b)<A(a,b)<M(a,b)<T(a,b)<S(a,b)$$

hold for $a,b>0$ with $a\neq b$.

Neuman and Sándor [12, 13] proved that inequalities

$$\frac{\pi}{4\log(1+\sqrt{2})}T(a,b)<M(a,b)<\frac{A(a,b)}{\log(1+\sqrt{2})},$$

$$\frac{\pi}{4\sinh^{-1}(1)}T(a,b)<M(a,b)<\frac{\pi}{2\sinh^{-1}(1)}P(a,b),$$

$$\sqrt{A(a,b)T(a,b)}<M(a,b)<\sqrt{A^2(a,b)+T^2(a,b)},$$

$$\frac{G(x,y)}{G(1-x,1-y)}<\frac{L(x,y)}{L(1-x,1-y)}<\frac{P(x,y)}{P(1-x,1-y)}$$
$$<\frac{A(x,y)}{A(1-x,1-y)}<\frac{M(x,y)}{M(1-x,1-y)}<\frac{T(x,y)}{T(1-x,1-y)}$$

hold for all $a,b>0$ and $x,y\in(0,1/2]$ with $a\neq b$ and $x\neq y$.

The following bounds for the Seiffert means $P(a,b)$ and $T(a,b)$ in terms of the power mean were presented by Jagers in [8]

$$M_{\frac{1}{2}}(a,b)<P(a,b)<M_{\frac{2}{3}}(a,b)$$

for all $a,b>0$ with $a\neq b$. Hästö [7] improved the results of [8] and found the sharp lower power mean bound for the Seiffert mean $P(a,b)$ as follows

$$P(a,b)>M_{\frac{\log 2}{\log\pi}}(a,b)$$

for all $a,b>0$ with $a\neq b$.

In [1], Alzer and Qiu proved

$$M_c(a,b)\leqslant\frac{1}{2}L(a,b)+\frac{1}{2}I(a,b)$$

for all $a,b>0$ with the best possible parameter $c=\frac{\log 2}{1+\log 2}$,

$$[G(a,b)]^{A(a,b)}<[L(a,b)]^{I(a,b)}<[A(a,b)]^{G(a,b)}$$

for $a,b\geqslant e$, and

$$[A(a,b)]^{G(a,b)}<[I(a,b)]^{L(a,b)}<[G(a,b)]^{A(a,b)}$$

for $0<a,\ b<e$.

In [11] and [9], the authors proved that the double inequalities

$$S^{\alpha_1}(a,b)A^{1-\alpha_1}(a,b)<M(a,b)<S^{\beta_1}(a,b)A^{1-\beta_1}(a,b),$$

$$\alpha_2 A(a,b)+(1-\alpha_2)G(a,b)<P(a,b)<\beta_2 A(a,b)+(1-\beta_2)G(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leqslant 1/3$, $\beta_1\geqslant 2(\log(2+\sqrt{2})-\log 3)/\log 2$, $\alpha_2\leqslant \pi/2$, $\beta_2\geqslant 2/3$, respectively.

In [5], it was shown that

$$H^{\alpha_3}(a,b)L^{1-\alpha_3}(a,b)\geqslant M_{\frac{1-4\alpha_3}{3}}(a,b),$$

$$H^{\beta_3}(a,b)L^{1-\beta_3}(a,b)\leqslant M_{\frac{1-4\beta_3}{3}}(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_3\in[\frac{1}{4},1)$, $\beta_3\in(0,\frac{3\sqrt{5}-5}{40}]$.

In [14], the authors proved that

$$\alpha_4 H(a,b)+(1-\alpha_4)L(a,b)>M_{\frac{1-4\alpha_4}{3}}(a,b),$$

$$\beta_4 H(a,b)+(1-\beta_4)L(a,b)<M_{\frac{1-4\beta_4}{3}}(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_4\in[\frac{1}{4},1)$, $\beta_4\in(0,3\sqrt{345}/80-11/16)$.

The aim of this paper is to find the least value α and the greatest value β such that the double inequality

$$\alpha L(a,b)+(1-\alpha)M(a,b)<P(a,b)<\beta L(a,b)+(1-\beta)M(a,b)$$

holds for all $a,b>0$ with $a\neq b$.

2. Lemmas

To establish our main result, we need several lemmas, which we present in this section.

LEMMA 2.1. *It holds that*

$$x+\frac{2-\beta}{6\beta}x^3+\frac{36\beta^2-49a+40}{360\beta^2}x^5>\tanh^{-1}x, \quad x\in(0,0.93), \tag{2.1}$$

$$x+\frac{1}{3}x^3+\frac{1}{5}x^5<\tanh^{-1}x, \quad x\in(0,0.52), \tag{2.2}$$

$$x^2+\frac{97}{720}x^6-\frac{1}{96}x^8+\frac{9}{1600}x^{10}<\left(x-\frac{1}{3}x^3+\frac{3}{160}x^5\right)\tanh^{-1}x, \quad x\in(0.52,0.72), \tag{2.3}$$

where $\beta=1-\frac{2}{\pi}\sinh^{-1}(1)\approx 0.4389$.

Proof. Let

$$f(x)=\tanh^{-1}(x)-\left(x+\frac{2-\beta}{6\beta}x^3+\frac{36\beta^2-49\beta+40}{360\beta^2}x^5\right).$$

Then we can get

$$f'(x)=\frac{1}{1-x^2}-\left(1+\frac{2-\beta}{2\beta}x^2+\frac{36\beta^2-49\beta+40}{72\beta^2}x^4\right)=\frac{x^2}{1-x^2}g(x), \tag{2.4}$$

where

$$g(x)=\frac{3\beta-2}{2\beta}+\frac{-72\beta^2+121\beta-40}{72\beta^2}x^2+\frac{36\beta^2-49\beta+40}{72\beta^2}x^4.$$

It is easy to verify that there exist $x_0\in(0,0.93)$, $x_0\approx 0.8634$, such that $g(x)<0$ for $x\in(0,x_0)$, $g(x_0)=0$, and $g(x)>0$ for $x\in(x_0,0.93)$. Thus, equation (2.4) implies that $f(x)$ is decrease on $(0,x_0)$ and increase on $(x_0,0.93)$. Therefore, $f(x)<0$ for $x\in(0,0.93)$ follows from the fact that $f(0)=0$, $f(0.93)<0$ and the monotonicity of $f(x)$. That means inequality (2.1) holds.

Observe that

$$\tanh^{-1}x=\sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}, \qquad -1<x<1.$$

So it is obvious that inequality (2.2) holds.

Let

$$h(x)=\left(x-\frac{1}{3}x^3+\frac{3}{160}x^5\right)\tanh^{-1}x-\left(x^2+\frac{97}{720}x^6-\frac{1}{96}x^8+\frac{9}{1600}x^{10}\right).$$

Then direct computation leads to

$$
\begin{aligned}
h(x) &= \Big(x-\frac{1}{3}x^3+\frac{3}{160}x^5\Big)\sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}-\Big(x^2+\frac{97}{720}x^6-\frac{1}{96}x^8+\frac{9}{1600}x^{10}\Big)\\
&= x^6\Big(-\frac{13}{480}+\frac{13}{140}x^2+\frac{157}{2548}x^4-\frac{1039}{30240}x^6+\frac{1}{480}x^8\Big)\\
&\quad+\Big(x-\frac{1}{3}x^3+\frac{3}{160}x^5\Big)\sum_{n=6}^{\infty}\frac{1}{2n+1}x^{2n+1}.
\end{aligned}
$$

Noting that $-\frac{13}{480}+\frac{13}{140}x^2+\frac{157}{2548}x^4-\frac{1039}{30240}x^6+\frac{1}{480}x^8>0$ and $x-\frac{1}{3}x^3+\frac{3}{160}x^5>0$ for $x\in(0.52,0.72)$, so $h(x)>0$ for $x\in(0.52,0.72)$ and the inequality (2.3) holds. □

LEMMA 2.2. *The inequalities*

$$x+\frac{1}{6}x^3+\frac{3}{40}x^5<\sin^{-1}x, \qquad x\in(0,0.93), \tag{2.5}$$

$$x+\frac{1}{6}x^3+\frac{9}{100}x^5>\sin^{-1}x, \qquad x\in(0,0.52), \tag{2.6}$$

$$x+\frac{1}{6}x^3+\frac{9}{80}x^5>\sin^{-1}x, \qquad x\in(0.52,0.72) \tag{2.7}$$

hold.

Proof. It is known that

$$\sin^{-1}x=x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1}.$$

So it's easy to see that inequality (2.5) holds.

Let $h_i(x)=f_i(x)-g(x)$, $F_i(x)/x^4=(\frac{f_i'(x)}{g'(x)})^2-1$, $i=1,2$, where

$$
\begin{aligned}
f_1(x) &= x+\frac{1}{6}x^3+\frac{9}{100}x^5,\\
f_2(x) &= x+\frac{1}{6}x^3+\frac{9}{80}x^5,
\end{aligned}
$$

and

$$g(x)=\sin^{-1}x.$$

Then it follows that

$$
\begin{aligned}
f_1'(x) &= 1+\frac{1}{2}x^2+\frac{9}{20}x^4,\\
f_2'(x) &= 1+\frac{1}{2}x^2+\frac{9}{16}x^4,\\
g'(x) &= \frac{1}{\sqrt{1-x^2}},
\end{aligned}
$$

and

$$F_1(x) = -\frac{81}{400}x^6 - \frac{99}{400}x^4 - \frac{7}{10}x^2 + \frac{3}{20},$$
$$F_2(x) = -\frac{81}{256}x^6 - \frac{63}{256}x^4 - \frac{13}{16}x^2 + \frac{3}{8}.$$

It's easy to see that there exists $x_1 \in (0, 0.52)$ and $x_2 \in (0.52, 0.72)$, such that $F_1(x_1) = 0$, $F_2(x_2) = 0$, $F_1(x)$ and $F_2(x)$ are strictly decrease in $(0, 0.52)$ and $(0.52, 0.72)$, respectively. Thus $h_1(x)$ and $h_2(x)$ are increase on $(0, x_1)$ and $(0.52, x_2)$, respectively, and decrease on $(x_1, 0.52)$ and $(x_2, 0.72)$, respectively. Therefore, inequalities (2.6) and (2.7) follow from the fact that $h_1(0) = 0$, $h_1(0.52) > 0$, $h_2(0.52) > 0$, $h_2(0.72) > 0$ and the monotonicity of $h_1(x)$ and $h_2(x)$, respectively. $\square$

LEMMA 2.3. *It holds that*

$$x - \frac{1}{6}x^3 + \frac{1}{16}x^5 < \sinh^{-1} x, \qquad x \in (0, 0.52), \tag{2.8}$$
$$x - \frac{1}{6}x^3 + \frac{1}{20}x^5 < \sinh^{-1} x, \qquad x \in (0.52, 0.72), \tag{2.9}$$
$$x - \frac{1}{6}x^3 + \frac{1}{10}x^5 > \sinh^{-1} x, \qquad x \in (0, 0.93). \tag{2.10}$$

Proof. Let $h_i(x) = f_i(x) - g(x)$, $F_i(x)/x^4 = (\frac{f_i'(x)}{g'(x)})^2 - 1$, $i = 1, 2, 3$, where

$$f_1(x) = x - \frac{1}{6}x^3 + \frac{1}{16}x^5,$$
$$f_2(x) = x - \frac{1}{6}x^3 + \frac{1}{20}x^5,$$
$$f_3(x) = x - \frac{1}{6}x^3 + \frac{1}{10}x^5,$$
$$g(x) = \sinh^{-1}(x).$$

Then direct computation lead to

$$f_1'(x) = 1 - \frac{1}{2}x^2 + \frac{5}{16}x^4,$$
$$f_2'(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4}x^4,$$
$$f_3'(x) = 1 - \frac{1}{2}x^2 + \frac{1}{2}x^4,$$
$$g'(x) = \frac{1}{\sqrt{1+x^2}},$$

and

$$\begin{aligned}F_1(x) &= \frac{25}{256}x^6 - \frac{55}{256}x^4 + \frac{9}{16}x^2 - \frac{1}{8},\\ F_2(x) &= \frac{1}{16}x^6 - \frac{3}{16}x^4 + \frac{1}{2}x^2 - \frac{1}{4},\\ F_3(x) &= \frac{1}{4}x^6 - \frac{1}{4}x^4 + \frac{3}{4}x^2 + \frac{1}{4}.\end{aligned}$$

It can be verified that there exists $x_1 \in (0,0.52)$ such that $F_1(x) < 0$ for $x \in (0,x_1)$ and $F_1(x) > 0$ for $x \in (x_1,0.52)$. So $h_1(x)$ is decrease on $(0,x_1)$ and increase on $(x_1,0.52)$. Therefore, inequality (2.8) follows from $h_1(0) = 0$, $h_1(0.52) < 0$ and the monotonicity of $h_1(x)$.

Observe that $F_2(x) < 0$ and $F_3(x) > 0$ for $x \in (0.52,0.72)$ and $x \in (0,0.93)$, respectively. It imply that $h_2(x)$ and $h_3(x)$ are decrease on $(0.52,0.72)$ and increase on $(0,0.93)$, respectively. Furthermore, considering $h_2(0.52) < 0$ and $h_3(0) = 0$, one has inequalities (2.9) and (2.10). □

LEMMA 2.4. *Let*

$$t(x) = (1-\beta)\sqrt{1-x^2}\tanh^{-1}x + 2(1-\beta)\sin^{-1}x - 2\sinh^{-1}x.$$

Then $t(x) > 0$ *for* $x \in (0.93,1)$, *where* $\beta = 1 - \frac{2}{\pi}\sinh^{-1}(1)$.

Proof. Direct computation lead to that

$$t'(x) = \frac{\phi(x)}{\sqrt{1-x^4}}, \tag{2.11}$$

where

$$\phi(x) = (1-\beta)\sqrt{1+x^2}(3 - x\tanh^{-1}x) - 2\sqrt{1-x^2}.$$

It follows that

$$\phi'(x) = -\frac{(1+\beta)x^2\tanh^{-1}x}{\sqrt{1+x^2}} + (1-\beta)\sqrt{1+x^2}\,\alpha(x) + \frac{x}{1-x^2}\beta(x), \tag{2.12}$$

where

$$\begin{aligned}\alpha(x) &= \frac{3x}{(1+x^2)} - \tanh^{-1}x,\\ \beta(x) &= 2\sqrt{1-x^2} - (1-\beta)\sqrt{1+x^2}.\end{aligned}$$

Noting that $\alpha(0.93) < 0$, $\beta(0.93) < 0$, and both

$$\alpha'(x) = 2\frac{1-4x^2+x^4}{(1+x^2)^2(1-x^2)} < 0$$

and

$$\beta'(x) = \frac{-2x[\sqrt{1+x^2}+(1-\beta)\sqrt{1-x^2}]}{\sqrt{1-x^4}} < 0$$

for $x \in (0.93,1)$, we can get both $\alpha(x) < 0$ and $\beta(x) < 0$ for $x \in (0.93,1)$. Thus, equation (2.12) implies that $\phi(x)$ is decrease on $(0.93,1)$. Considering that $\phi(0.93) > 0$ and $\phi(1^-) < 0$, it is easy to see that there exist a point $\lambda \in (0.93,1)$, such that $\phi(x)$ is increase on $(0.93,\lambda)$ and decrease on $(\lambda,1)$. Equation (2.11) implies that $\phi(x)$ and $t(x)$ have same monotonicity on $(0.93,1)$. Therefore, $t(x) > 0$ for $x \in (0.93,1)$ follows from $t(0.93) > 0$, $t(1^-) = 0$ and its monotonicity. □

LEMMA 2.5. *For $x \in (0.72,0.9)$, the following inequalities hold:*

$$\frac{1}{\sinh^{-1}x} < a_1x+b_1, \tag{2.13}$$

$$\frac{2}{\tanh^{-1}x} < \frac{9}{2}x+\frac{109}{20}, \tag{2.14}$$

$$\frac{3}{\sin^{-1}x} > \Big(\frac{9}{2}+a_1\Big)x+\Big(\frac{109}{20}+b_1\Big), \tag{2.15}$$

where $a_1 = \frac{50}{9}\big(\frac{1}{\sinh^{-1}(0.9)} - \frac{1}{\sinh^{1}(0.72)}\big)$ *and* $b_1 = \frac{5}{\sinh^{-1}(0.72)} - \frac{4}{\sinh^{-1}(0.9)}$.

Proof. Simple computation deduce that

$$\Big(\frac{1}{\sinh^{-1}x}\Big)'' = \frac{2\sinh^{-1}x + \frac{x(\sinh^{-1}x)^2}{\sqrt{1+x^2}}}{(\sinh^{-1}x)^4(1+x^2)} > 0$$

for any $x \in (0,1)$. So $1/\sinh^{-1}x$ is convex on $(0.72,0.9)$. Observe that the line $y = a_1x+b_1$ intersects the curve $y = 1/\sinh^{-1}x$ at two points which abscissas are 0.72 and 0.9. Thus the geometric property of convex function deduce the inequality (2.13).

Let

$$f_1(x) = \tanh^{-1}x - \frac{2}{\frac{9}{2}x+\frac{109}{20}},$$

$$f_2(x) = \frac{3}{(\frac{9}{2}+a_1)x+(\frac{109}{20}+b_1)} - \sin^{-1}x.$$

It follows that

$$f_1'(x) = \frac{1}{1-x^2} + \frac{9}{(\frac{9}{2}x+\frac{109}{20})^2} > 0,$$

$$f_2'(x) = -\frac{3(\frac{9}{2}+a_1)}{[(\frac{9}{2}+a_1)x+(\frac{109}{20}+b_1)]^2} - \frac{1}{\sqrt{1-x^2}} < 0.$$

Considering that $f_1(0.72) > 0$ and $f_2(0.9) > 0$, respectively, we can get inequalities (2.14) and (2.15). □

LEMMA 2.6. *For $x \in (0.9,1)$, the following inequalities hold:*

$$\frac{1}{\sinh^{-1} x} < a_2 x + b_2, \tag{2.16}$$

$$\frac{3}{\sin^{-1} x} > a_3 x + b_3, \tag{2.17}$$

$$\frac{2}{\tanh^{-1} x} < a_4 x + b_4, \tag{2.18}$$

where $a_2 = 10(\frac{1}{\sinh^{-1}(1)} - \frac{1}{\sinh^{-1}(0.9)})$, $b_2 = \frac{10}{\sinh^{-1}(0.9)} - \frac{9}{\sinh^{-1}(1)}$, $a_3 = 30(\frac{1}{\sin^{-1}(1)} - \frac{1}{\sin^{-1}(0.9)})$, $b_3 = \frac{30}{\sin^{-1}(0.9)} - \frac{27}{\sin^{-1}(1)}$, $a_4 = a_3 - a_2$, $b_4 = b_3 - b_2$.

Proof. The proof of inequality (2.16) is same as that of inequality (2.13).

Let $g(x) = \alpha(x) - \beta(x)$ and $f(x) = (\frac{\alpha'(x)}{\beta'(x)})^2 - 1$, where

$$\alpha(x) = \sin^{-1} x,$$
$$\beta(x) = \frac{3}{a_3 x + b_3}.$$

Then direct computation lead to

$$f(x) = \frac{10a_3^2 x^2 + 2a_3 b_3 x + b_3^2 - 9a_3^2}{a_3^2(1-x^2)}.$$

Observe that $10a_3^2x^2 + 2a_3b_3x + b_3^2 - 9a_3^2$ is increase on $(0.9,1)$, $f(0.9) < 0$, and $f(1) > 0$. Thus $g(x)$ is decrease firstly and then increase on $(0.9,1)$. Furthermore, it is clear that $g(0.9) = g(1) = 0$. Therefore, inequality (2.17) holds.

Let

$$h(x) = \tanh^{-1} x - \frac{2}{a_4 x + b_4}.$$

It follows that $h(0.9) > 0$ and

$$h'(x) = \frac{1}{1-x^2} - \frac{a_4}{(a_4 x + b_4)^2} = \frac{(a_4^2 + a_4)x^2 + 2a_4 b_4 x + b_4^2 - a_4}{(1-x^2)(a_4 x + b_4)^2} > 0.$$

Thus inequality (2.18) holds. □

LEMMA 2.7. *Let*

$$g(x) = [(1-\beta)\sin^{-1} x - \sinh^{-1} x]\tanh^{-1} x,$$

where $\beta = 1 - \frac{2}{\pi}\sinh^{-1}(1)$. *Then* $g(x)$ *is increase on* $(0.93,1)$.

Proof. Direct computation deduce that

$$
\begin{aligned}
g'(x) &= \Big(\frac{1-\beta}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1+x^2}}\Big)\tanh^{-1}x + \frac{(1-\beta)\sin^{-1}x - \sinh^{-1}x}{1-x^2},\\
g''(x) &= \Big(\frac{(1-\beta)x}{\sqrt{(1-x^2)^3}} - \frac{x}{\sqrt{(1+x^2)^3}}\Big)\tanh^{-1}x\\
&\quad + \frac{2x}{(1-x^2)^2}((1-\beta)\sin^{-1}x - \sinh^{-1}x)\\
&\quad + \Big(\frac{1-\beta}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1+x^2}}\Big)\frac{2}{1-x^2}\\
&= \frac{x[(1-\beta)\sqrt{1-x^2}\tanh^{-1}x + 2(1-\beta)\sin^{-1}x - 2\sinh^{-1}x]}{(1-x^2)^2}\\
&\quad + \Big(\frac{1-\beta}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1+x^2}}\Big)\frac{2}{1-x^2}.
\end{aligned}
$$

Observe that $\frac{1-\beta}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1+x^2}} > 0$ for $x \in (0.93, 1)$. Considering Lemma 2.4, we get $g''(x) > 0$ for $x \in (0.93, 1)$. Noting $g'(0.93) > 0$, it is easy to see that $g'(x) > 0$ for $x \in (0.93, 1)$. $\square$

3. Main result

THEOREM 3.1. *The double inequality*

$$
\alpha L(a,b) + (1-\alpha)M(a,b) < P(a,b) < \beta L(a,b) + (1-\beta)M(a,b) \tag{3.1}
$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha \geqslant \frac{2}{3}$ *and* $\beta \leqslant 1 - \frac{2}{\pi}\sinh^{-1}(1) = 0.4389\ldots$.

Proof. Because $P(a,b)$, $M(a,b)$ and $T(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $p \in (0,1)$, $x = \frac{a-b}{a+b} \in (0,1)$ and $\lambda = 1 - \frac{2}{\pi}\sinh^{-1}(1)$. Then by (1.1), (1.2) and (1.3), direct computations lead to

$$
\begin{aligned}
\frac{L(a,b)}{A(a,b)} &= \frac{x}{\tanh^{-1}x},\\
\frac{M(a,b)}{A(a,b)} &= \frac{x}{\sinh^{-1}x},\\
\frac{P(a,b)}{A(a,b)} &= \frac{x}{\sin^{-1}x}.
\end{aligned}
$$

Then

$$
\begin{aligned}
D_p(x) &:= \frac{pL(a,b) + (1-p)M(a,b) - P(a,b)}{A(a,b)}\\
&= \frac{x}{\tanh^{-1}x} + (1-p)\frac{x}{\sinh^{-1}x} - \frac{x}{\sin^{-1}x}.
\end{aligned} \tag{3.2}
$$

From inequalities (2.2), (2.6) and (2.8), we can get

$$\begin{aligned}
-3D_{\frac{2}{3}}(x) &= \frac{3}{\sin^{-1}x} - \frac{2}{\tanh^{-1}x} - \frac{1}{\sinh^{-1}x} \\
&> \frac{3}{x+\frac{1}{6}x^3+\frac{9}{100}x^5} - \frac{2}{x+\frac{1}{3}x^3+\frac{1}{5}x^5} - \frac{1}{x-\frac{1}{6}x^3+\frac{1}{16}x^5} \\
&= \frac{\frac{31}{1200}-\frac{11}{120}x^2+\frac{33}{4000}x^4}{(x+\frac{1}{6}x^3+\frac{9}{100}x^5)(x+\frac{1}{3}x^3+\frac{1}{5}x^5)(x-\frac{1}{6}x^3+\frac{1}{16}x^5)}x^6 \\
&> 0 \qquad (3.3)
\end{aligned}$$

for $x \in (0, 0.52)$.

From inequalities (2.3), (2.7), and (2.9), we obtain

$$\begin{aligned}
-3D_{\frac{2}{3}}(x) &= \frac{3}{\sin^{-1}x} - \frac{2}{\tanh^{-1}x} - \frac{1}{\sinh^{-1}x} \\
&> \frac{3}{x+\frac{1}{6}x^3+\frac{9}{80}x^5} - \frac{2}{\tanh^{-1}x} - \frac{1}{x-\frac{1}{6}x^3+\frac{1}{20}x^5} \\
&= \frac{(2x-\frac{2}{3}x^3+\frac{3}{80}x^5)\tanh^{-1}x-(x^2+\frac{97}{720}x^6-\frac{1}{96}x^8+\frac{9}{1600}x^{10})}{(x^2+\frac{97}{720}x^6-\frac{1}{96}x^8+\frac{9}{1600}x^{10})\tanh^{-1}x} \\
&> 0 \qquad (3.4)
\end{aligned}$$

for $x \in (0.52, 0.72)$.

By Lemma 2.5, we get

$$\begin{aligned}
-3D_{\frac{2}{3}}(x) &= \frac{3}{\sin^{-1}x} - \frac{2}{\tanh^{-1}x} - \frac{1}{\sinh^{-1}x} \\
&> \left(\frac{9}{2}+a_1\right)x+\frac{109}{20}+b_1-\left(\frac{9}{2}x+\frac{109}{20}\right)-(a_1x+b_1)=0 \qquad (3.5)
\end{aligned}$$

for $x \in (0.72, 0.9)$ as well as Lemma 2.6 deduce that

$$\begin{aligned}
-3D_{\frac{2}{3}}(x) &= \frac{3}{\sin^{-1}x} - \frac{2}{\tanh^{-1}x} - \frac{1}{\sinh^{-1}x} \\
&> a_3x+b_3-[(a_3-a_2)x+b_3-b_2]-(a_2x+b_2)=0. \qquad (3.6)
\end{aligned}$$

for $x \in (0.9, 1)$.

Therefore, it follows from inequalities (3.3)–(3.6) that

$$\frac{2}{3}L(a,b)+\frac{1}{3}M(a,b) < P(a,b) \qquad (3.7)$$

holds for all $a, b > 0$ with $a \neq b$.

From inequalities (2.1), (2.5), and (2.10), we have

$$
\begin{aligned}
D_\lambda(x) &= \frac{\lambda}{\tanh^{-1}x} + \frac{1-\lambda}{\sinh^{-1}x} - \frac{1}{\sin^{-1}x} \\
&> \frac{\lambda}{x+\frac{2-\lambda}{6\lambda}x^3+\frac{36\lambda^2 49\lambda+40}{360\lambda^2}x^5} + \frac{1-\lambda}{x-\frac{1}{6}x^3+\frac{1}{10}x^5} - \frac{1}{x+\frac{1}{6}x^3+\frac{3}{40}x^5} \\
&= x^8 \frac{\frac{80-156\lambda+76\lambda^2}{2160\lambda^2}+\frac{111\lambda^2-71\lambda-40}{14400\lambda^2}x^2}{(x+\frac{2-\lambda}{6\lambda}x^3+\frac{36\lambda^2 49\lambda+40}{360\lambda^2}x^5)(x-\frac{1}{6}x^3+\frac{1}{10}x^5)(x+\frac{1}{6}x^3+\frac{3}{40}x^5)} \\
&> 0
\end{aligned}
\tag{3.8}
$$

for $x \in (0, 0.93)$.

Simple computation lead to

$$
D_\lambda(x) = \frac{F(x)}{(\tanh^{-1}x)(\sinh^{-1}x)(\sin^{-1}x)}, \tag{3.9}
$$

where

$$
F(x) = \lambda \sin^{-1}x \sinh^{-1}x + [(1-\lambda)\sin^{-1}x - \sinh^{-1}x]\tanh^{-1}x.
$$

It is obvious that $\lambda \sin^{-1}x \sinh^{-1}x$ is increase on (0.93,1). Considering Lemma 2.7, we get that $F(x)$ is increase on $(0.93, 1)$. Noting that $F(0.93) > 0$. Thus equation (3.9) implies that

$$
D_\lambda(x) > 0 \tag{3.10}
$$

for $x \in (0.93, 1)$.

Therefore, it follows from inequalities (3.8) and (3.10) that for $x \in (0,1)$

$$
P(a,b) < \beta L(a,b) + (1-\beta)M(a,b) \tag{3.11}
$$

holds for all $a, b > 0$ with $a \neq b$.

Finally, by easy computations, equations(1.1), (1.2) and (1.3) lead to

$$
\frac{P(a,b)-M(a,b)}{L(a,b)-M(a,b)} = \frac{x/\sin^{-1}(x) - x/\sinh^{-1}(x)}{x/\tanh^{-1}(x) - x/\sinh^{-1}(x)}, \tag{3.12}
$$

$$
\lim_{x\to 0^+} \frac{x/\sin^{-1}(x) - x/\sinh^{-1}(x)}{x/\tanh^{-1}(x) - x/\sinh^{-1}(x)} = \frac{2}{3}, \tag{3.13}
$$

$$
\lim_{x\to 1^-} \frac{x/\sin^{-1}(x) - x/\sinh^{-1}(x)}{x/\tanh^{-1}(x) - x/\sinh^{-1}(x)} = \lambda. \tag{3.14}
$$

Thus, we have the following claims.

Claims 1. If $\alpha < \frac{2}{3}$, then (3.12) and (3.13) imply that there exists $\sigma \in (0,1)$ such that $\alpha L(a,b) + (1-\alpha)M(a,b) > P(a,b)$ for all a, b with $(a-b)/(a+b) \in (0, \sigma)$.

Claims 2. If $\beta > \lambda$, then (3.12) and (3.14) imply that there exists $\varsigma \in (0,1)$ such that $\beta L(a,b) + (1-\beta)M(a,b) < P(a,b)$ for all a, b with $(a-b)/(a+b) \in (1-\varsigma, 1)$.

Inequalities (3.7) and (3.11) in conjunction with the above two claims mean the proof is completed. $\square$

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Jian-Jun Lei
School of Mathematical Science
Anhui University
Hefei 230601, China
e-mail: jianjunlei@126.com

Jing-Jing Chen
School of Mathematical Science
Anhui University
Hefei 230601, China
e-mail: jingjingchennew@163.com

Bo-Yong Long
School of Mathematical Science
Anhui University
Hefei 230601, China
e-mail: longboyong@ahu.edu.cn