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THE DUAL ORLICZ–BRUNN–MINKOWSKI INEQUALITY FOR CONCAVE FUNCTIONS

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Abstract. In this paper, we define the Orlicz radial sum and dual Orlicz mixed quermassintegral for concave functions, and then establish the dual Orlicz-Brunn-Minkowski inequality for concave functions.

1. Introduction

Convex geometry analysis has made great achievement in Orlicz space (see, e.g., [4, 7–9, 13, 16, 25, 26, 29, 30, 32, 33, 36]). It is worth mentioning that the (dual) Orlicz-Brunn-Minkowski theory is an extenuation of the (dual) Brunn-Minkowski theory (see, e.g., [1–3, 5, 10, 11, 14, 15, 17, 21–24, 27, 31, 34, 35]). Recently, Zhu, Zhou and Xu [32] defined an Orlicz radial sum for convex functions for star bodies, and established the dual Orlicz-Brunn-Minkowski inequality for convex functions for star bodies. Let $\phi : (0,+\infty) \to (0,+\infty)$ be a convex and strictly decreasing function such that $\lim_{t\to\infty}\phi(t) = 0$, $\lim_{t\to 0}\phi(t) = \infty$, and $\phi(0) = \infty$. Let K and L be two star bodies about the origin in $\mathbb{R}^n$. The Orlicz radial sum $K\widetilde{+}_\phi L$ of two star bodies K and L is defined by

$$\rho_{K\widetilde{+}_\phi L}(u) = \sup\left\{t > 0 : \ \phi\left(\frac{\rho_K(u)}{t}\right) + \phi\left(\frac{\rho_L(u)}{t}\right) \leqslant \phi(1)\right\},\ \forall u \in S^{n-1}. \tag{1.1}$$

The case $\phi(t) = t^{-p}(p \geqslant 1)$ of the Orlicz radial sum is the L_p harmonic radial sum which was defined by Lutwak [20].

Zhu, Zhou and Xu [32] established the dual Orlicz-Brunn-Minkowski inequality for convex functions.

THEOREM A. ([32]) *If K and L are two star bodies about the origin in $\mathbb{R}^n$, then*

$$\phi(1) \geqslant \phi\left(\left(\frac{V(K)}{V(K\widetilde{+}_\phi L)}\right)^{\frac{1}{n}}\right) + \phi\left(\left(\frac{V(L)}{V(K\widetilde{+}_\phi L)}\right)^{\frac{1}{n}}\right), \tag{1.2}$$

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with equality if and only if K and L are dilates.

In [18], Ludwig introduced two families of general affine surface areas. One family of general affine surface areas is for concave functions, while the other is for convex functions. Inspired by Ludwig's work, in this paper, we will define an Orlicz radial sum for concave functions. Let $\psi:[0,+\infty)\to[0,+\infty)$ be a concave and strictly increasing function such that $\psi(0)=0$ and $\lim_{t\to\infty}\psi(t)=\infty$. Let K and L be two star bodies about the origin in $\mathbb{R}^n$. The Orlicz radial sum $K\widetilde{+}_\psi L$ of two star bodies K and L is defined by

$$\rho_{K\widetilde{+}_\psi L}(u)=\inf\Big\{t>0:\ \psi\Big(\frac{\rho_K(u)}{t}\Big)+\psi\Big(\frac{\rho_L(u)}{t}\Big)\leqslant\psi(1)\Big\},\qquad u\in S^{n-1}. \tag{1.3}$$

Let $\mathscr{V}^+$ be the set of concave and strictly increasing functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\psi(0)=0$ and $\lim_{t\to\infty}\psi(t)=\infty$. We will establish the following dual Orlicz-Brunn-Minkowski inequality for concave functions.

THEOREM 1.1. *Let K and L be two star bodies about the origin in $\mathbb{R}^n$. If $\psi\in\mathscr{V}^+$, then*

$$\psi(1)\leqslant\psi\Big(\Big(\frac{V(K)}{V(K\widetilde{+}_\psi L)}\Big)^{\frac{1}{n}}\Big)+\psi\Big(\Big(\frac{V(L)}{V(K\widetilde{+}_\psi L)}\Big)^{\frac{1}{n}}\Big), \tag{1.4}$$

with equality if and only if K and L are dilates.

2. Notation and background material

The unit ball and its surface in $\mathbb{R}^n$ are denoted by B and S^{n-1}, respectively. We write $V(K)$ for the volume of a compact set K in $\mathbb{R}^n$. Let $GL(n)$ denote the general linear group of degree n. For $A\in GL(n)$ write A^t for the transpose of A and A^{-t} for the inverse of the transpose of A. The absolute value of the determinant of A is denoted by $|A|$.

We say a sequence $\{\psi_i\}\subset\mathscr{V}^+$ is such that $\psi_i\to\psi_0\in\mathscr{V}^+$, provided

$$|\psi_i-\psi_0|_I:=\max_{s\in I}|\psi_i(s)-\psi_0(s)|\to 0,$$

for every compact interval $I\in[0,\infty)$.

The radial function $\rho_K(u):S^{n-1}\to[0,\infty)$ of a compact star-shaped about the origin $K\in\mathbb{R}^n$ is defined, for $u\in S^{n-1}$, by

$$\rho_K(u)=\max\{\lambda\geqslant 0:\ \lambda u\in K\}.$$

If $\rho_K(\cdot)$ is positive and continuous, then K is called a star body about the origin. The set of star bodies about the origin in $\mathbb{R}^n$ is denoted by $\mathscr{S}_0^n$. Obviously, for $K,L\in\mathscr{S}_0^n$,

$$K\subseteq L\Leftrightarrow\rho_K(u)\leqslant\rho_L(u),\quad \text{for all } u\in S^{n-1}. \tag{2.1}$$

If $\dfrac{\rho_K(u)}{\rho_L(u)}$ is independent of $u\in S^{n-1}$, then we say star bodies K and L are dilates. If $s>0$, we have

$$\rho_{sK}(u)=s\rho_K(u),\quad \text{for all } u\in S^{n-1}. \tag{2.2}$$

More generally, for $A \in GL(n)$, we have

$$\rho_{AK}(u) = \rho_K(A^{-1}u), \quad \text{for all } u \in S^{n-1}. \tag{2.3}$$

For $K \in \mathscr{S}_0^n$, defined the real numbers R_K and r_K by

$$R_K = \max_{u \in S^{n-1}} \rho_K(u) \text{ and } r_K = \min_{u \in S^{n-1}} \rho_K(u).$$

Note that $0 < r_K \leqslant R_K < \infty$, for all $K \in \mathscr{S}_0^n$.

A sequence $\{K_i\}$ of star bodies is said to be convergent to K if

$$\widetilde{\delta}(K_i, K) \to 0, \text{ as } i \to \infty,$$

where $\widetilde{\delta}(K_i, K) = \max_{u \in S^{n-1}} |\rho_{K_i}(u) - \rho_K(u)|$.

Therefore, a sequence of star bodies K_i converges to K if and only if the sequence of radial function $\rho(K_i, \cdot)$ converges uniformly to $\rho(K, \cdot)$ on S^{n-1}.

For $p \neq 0$. If $K, L \in \mathscr{S}_0^n$ and $a, b \geqslant 0$ (not both 0), the L_p radial sum $a \cdot K \widetilde{+}_p b \cdot L$ is defined by [6]

$$\rho^p_{a \cdot K \widetilde{+}_p b \cdot L}(u) = a\rho_K^p(u) + b\rho_L^p(u), \ \forall u \in S^{n-1}. \tag{2.4}$$

Let $K, L \in \mathscr{S}_0^n$ and $0 \leqslant i < n-1$, the dual L_p mixed quermassintegral $\widetilde{W}_{p,i}(K, L)$ is defined by [28]

$$\widetilde{W}_{p,i}(K, L) = \frac{p}{n-i} \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K \widetilde{+}_p \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}.$$

The dual L_p mixed quermassintegral $\widetilde{W}_{p,i}(K, L)$ has the following integral representation [28]

$$\widetilde{W}_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i-p}(u) \rho_L^p(u) dS(u). \tag{2.5}$$

By using the Hölder inequality, we can obtain the dual L_p Minkowski inequality [28]. Let $K, L \in \mathscr{S}_0^n$ and $0 \leqslant i < n-1$. If $0 < p < n-i$, then

$$\widetilde{W}_{p,i}(K, L) \leqslant \widetilde{W}_i(K)^{\frac{n-i-p}{n}} \widetilde{W}_i(L)^{\frac{p}{n}}, \tag{2.6}$$

with equality if and only if K and L are dilates.

By using the Minkowski's integral inequality, we can obtain the dual L_p Brunn-Minkowski inequality. Let $K, L \in \mathscr{S}_0^n$, $0 \leqslant i < n-1$ and $a, b \geqslant 0$. If $0 < p < n-i$, then

$$\widetilde{W}_i(a \cdot K \widetilde{+}_p b \cdot L)^{\frac{p}{n-i}} \leqslant a\widetilde{W}_i(K)^{\frac{p}{n-i}} + b\widetilde{W}_i(L)^{\frac{p}{n-i}}, \tag{2.7}$$

with equality if and only if K and L are dilates.

The cases $p = 1$ of the dual L_p Minkowski inequality and the dual L_p Brunn-Minkowski inequality were established by Lutwak [19].

Suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-intergrable function, where I is a possibly infinite interval. Jessen's inequality states that if $\psi : X \to I \subset \mathbb{R}$ is a concave function, then

$$\int_X \psi(g(x))d\mu(x) \leqslant \psi(\int_X g(x)d\mu(x)). \tag{2.8}$$

If ψ is strictly concave, equality holds if and only if $g(x)$ is a constant for μ-almost all $x \in X$, (see [12]).

3. Orlicz radial sum for concave functions

DEFINITION 3.1. Let $K, L \in \mathscr{S}_0^n$. For $a, b \geqslant 0$ and $\psi \in \mathscr{V}^+$, define the Orlicz radial sum $a \cdot K \widetilde{+}_\psi b \cdot L$ of K and L by

$$\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u) = \inf\Big\{t > 0 :\ a\psi\Big(\frac{\rho_K(u)}{t}\Big) + b\psi\Big(\frac{\rho_L(u)}{t}\Big) \leqslant \psi(1)\Big\},\ \ u \in S^{n-1}. \tag{3.1}$$

REMARK 3.1. The case $\psi(t) = t^p$ $(0 < p \leqslant 1)$ of the Orlicz radial sum reduces to L_p radial sum.

By the assumption that ψ is strictly increasing and concave, the function

$$t \to a\psi\Big(\frac{\rho_K}{t}\Big) + b\psi\Big(\frac{\rho_L}{t}\Big)$$

is strictly decreasing and continuous on $[0, \infty)$. Thus, we have

LEMMA 3.1. *Let $K, L \in \mathscr{S}_0^n$ and $u \in S^{n-1}$. If $\psi \in \mathscr{V}^+$, then*

$$a\psi\Big(\frac{\rho_K(u)}{t}\Big) + b\psi\Big(\frac{\rho_L(u)}{t}\Big) = \psi(1)$$

if and only if

$$\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u) = t.$$

If $K, L \in \mathscr{S}_0^n$, let $R = \max\{R_K, R_L\}$ and $r = \min\{r_K, r_L\}$. For $a, b \geqslant 0$, let $c = a + b$. Since ψ is continuous and strictly increasing on $[0, \infty)$, hence the inverse ψ^{-1} is also continuous and increasing on $[0, \infty)$.

LEMMA 3.2. *Let $K, L \in \mathscr{S}_0^n$. If $\psi \in \mathscr{V}^+$, then*

$$\frac{r}{\psi^{-1}(\frac{\psi(1)}{c})} \leqslant \rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u) \leqslant \frac{R}{\psi^{-1}(\frac{\psi(1)}{c})}.$$

Proof. Suppose $u \in S^{n-1}$ and let $\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u) = t$. By Lemma 3.1 and the fact that ψ is strictly increasing on $[0,\infty)$, one can obtain that

$$\begin{aligned}\psi(1) &= a\psi\Big(\frac{\rho_K(u)}{t}\Big)+b\psi\Big(\frac{\rho_L(u)}{t}\Big)\\ &\geqslant a\psi\Big(\frac{r_K(u)}{t}\Big)+b\psi\Big(\frac{r_L(u)}{t}\Big)\\ &\geqslant a\psi\Big(\frac{r}{t}\Big)+b\psi\Big(\frac{r}{t}\Big)\\ &= c\psi\Big(\frac{r}{t}\Big).\end{aligned}$$

Since the inverse ψ^{-1} of ψ is strictly increasing on $[0,\infty)$, we have

$$t \geqslant \frac{r}{\psi^{-1}\big(\frac{\psi(1)}{c}\big)}.$$

On the other hand, from Lemma 3.1, together with the concavity and the strictly increasing on $[0,\infty)$ of ψ, one can obtain that

$$\begin{aligned}\frac{\psi(1)}{a+b} &= \frac{a}{a+b}\psi\Big(\frac{\rho_K(u)}{t}\Big)+\frac{b}{a+b}\psi\Big(\frac{\rho_L(u)}{t}\Big)\\ &\leqslant \frac{a}{a+b}\psi\Big(\frac{R_K}{t}\Big)+\frac{b}{a+b}\psi\Big(\frac{R_L}{t}\Big)\\ &\leqslant \psi\Big(\frac{\frac{a}{a+b}R_K+\frac{b}{a+b}R_L}{t}\Big)\\ &\leqslant \psi\Big(\frac{R}{t}\Big).\end{aligned}$$

Thus we obtain

$$t \leqslant \frac{R}{\psi^{-1}\big(\frac{\psi(1)}{c}\big)}. \quad \square$$

By using the same method in [32], we can prove the following Lemmas.

LEMMA 3.3. *Let $K,L \in \mathscr{S}_0^n$ and $a,b \geqslant 0$. If $\psi \in \mathscr{V}^+$, then for $A \in GL(n)$,*

$$A(a\cdot K\widetilde{+}_\psi b\cdot L) = a\cdot AK\widetilde{+}_\psi b\cdot AL.$$

Proof. For $u \in S^{n-1}$, by (3.1) and (2.3)

$$\begin{aligned}\rho_{a\cdot AK\widetilde{+}_\psi b\cdot AL}(u) &= \inf\Big\{t>0:\ a\psi\Big(\tfrac{\rho_{AK}(u)}{t}\Big)+b\psi\Big(\tfrac{\rho_{AL}(u)}{t}\Big)\leqslant \psi(1)\Big\}\\ &= \inf\Big\{t>0:\ a\psi\Big(\tfrac{\rho_K(A^{-1}u)}{t}\Big)+b\psi\Big(\tfrac{\rho_L(A^{-1}u)}{t}\Big)\leqslant \psi(1)\Big\}\\ &= \rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(A^{-1}u)\\ &= \rho_{A(a\cdot K\widetilde{+}_\psi b\cdot L)}(u). \quad \square\end{aligned}$$

LEMMA 3.4. *Let $K,L \in \mathscr{S}_0^n$ and $a,b \geqslant 0$ (not both 0). If $\psi \in \mathscr{V}^+$, then $a\cdot K\widetilde{+}_\psi b\cdot L \in \mathscr{S}_0^n$.*

Proof. Let $u \in S^{n-1}$, for any subsequence $\{u_i\} \subset S^{n-1}$ such that $u_i \to u_0$ as $i \to \infty$, we need to show

$$\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u_i) \to \rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u_0), \ \text{ as } \ i\to\infty.$$

Let $\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u_i) = t_i$. By Lemma 3.2, we have

$$\frac{r}{\psi^{-1}\big(\frac{\psi(1)}{c}\big)} \leqslant t_i \leqslant \frac{R}{\psi^{-1}\big(\frac{\psi(1)}{c}\big)}.$$

Since $K,L\in\mathscr{S}_0^n$, it follows that $0<r_K\leqslant R_K<\infty$, $0<r_L\leqslant R_L<\infty$. Thus, there exist λ,μ such that $0<\lambda\leqslant t_i\leqslant\mu<\infty$, for all i. To show that the bounded sequence $\{t_i\}$ converges to $\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u_0)$, we show that every convergent subsequence of $\{t_i\}$ converges to $\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u_0)$. Denote an arbitrary convergent subsequence of $\{t_i\}$ by $\{t_i\}$ as well, and suppose that for this subsequence $t_i \to t_0$.

It is clear that $\lambda \leqslant t_0 \leqslant \mu$. From Lemma 3.1 and note the fact $\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u_i)=t_i$, we have

$$\psi\Big(\frac{\rho_K(u_i)}{t_i}\Big)+b\psi\Big(\frac{\rho_L(u_i)}{t_i}\Big)=\psi(1).$$

From the continuities of these functions ψ, ρ_K, ρ_L and the fact $t_i\to t_0$, we have

$$\psi\Big(\frac{\rho_K(u_0)}{t_0}\Big)+b\psi\Big(\frac{\rho_L(u_0)}{t_0}\Big)=\psi(1).$$

By Lemma 3.1, it follows that $\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u_0)=t_0$. This means

$$\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u_i) \to \rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u_0).$$

Therefore, the continuity of $\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}$ is proved and $a\cdot K\widetilde{+}_\psi b\cdot L\in\mathscr{S}_0^n$. $\square$

LEMMA 3.5. *Let $\psi \in \mathscr{V}^+$. If $K_i, L_i \in \mathscr{S}_0^n$ and $K_i \to K \in \mathscr{S}_0^n$, $L_i \to L \in \mathscr{S}_0^n$, as $i \to \infty$, then*

$$a \cdot K_i \widetilde{+}_\psi b \cdot L_i \to a \cdot K \widetilde{+}_\psi b \cdot L, \text{ as } i \to \infty$$

for all a and b.

Proof. We will show that, for $u \in S^{n-1}$,

$$\rho_{a\cdot K_i \widetilde{+}_\psi b\cdot L_i}(u) \to \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u), \text{ as } i \to \infty. \tag{3.2}$$

Let $\rho_{a\cdot K_i \widetilde{+}_\psi b\cdot L_i}(u) = t_i$. We set $R_i = \max\{R_{K_i}, R_{L_i}\}$ and $r_i = \max\{r_{K_i}, r_{L_i}\}$. Then Lemma 3.2 gives

$$\frac{r_i}{\psi^{-1}\left(\frac{\psi(1)}{c}\right)} \leqslant t_i \leqslant \frac{R_i}{\psi^{-1}\left(\frac{\psi(1)}{c}\right)}.$$

Since $K_i \to K \in \mathscr{S}_0^n$ and $L_i \to L \in \mathscr{S}_0^n$, it follows that $R_{K_i} \to R_K < \infty$, $R_{L_i} \to R_L < \infty$ and $r_{K_i} \to r_K > 0$, $r_{L_i} \to r_L > 0$. For the functions $R = \max\{R_K, R_L\}$ and $r = \min\{r_K, r_L\}$ are continuous, we have $R_i \to R < \infty$ and $r_i \to r > 0$. Thus, there exist λ, μ such that

$$0 < \lambda \leqslant \mu < \infty, \text{ for all } i. \tag{3.3}$$

To show that the bounded sequence $\{t_i\}$ converges to $\rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u)$, we need to show that every convergent subsequence of $\{t_i\}$ converges to $\rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u)$. Denote an arbitrary convergent subsequence of $\{t_i\}$ by $\{t_i\}$ as well, and suppose that for this subsequence we have $t_i \to t_0$.

It is clear that $\lambda < t_0 < \mu$. We set $\widetilde{K_i} = t_i^{-1} K_i$ and $\widetilde{L_i} = t_i^{-1} L_i$. Since $K_i \to K_0$, $L_i \to L_0$ and $t_i \to t_0$, it follows that $\widetilde{K_i} \to K_0$ and $\widetilde{L_i} \to L_0$. From Lemma 3.3 and the fact $\rho_{a\cdot K_i \widetilde{+}_\psi b\cdot L_i}(u) = t_i$, we have $\rho_{a\cdot \widetilde{K_i} \widetilde{+}_\psi b\cdot \widetilde{L_i}}(u) = 1$. That is

$$a\varphi(\rho_{\widetilde{K_i}}(u)) + b\varphi(\rho_{\widetilde{L_i}}(u)) = \varphi(1), \text{ for all } i.$$

Since $\widetilde{K_i} \to K_0$ and $\widetilde{L_i} \to L_0$, together with the continuity of φ, and (2.2), it follows that

$$a\varphi\Big(\frac{\rho_K(u)}{t_0}\Big) + b\varphi\Big(\frac{\rho_L(u)}{t_0}\Big) = \varphi(1).$$

By Lemma 3.1, we have

$$t_0 = \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u).$$

This means

$$\rho_{a\cdot K_i \widetilde{+}_\psi b\cdot L_i}(u) \to \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u).$$

The pointwise convergence (3.2) has been proved.

Next, we will show that the convergence (3.2) is uniform for any $u \in S^{n-1}$. Assume that $\rho_{a\cdot K_i \widetilde{+}_\psi b\cdot L_i}$ does not converge uniformly to $\rho_{a\cdot K \widetilde{+}_\psi b\cdot L}$. Then, there exists a $\delta_0 > 0$ such that, for $i \geqslant N_0 \in \mathbb{N}$,

$$|\rho_{a\cdot K_i \widetilde{+}_\psi b\cdot L_i}(u_i) - \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u_i)| \geqslant \delta_0. \tag{3.4}$$

Since S^{n-1} is compact, for some $u_0 \in S^{n-1}$, there exists a subsequence $\{u_i\} \subset S^{n-1}$ such that $u_i \to u_0 \in S^{n-1}$ as $i \to \infty$.

From Lemma 3.2, there exist an $N_0 \in \mathbb{N}$ and positive λ, μ such that (3.3) holds for $i \geqslant N_0$. Then, there exists a positive s_0 such that

$$\rho_{a\cdot K_i \widetilde{+}_\psi b\cdot L_i}(u_i) \to s_0.$$

By (3.4), we have

$$|s_0 - \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u_0)| \geqslant \delta_0.$$

This implies

$$s_0 \neq \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u_0). \tag{3.5}$$

Let $s_i = \rho_{a\cdot K_i \widetilde{+}_\psi b\cdot L_i}(u_i)$. By Lemma 3.1, it follows that

$$a\varphi\Big(\frac{\rho_{K_i}(u_i)}{s_i}\Big) + b\varphi\Big(\frac{\rho_{L_i}(u_i)}{s_i}\Big) = \varphi(1).$$

Applying with the facts that $K_i \to K, L_i \to L$ and $s_i \to s_0$, we have

$$a\varphi\Big(\frac{\rho_K(u_0)}{s_0}\Big) + b\varphi\Big(\frac{\rho_L(u_0)}{s_0}\Big) = \varphi(1).$$

By Lemma 3.1 again, we have

$$s_0 = \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u_0).$$

This contradicts to (3.5). Therefore,

$$\rho_{a\cdot K_i \widetilde{+}_\psi b\cdot L_i} \to \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}$$

uniformly on S^{n-1} and hence

$$a\cdot K_i \widetilde{+}_\psi b\cdot L_i \to a\cdot K \widetilde{+}_\psi b\cdot L. \quad \square$$

LEMMA 3.6. *Let $\psi \in \mathscr{V}^+$. If $a_i \to a, b_i \to b$, as $i \to \infty$, then*

$$a_i \cdot K \widetilde{+}_\psi b_i \cdot L \to a\cdot K \widetilde{+}_\psi b\cdot L, \text{ as } i \to \infty$$

for all $K, L \in \mathscr{S}_0^n$.

Proof. For $u \in S^{n-1}$ and $K, L \in \mathscr{S}_0^n$, we will show that

$$\rho_{a_i\cdot K \widetilde{+}_\psi b_i\cdot L}(u) \to \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u), \text{ as } i \to \infty. \tag{3.6}$$

Let $\rho_{a_i\cdot K \widetilde{+}_\psi b_i\cdot L}(u) = t_i$. By Lemma 3.2, one can obtain that

$$\frac{r}{\psi^{-1}\big(\frac{\psi(1)}{2C_i}\big)} \leqslant t_i \leqslant \frac{R}{\psi^{-1}\big(\frac{\psi(1)}{c_i}\big)}.$$

Since $a_i \to a, b_i \to b$ and the facts that the functions $C_i = \max\{a_i, b_i\}$ and $c_i = a_i + b_i$ are continuous, we have $C_i \to C$ and $c_i \to c$. Note that the inverse φ^{-1} of φ is also continuous and decreasing in $(0,\infty)$, there exist λ, μ such that $0 < \lambda \leqslant t_i \leqslant \mu < \infty$, for all i. To show that the bounded sequence $\{t_i\}$ converges to $\rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u)$, we show that every convergent subsequence of $\{t_i\}$ converges to $\rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u)$. Denote an arbitrary convergent subsequence of $\{t_i\}$ by $\{t_i\}$ as well, and suppose that for this subsequence we have $t_i \to t_0$.

It is clear that $0 < \lambda \leqslant t_0 \leqslant \mu$. By Lemma 3.1 and the fact $\rho_{a_i\cdot K \widetilde{+}_\psi b_i\cdot L}(u) = t_i$, we have

$$a_i \varphi\left(\frac{\rho_K(u)}{t_i}\right) + b_i \varphi\left(\frac{\rho_L(u)}{t_i}\right) = \varphi(1).$$

Applying with the facts that $a_i \to a, b_i \to b$, $t_i \to t_0$ and the continuity of ψ, one can obtain that

$$a \varphi\left(\frac{\rho_K(u)}{t_0}\right) + b \varphi\left(\frac{\rho_L(u)}{t_0}\right) = \varphi(1).$$

By Lemma 3.1, we have

$$t_0 = \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u).$$

This means

$$\rho_{a_i\cdot K \widetilde{+}_\psi b_i\cdot L}(u) \to \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u).$$

The pointwise convergence (3.6) has been proved.

Next, we will show that the convergence (3.6) is uniform for any $u \in S^{n-1}$. Assume that $\rho_{a_i\cdot K \widetilde{+}_\psi b_i\cdot L}$ does not converge uniformly to $\rho_{a\cdot K \widetilde{+}_\psi b\cdot L}$. Then, there exists a $\delta_0 > 0$ such that, for $i \geqslant N_0 \in \mathbb{N}$,

$$|\rho_{a_i\cdot K \widetilde{+}_\psi b_i\cdot L}(u_i) - \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u_i)| \geqslant \delta_0. \tag{3.7}$$

Since S^{n-1} is compact, for $u_0 \in S^{n-1}$, there exists a subsequence $\{u_i\} \subset S^{n-1}$ such that $u_i \to u_0 \in S^{n-1}$ as $i \to \infty$.

From Lemma 3.2, there exist an $N_0 \in \mathbb{N}$ and positive λ, μ such that, for $i \geqslant N_0$,

$$0 < \lambda \leqslant \rho_{a_i\cdot K \widetilde{+}_\psi b_i\cdot L}(u_i) \leqslant \mu < \infty.$$

Then, there exists a positive s_0 such that, for $i \geqslant N_0$,

$$\rho_{a_i\cdot K \widetilde{+}_\psi b_i\cdot L}(u_i) \to s_0.$$

By (3.7), it follows that

$$|s_0 - \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u_0)| \geqslant \delta_0.$$

This implies

$$s_0 \neq \rho_{a\cdot K \widetilde{+}_\psi b\cdot L}(u_0). \tag{3.8}$$

Let $s_i = \rho_{a_i\cdot K \widetilde{+}_\psi b_i\cdot L}(u_i)$. From Lemma 3.1, we have

$$a_i \varphi\left(\frac{\rho_K(u_i)}{s_i}\right) + b_i \varphi\left(\frac{\rho_L(u_i)}{s_i}\right) = \varphi(1).$$

Applying with the facts that $a_i \to a, b_i \to b$ and $s_i \to s_0$, one can obtain that

$$a\varphi\Big(\frac{\rho_K(u_0)}{s_0}\Big)+b\varphi\Big(\frac{\rho_L(u_0)}{s_0}\Big)=\varphi(1).$$

By Lemma 3.1 again, we have

$$s_0=\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}(u_0).$$

This contradicts to (3.8). Therefore,

$$\rho_{a_i\cdot K\widetilde{+}_\psi b_i\cdot L}\to\rho_{a\cdot K\widetilde{+}_\psi b\cdot L}$$

uniformly on S^{n-1} and hence

$$a_i\cdot K\widetilde{+}_\psi b_i\cdot L\to a\cdot K\widetilde{+}_\psi b\cdot L. \quad \square$$

We will show that the Orlicz radial sum is monotone with respect to set inclusion.

LEMMA 3.7. *Let $L_1, L_2 \in \mathscr{S}_0^n$ and $\psi \in \mathscr{V}^+$. If $L_1 \subseteq L_2$, then*

$$K\widetilde{+}_\psi L_1 \subseteq K\widetilde{+}_\psi L_2,$$

for all $K \in \mathscr{S}_0^n$.

Proof. For $\forall u \in S^{n-1}$, let $\rho_{K\widetilde{+}_\psi L_1}(u)=t_1$ and $\rho_{K\widetilde{+}_\psi L_2}(u)=t_2$. By Lemma 3.1, we have

$$\psi\Big(\frac{\rho_K(u)}{t_1}\Big)+\psi\Big(\frac{\rho_{L_1}(u)}{t_1}\Big)=\psi(1), \tag{3.9}$$

and

$$\psi\Big(\frac{\rho_K(u)}{t_2}\Big)+\psi\Big(\frac{\rho_{L_2}(u)}{t_2}\Big)=\psi(1). \tag{3.10}$$

We assume $t_1 > t_2$. Since ψ is strictly increasing on $[0,\infty)$, it follows that

$$\psi\Big(\frac{\rho_K(u)}{t_1}\Big)<\psi\Big(\frac{\rho_K(u)}{t_2}\Big). \tag{3.11}$$

By (3.9), (3.10) and (3.11), we have

$$\psi\Big(\frac{\rho_{L_1}(u)}{t_1}\Big)>\psi\Big(\frac{\rho_{L_2}(u)}{t_2}\Big).$$

Since ψ is strictly increasing on $[0,\infty)$, it follows that

$$\frac{\rho_{L_1}(u)}{t_1}>\frac{\rho_{L_2}(u)}{t_2}.$$

Thus,

$$\frac{\rho_{L_1}(u)}{\rho_{L_2}(u)}>\frac{t_1}{t_2}>1,$$

and it means that $L_1 \supset L_2$. However, this is a contradiction with the condition. $\square$

We will show that the Orlicz radial sum for concave functions is closely related with the raidal sum.

LEMMA 3.8. *Let $K,L \in \mathscr{S}_0^n$. For $0<\lambda<1$, if $\psi \in \mathscr{V}^+$, then*

$$(1-\lambda)\cdot K\widetilde{+}_\psi \lambda \cdot L \subseteq (1-\lambda)K\widetilde{+}\lambda L. \tag{3.12}$$

If ψ is strictly concave, equality holds if and only if K and L are dilates.

Proof. Let $K_\lambda = (1-\lambda)\cdot K\widetilde{+}_\psi \lambda \cdot L$. By Lemma 3.1 and concavity of ψ, we have

$$\begin{aligned}\psi(1) &= (1-\lambda)\psi\Big(\tfrac{\rho_K(u)}{\rho_{K_\lambda}(u)}\Big) + \lambda\psi\Big(\tfrac{\rho_L(u)}{\rho_{K_\lambda}(u)}\Big)\\ &\leqslant \psi\Big(\frac{(1-\lambda)\rho_K(u)+\lambda\rho_L(u)}{\rho_{K_\lambda}(u)}\Big)\\ &= \psi\Big(\frac{\rho_{(1-\lambda)K\widetilde{+}\lambda L}(u)}{\rho_{K_\lambda}(u)}\Big).\end{aligned}$$

Since ψ is strictly increasing on $[0,\infty)$, then, for $\forall u \in S^{n-1}$ we have

$$\rho_{K_\lambda}(u) \leqslant \rho_{(1-\lambda)K\widetilde{+}\lambda L}(u).$$

Thus,

$$K_\lambda \subseteq (1-\lambda)K\widetilde{+}\lambda L.$$

From the equality condition in Jensen's inequality (2.7), if ψ is strictly convex, then equation holds in (3.12) if and only if K and L are dilates. $\square$

4. Dual Orlicz mixed volume for concave functions

DEFINITION 4.1. Let $K,L \in \mathscr{S}_0^n$ and $\psi \in \mathscr{V}^+$, for $0 \leqslant i < n-1$ and $0<p<n-i$, the dual Orlicz mixed quermassintegral $\widetilde{W}_{\psi,i}(K,L)$ for concave functions is defined by

$$\widetilde{W}_{\psi,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \psi\Big(\frac{\rho_L(u)}{\rho_K(u)}\Big)\rho_K^{n-i}(u)dS(u). \tag{4.1}$$

REMARK 4.1. The case $\psi(t)=t^p$ $(0<p\leqslant 1)$ of the dual Orlicz mixed quermassintegral $\widetilde{W}_{\psi,i}(K,L)$ is the dual L_p mixed quermassintegral $\widetilde{W}_{p,i}(K,L)$.

We denote the left derivative of a real-valued function f by f'_-. For $\psi \in \mathscr{V}^+$, there is $\psi'_-(1)>0$ because ψ is concave and strictly increasing.

LEMMA 4.1. *Let $K,L \in \mathscr{S}_0^n$ and $\psi \in \mathscr{V}^+$. Then*

$$\lim_{\varepsilon\to 0^+} \frac{\rho_{K\widetilde{+}_\psi \varepsilon\cdot L}(u) - \rho_K(u)}{\varepsilon} = \frac{\rho_K(u)}{\psi'_-(1)}\psi\Big(\frac{\rho_L(u)}{\rho_K(u)}\Big), \tag{4.2}$$

uniformly for all $u \in S^{n-1}$.

Proof. Suppose $\varepsilon>0$, $K,L \in \mathscr{S}_0^n$, and $u \in S^{n-1}$. Let

$$t_\varepsilon = \rho_{K\widetilde{+}_\psi \varepsilon\cdot L}(u).$$

Then, by Lemma 3.6, we have

$$t_\varepsilon \to \rho_K(u) \ \text{ as } \varepsilon \to 0. \tag{4.3}$$

By Lemma 3.1, we have

$$\psi\Big(\frac{\rho_K(u)}{t_\varepsilon}\Big) + \varepsilon\psi\Big(\frac{\rho_L(u)}{t_\varepsilon}\Big) = \psi(1).$$

Then

$$\frac{\rho_K(u)}{t_\varepsilon} = \psi^{-1}\Big(\psi(1) - \varepsilon\psi\Big(\frac{\rho_L(u)}{t_\varepsilon}\Big)\Big).$$

Let

$$s = \psi^{-1}\Big(\psi(1) - \varepsilon\psi\Big(\frac{\rho_L(u)}{t_\varepsilon}\Big)\Big). \tag{4.4}$$

Since the inverse ψ^{-1} of ψ is strictly increasing on $[0,\infty)$, we have that $s \to 1^-$ as $\varepsilon \to 0^+$. Thus

$$\frac{t_\varepsilon - \rho_K(u)}{t_\varepsilon} = 1 - \frac{\rho_K(u)}{t_\varepsilon} = 1 - s. \tag{4.5}$$

Combining (4.3), (4.5) and (4.4), we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{\rho_{K\widetilde{+}_\psi \varepsilon\cdot L}(u) - \rho_K(u)}{\varepsilon} &= \lim_{\varepsilon\to 0^+} \frac{t_\varepsilon}{\varepsilon}\cdot\frac{t_\varepsilon - \rho_K(u)}{t_\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} t_\varepsilon\cdot\psi\Big(\frac{\rho_L(u)}{t_\varepsilon}\Big)\cdot\frac{\frac{t_\varepsilon-\rho_K(u)}{t_\varepsilon}}{\psi(1) - (\psi(1) - \varepsilon\psi(\frac{\rho_L(u)}{t_\varepsilon}))} \\
&= \rho_K(u)\cdot\psi\Big(\frac{\rho_L(u)}{\rho_K(u)}\Big)\cdot\lim_{s\to 1^-}\frac{1-s}{\psi(1)-\psi(s)} \\
&= \frac{\rho_K(u)}{\psi'_-(1)}\psi\Big(\frac{\rho_L(u)}{\rho_K(u)}\Big).
\end{aligned} \tag{4.6}$$

Then the pointwise limit (4.2) has been proved.

Moreover, the convergence is uniform for any $u \in S^{n-1}$. Indeed, by (4.4) and (4.6), it suffices to recall that by Lemma 3.6,

$$\lim_{\varepsilon\to 0^+} \rho_{K\widetilde{+}_\psi \varepsilon\cdot L}(u) = \rho_K(u), \tag{4.7}$$

uniformly for $u \in S^{n-1}$. $\square$

We are ready to derive the variational formula of volume for the Orlicz radial sum.

THEOREM 4.1. *Let* $K,L\in\mathscr{S}_0^n$, $0\leqslant i<n-1$ *and* $\psi\in\mathscr{V}^+$. *Then*

$$\lim_{\varepsilon\to0^+}\frac{\widetilde{W}_i(K\widetilde{+}_\psi\varepsilon\cdot L)-\widetilde{W}_i(K)}{\varepsilon}=\frac{n-i}{n\psi'_-(1)}\int_{S^{n-1}}\psi\Big(\frac{\rho_L(u)}{\rho_K(u)}\Big)\rho_K^{n-i}(u)dS(u).$$

Proof. Suppose $\varepsilon>0$, $K,L\in\mathscr{S}_0^n$, and $u\in S^{n-1}$. By Lemma 4.1 and (4.3), it follows that

$$\begin{aligned}\lim_{\varepsilon\to0^+}\frac{\rho^{n-i}_{K\widetilde{+}_\psi\varepsilon\cdot L}(u)-\rho_K^{n-i}(u)}{\varepsilon}&=(n-i)\rho^{n-i-1}_{K\widetilde{+}_\psi\varepsilon\cdot L}(u)|_{\varepsilon=0}\cdot\lim_{\varepsilon\to0^+}\frac{\rho_{K\widetilde{+}_\psi\varepsilon\cdot L}(u)-\rho_K(u)}{\varepsilon}\\&=\frac{(n-i)\rho_K^{n-i}(u)}{\psi'_-(1)}\psi\Big(\frac{\rho_L(u)}{\rho_K(u)}\Big),\end{aligned}$$

uniformly on S^{n-1}.

Hence

$$\begin{aligned}\lim_{\varepsilon\to0^+}\frac{\widetilde{W}_i(K\widetilde{+}_\psi\varepsilon\cdot L)-\widetilde{W}_i(K)}{\varepsilon}&=\lim_{\varepsilon\to0^+}\Big(\frac{1}{n}\int_{S^{n-1}}\frac{\rho^{n-i}_{K\widetilde{+}_\psi\varepsilon\cdot L}(u)-\rho_K^{n-i}(u)}{\varepsilon}dS(u)\Big)\\&=\frac{1}{n}\int_{S^{n-1}}\lim_{\varepsilon\to0^+}\frac{\rho^{n-i}_{K\widetilde{+}_\psi\varepsilon\cdot L}(u)-\rho_K^{n-i}(u)}{\varepsilon}dS(u)\\&=\frac{n-i}{n\psi'_-(1)}\int_{S^{n-1}}\psi\Big(\frac{\rho_L(u)}{\rho_K(u)}\Big)\rho_K^{n-i}(u)dS(u).\end{aligned}$$

We complete the proof of Theorem 4.1. □

From the definition (4.1) and the variational formula of volume in Theorem 4.1, we have

$$\frac{n-i}{\psi'_-(1)}\widetilde{W}_{\psi,i}(K,L)=\lim_{\varepsilon\to0^+}\frac{\widetilde{W}_i(K\widetilde{+}_\psi\varepsilon\cdot L)-\widetilde{W}_i(K)}{\varepsilon}.\tag{4.8}$$

An immediate consequence of Lemma 3.3 and (4.8) is the invariance of the dual Orlicz mixed quermassintegral under simultaneous orthogonal transforms.

COROLLARY 4.1. *Let* $K,L\in\mathscr{S}_0^n$, $0\leqslant i<n-1$ *and* $\psi\in\mathscr{V}^+$. *Then for* $A\in O(n)$,

$$\widetilde{W}_{\psi,i}(AK,AL)=\widetilde{W}_{\psi,i}(K,L).$$

Proof. From Lemma 3.3 and (4.8), we have, for $A \in O(n)$,

$$\begin{aligned}\widetilde{W}_{\psi,i}(AK,AL) &= \frac{\psi'_-(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(AK\widetilde{+}_\psi\varepsilon\cdot AL)-\widetilde{W}_i(AK)}{\varepsilon}\\ &= \frac{\psi'_-(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(A(K\widetilde{+}_\psi\varepsilon\cdot L))-\widetilde{W}_i(K)}{\varepsilon}\\ &= \frac{\psi'_-(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widetilde{+}_\psi\varepsilon\cdot L)-\widetilde{W}_i(K)}{\varepsilon}\\ &= \widetilde{W}_{\psi,i}(K,L). \quad \square\end{aligned}$$

5. Dual Orlicz-Brunn-Minkowski inequality for concave functions

For $K \in \mathscr{S}_0^n$, since

$$\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)dS(u) = \widetilde{W}_i(K), \tag{5.1}$$

the measure $\dfrac{\rho_K^{n-i}(\cdot)dS(\cdot)}{n\widetilde{W}_i(K)}$ is a probability measure on S^{n-1}.

THEOREM 5.1. *Let $K,L \in \mathscr{S}_0^n$, $0 \leqslant i < n-1$ and $\psi \in \mathscr{V}^+$. Then*

$$\widetilde{W}_{\psi,i}(K,L) \leqslant \widetilde{W}_i(K)\psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big), \tag{5.2}$$

with equality if and only if K and L are dilates.

Proof. By (4.1), (2.7), (2.5) and the fact that ψ is concave and increasing on $[0,\infty)$, we obtain

$$\begin{aligned}\frac{\widetilde{W}_{\psi,i}(K,L)}{\widetilde{W}_i(K)} &= \frac{1}{n\widetilde{W}_i(K)}\int_{S^{n-1}}\psi\Big(\frac{\rho_L(u)}{\rho_K(u)}\Big)\rho_K^{n-i}(u)dS(u)\\ &\leqslant \psi\Big(\frac{1}{n\widetilde{W}_i(K)}\int_{S^{n-1}}\frac{\rho_L(u)}{\rho_K(u)}\rho_K^{n-i}(u)dS(u)\Big)\\ &= \psi\Big(\frac{\widetilde{W}_i(K,L)}{\widetilde{W}_i(K)}\Big)\\ &\leqslant \psi\Big(\frac{\widetilde{W}_i(K)^{\frac{n-i-1}{n-i}}\widetilde{W}_i(L)^{\frac{1}{n-i}}}{\widetilde{W}_i(K)}\Big)\\ &= \psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big).\end{aligned}$$

This gives the desired inequality. Since ψ is strictly increasing, from the equality conditions of the dual Minkowski inequality (2.5), we have that the equality in (5.2) holds if and only if K and L are dilates.

Conversely, when $L = \lambda K$, by (4.1), we have

$$\widetilde{W}_{\psi,i}(K,L) = \widetilde{W}_i(K)\psi(\lambda) = \widetilde{W}_i(K)\psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big). \quad \square$$

The following uniqueness is a direct consequence of Theorem 5.1.

COROLLARY 5.1. *Suppose* $\psi \in \mathscr{V}^+$, $0 \leqslant i < n-1$ *and* $\mathscr{M} \subset \mathscr{S}_0^n$ *such that* $K, L \in \mathscr{M}$. *If*

$$\widetilde{W}_{\psi,i}(M,K) = \widetilde{W}_{\psi,i}(M,L), \ \textit{for all } M \in \mathscr{M}, \tag{5.3}$$

or

$$\frac{\widetilde{W}_{\psi,i}(K,M)}{\widetilde{W}_i(K)} = \frac{\widetilde{W}_{\psi,i}(L,M)}{\widetilde{W}_i(L)}, \ \textit{for all } M \in \mathscr{M}, \tag{5.4}$$

then $K = L$.

Proof. Suppose (5.3) holds. If we take K for M, then from (4.1), we obtain

$$\psi(1)\widetilde{W}_i(K) = \widetilde{W}_{\psi,i}(K,K) = \widetilde{W}_{\psi,i}(K,L).$$

Hence, from the dual Orlicz-Minkowski inequality (5.2), we have

$$\psi(1) \leqslant \psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n}}\Big),$$

with equality if and only if K and L are dilates. Since ψ is strictly increasing on $[0,\infty)$, we have

$$\widetilde{W}_i(L) \geqslant \widetilde{W}_i(K),$$

with equality if and only if K and L are dilates. If we take L for M, we similarly have $\widetilde{W}_i(L) \leqslant \widetilde{W}_i(K)$. Hence, $\widetilde{W}_i(K) = \widetilde{W}_i(L)$ and from the equality conditions we can conclude that K and L are dilates. However, since they have the same volume they must be equal.

Next, suppose (5.4) holds. If we take K for M, then from (4.1), we obtain

$$\psi(1) = \frac{\widetilde{W}_{\psi,i}(K,K)}{\widetilde{W}_i(K)} = \frac{\widetilde{W}_{\psi,i}(L,K)}{\widetilde{W}_i(L)}.$$

Then, from the dual Orlicz-Minkowski inequality (5.2), we have

$$\psi(1) \leqslant \psi\Big(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}}\Big),$$

with equality if and only if K and L are dilates. Since ψ is strictly increasing on $[0,\infty)$, we have

$$\widetilde{W}_i(K) \geqslant \widetilde{W}_i(L),$$

with equality if and only if K and L are dilates. If we take L for M, we similarly have $\widetilde{W}_i(K) \leqslant \widetilde{W}_i(L)$. Hence, $\widetilde{W}_i(K) = \widetilde{W}_i(L)$ and from the equality conditions we can conclude that K and L are dilates. However, since they have the same volume they must be equal. □

We define the dual Orlicz mixed surface area $\widetilde{S}_{\psi,i}(K)$ of $K \in \mathscr{S}_0^n$ is defined by

$$\widetilde{S}_{\psi,i}(K) = \lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(K \widetilde{+}_\psi \varepsilon \cdot B) - \widetilde{W}_i(K)}{\varepsilon}. \tag{5.5}$$

An immediate consequence of Theorem 5.1 is the following dual Orlicz isoperimetric inequality.

THEOREM 5.2. *Let $K \in \mathscr{S}_0^n$, $0 \leqslant i < n-1$ and $\psi \in \mathscr{V}^+$. If $\widetilde{W}_i(K) = \widetilde{W}_i(B)$, then*

$$\widetilde{S}_{\psi,i}(K) \leqslant \widetilde{S}_{\psi,i}(B),$$

with equality if and only if K is a ball (centered at the origin).

Proof. By (5.5), (4.8), (5.2) and $\widetilde{W}_i(K) = \widetilde{W}_i(B)$, it follows that

$$\begin{aligned}\widetilde{S}_{\psi,i}(K) &= \frac{n-i}{\psi'_-(1)} \widetilde{W}_{\psi,i}(K,B) \leqslant \frac{n-i}{\psi'_-(1)} \widetilde{W}_i(K)\psi\Big(\Big(\frac{\widetilde{W}_i(B)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big)\\ &= \frac{n-i}{\psi'_-(1)} \psi(1)\widetilde{W}_i(B) = \widetilde{S}_{\psi,i}(B),\end{aligned}$$

and equality holds if and only if K is a ball (centered at the origin). □

Taking $i=0$ in Theorem 5.2, one can obtain

COROLLARY 5.2. *Let $K \in \mathscr{S}_0^n$ and $\psi \in \mathscr{V}^+$. If $V(K) = V(B)$, then*

$$\widetilde{S}_\psi(K) \leqslant \widetilde{S}_\psi(B),$$

with equality if and only if K is a ball (centered at the origin).

From the dual Orlicz-Minkowski inequality, we will prove the following dual Orlicz-Brunn-Minkowski inequality which is more general than Theorem 1.1.

THEOREM 5.3. *Let $K, L \in \mathscr{S}_0^n$, $0 \leqslant i < n-1$ and $a, b > 0$. If $\psi \in \mathscr{V}^+$, then*

$$\psi(1) \leqslant a\psi\Big(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(a\cdot K \widetilde{+}_\psi b\cdot L)}\Big)^{\frac{1}{n-i}}\Big) + b\psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(a\cdot K \widetilde{+}_\psi b\cdot L)}\Big)^{\frac{1}{n-i}}\Big), \tag{5.6}$$

with equality if and only if K and L are dilates.

Proof. Let $K_\psi = a\cdot K\widetilde{+}_\psi b\cdot L$. From (2.4), Lemma 3.1 and (5.2), it follows that

$$\begin{aligned}
\psi(1) &= \frac{1}{n\widetilde{W}_i(K_\psi)}\int_{S^{n-1}}\psi(1)\rho_{K_\psi}^{n-i}(u)dS(u)\\
&= \frac{1}{n\widetilde{W}_i(K_\psi)}\int_{S^{n-1}}\left[a\psi\Big(\frac{\rho_K(u)}{\rho_{K_\psi}(u)}\Big)+b\psi\Big(\frac{\rho_L(u)}{\rho_{K_\psi}(u)}\Big)\right]\rho_{K_\psi}^{n-i}(u)dS(u)\\
&= \frac{a}{n\widetilde{W}_i(K_\psi)}\int_{S^{n-1}}\psi\Big(\frac{\rho_K(u)}{\rho_{K_\psi}(u)}\Big)\rho_{K_\psi}^{n-i}(u)dS(u)\\
&\quad+\frac{b}{n\widetilde{W}_i(K_\psi)}\int_{S^{n-1}}\psi\Big(\frac{\rho_L(u)}{\rho_{K_\psi}(u)}\Big)\rho_{K_\psi}^{n-i}(u)dS(u)\\
&= \frac{a}{\widetilde{W}_i(K_\psi)}\widetilde{W}_{\psi,i}(K_\psi,K)+\frac{b}{\widetilde{W}_i(K_\psi)}\widetilde{W}_{\psi,i}(K_\psi,L)\\
&\leqslant a\psi\Big(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\psi)}\Big)^{\frac{1}{n-i}}\Big)+b\psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K_\psi)}\Big)^{\frac{1}{n-i}}\Big)
\end{aligned}$$

From the equality conditions of the dual Orlicz-Minkowski inequality (5.2), we have that equality in (5.6) holds if and only if K and L are dilates. □

REMARK 5.1. The case $i=0$ of Theorem 5.3 was established by Gardner, Hug, Weil and Ye [3].

COROLLARY 5.3. *Let $K,L\in\mathscr{S}_0^n$, $0\leqslant i<n-1$ and $0<\lambda<1$. If $\psi\in\mathscr{V}^+$ and $\widetilde{W}_i(K)=\widetilde{W}_i(L)$, then*

$$\widetilde{W}_i((1-\lambda)\cdot K\widetilde{+}_\psi\lambda\cdot L)\leqslant\widetilde{W}_i(K),$$

with equality if and only if $K=L$.

Proof. By Lemma 3.8, (2.6) and $\widetilde{W}_i(K)=\widetilde{W}_i(L)$, we have

$$\begin{aligned}
\widetilde{W}_i((1-\lambda)\cdot K\widetilde{+}_\psi\lambda\cdot L)^{\frac{1}{n-i}} &\leqslant \widetilde{W}_i((1-\lambda)\cdot K\widetilde{+}\lambda\cdot L)^{\frac{1}{n-i}}\\
&\leqslant (1-\lambda)\widetilde{W}_i(K)^{\frac{1}{n-i}}+\lambda\widetilde{W}_i(L)^{\frac{1}{n-i}}\\
&= \widetilde{W}_i(K)^{\frac{1}{n-i}}.
\end{aligned}$$

The equality condition can be obtained from equality condition of the dual Brunn-Minkowski inequality (2.6). □

Indeed, we also can prove the dual Orlicz-Minkowski inequality by the dual Orlicz-Brunn-Minkowski inequality.

Proof. For $\varepsilon \geqslant 0$, let $K_\varepsilon = K\widetilde{+}_\psi \varepsilon \cdot L$. From Lemma 3.6, we have

$$K_\varepsilon \to K, \text{ as } \varepsilon \to 0^+. \tag{5.7}$$

By the dual Orlicz-Brunn-Minkowski inequality, the following function

$$G(\varepsilon) = \psi\Big(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{\frac{1}{n-i}}\Big) + \varepsilon\psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{\frac{1}{n}}\Big) - \psi(1) \tag{5.8}$$

is non-negative. Obviously, $G(0) = 0$. Thus

$$\lim_{\varepsilon\to 0^+} \frac{G(\varepsilon) - G(0)}{\varepsilon} \geqslant 0. \tag{5.9}$$

On the other hand, by (5.8) and (5.7), we have

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{G(\varepsilon)-G(0)}{\varepsilon} &= \lim_{\varepsilon\to 0^+} \frac{\psi((\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)})^{\frac{1}{n-i}}) + \varepsilon\psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{\frac{1}{n-i}}\Big) - \psi(1)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{\psi((\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)})^{\frac{1}{n-i}}) - \psi(1)}{\varepsilon} + \psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big) \\
&= \lim_{\varepsilon\to 0^+} \frac{\psi((\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)})^{\frac{1}{n-i}}) - \psi(1)}{(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)})^{\frac{1}{n-i}} - 1} \cdot \lim_{\varepsilon\to 0^+} \frac{(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)})^{\frac{1}{n-i}} - 1}{\varepsilon} \\
&\quad + \psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big).
\end{aligned} \tag{5.10}$$

Let $s = \Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{\frac{1}{n-i}}$ and note that $s \to 1^-$ as $\varepsilon \to 0^+$. Consequently,

$$\lim_{\varepsilon\to 0^+} \frac{\psi((\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)})^{\frac{1}{n-i}}) - \psi(1)}{(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)})^{\frac{1}{n-i}} - 1} = \lim_{s\to 1^-} \frac{\psi(s) - \psi(1)}{s-1} = \psi'_-(1). \tag{5.11}$$

By (5.7) and (4.8), we have

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{((\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)})^{\frac{1}{n-i}} - 1)}{\varepsilon} &= -\lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(K_\varepsilon)^{\frac{1}{n-i}} - \widetilde{W}_i(K)^{\frac{1}{n-i}}}{\varepsilon} \cdot \lim_{\varepsilon\to 0^+} \widetilde{W}_i(K_\varepsilon)^{-\frac{1}{n-i}} \\
&= -\frac{1}{n}\widetilde{W}_i(K)^{\frac{1}{n-i}-1} \cdot \lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(K_\varepsilon) - \widetilde{W}_i(K)}{\varepsilon} \cdot \widetilde{W}_i(K)^{-\frac{1}{n-i}} \\
&= -\frac{1}{\psi'_-(1)} \frac{\widetilde{W}_{\psi,i}(K,L)}{\widetilde{W}_i(K)}.
\end{aligned} \tag{5.12}$$

From (5.10), (5.11) and (5.12), it follows that

$$\lim_{\varepsilon\to 0^+}\frac{G(\varepsilon)-G(0)}{\varepsilon}=-\frac{\widetilde{W}_{\psi,i}(K,L)}{\widetilde{W}_i(K)}+\psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big) \tag{5.13}$$

Combing (5.9) and (5.13), we have

$$-\frac{\widetilde{W}_{\psi,i}(K,L)}{\widetilde{W}_i(K)}+\psi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big)\geqslant 0. \tag{5.14}$$

Therefore, the equality holds in (5.14) if and only if $G(\varepsilon)=G(0)=0$, this implies that K and L are dilates. □

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