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ON APPROXIMATION PROPERTIES OF SOME CLASS POSITIVE LINEAR OPERATORS IN q–ANALYSIS

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Abstract. This paper is concerned with some sequences of the positive linear operators based on q-Calculus. The approximation properties and the rate of convergence of these sequences of q-discrete type is established by means of the modulus of continuity. Moreover we give Voronovskaya-type theorems. Finally we present some applications such as q-Bernstein operators and q-Meyer-König and Zeller operators.

1. Introduction

First, let us provide some background information regarding what we know about q-calculus formulae, the study of which was initiated by Euler in the eighteenth century. Following this, many remarkable results in the field were obtained in the nineteenth century. In 1908, F. H. Jackson [9] introduced q-functions. He was also the first to develop q-calculus in a systematic way. Below, we present the outlines of q-integers, q-factorials, q-binomial coefficients, and q-differentiations. The definitions used in this study are based on terminology and notations as is seen in [3], [21] and [18].

The required theorems and definitions in q-Calculus are as outlined below, where $q > 0$. For $n \in \mathbb{N}$, the q-analogue of the integer n, called q-integer, is defined by

$$[n]_q := \frac{q^n - 1}{q - 1}, \qquad q \neq 1; \qquad [n]_1 := n.$$

Also $[0]_q := 0$. Similarly, the q-analogue of the factorial of n is defined by

$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q, \qquad n = 1, 2, 3, \cdots; \qquad [0]_q! := 1$$

Now, let us obtain the q-analogue of the Gauss binomial formula. The q-analogues of $(a+b)^n$ are given by

$$(a \oplus b)_q^n := \prod_{s=0}^{n-1} (a + q^s b); \qquad (a \oplus b)_q^0 := 1.$$

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By simple calculations, it follows that

$$(a \oplus b)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} b^k a^{n-k},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad 0 \leqslant k \leqslant n$$

is the q-binomial formula. All the concepts defined above, become their classical cases if q tends to 1.

The q-derivative of a function f, denoted by $D_q f$ is given by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \qquad x \neq 0$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. If f is differentiable at a point $x \neq 0$, we have

$$\lim_{q \to 1} (D_q f)(x) = f'(x).$$

Let us define the q-partial derivatives of a function $f(x,y)$ of two variables. The q-partial derivative of $f(x,y)$ with respect to x is defined by

$$\frac{\partial_q f(x,y)}{\partial_q x} = \frac{f(qx,y) - f(x,y)}{(q-1)x}, \qquad x \neq 0.$$

Likewise, the q-partial derivative of $f(x;y)$ with respect to y can be defined.

The first q-analogue of positive linear operators (actually, Bernstein polynomials) was adapted by A. Lupas [6] in 1987. In 1997, G. M. Phillips [11] proposed another q-version of Bernstein polynomials. For interesting properties of q-Bernstein polynomials and their distinct variants refer to [24], [5], [22] and, [12]. In addition to the information from the studies we have just referenced, there are a great number of mathematicians who constructed and investigated the q-analogues of positive linear operators of the discrete type, which are very important to approximation theory: Some of these are q-Baskakov operators [4], [25], q-Meyer-König-Zeller operators [19], [23] and q-Bleimann-Butzer-Hahn operators [1], [15], and [14].

The first general positive linear operators of the discrete type were constructed by V. A. Baskakov [20] in 1957. In 1966, F. Schurer [10] investigated some approximation properties of these operators. Our first encounter with a study on q-analogues of the operators based on generating functions was in C. Radu's paper [7] in 2009 and then in [8].

Recently the statistical approximation properties have also been investigated for q-analogue of several operators. In [29] the authors constructed a new family of operators with the help of q-analogue of Chan-Chyan-Srivastava polynomials, and they studied the statistical approximation properties via A-statistical convergence. In [31], a Korovkin type theorem based upon the statistical summability involving the idea of the generalized de la Vallee Poussin mean of positive linear operators in the space $C_{2\pi}(\mathbb{R})$

have been investigated. In [27] the authors extended the notions of statistical summability and statistical convergence by the help of some weighted regular methods and they established some important approximation theorems related to statistically weighted $\mathscr{B}$-summability which effectively extend and improve all of the existing results depending on the choice of sequence of infinite matrices. In [26], the authors introduced a new Λ^2-weighted statistical convergence. Based upon this definition, they proved some Korovkin type theorems. In the paper [30], a Kantorovich type generalization of q-Bernstein-Stancu operators was introduced and studied their convergence properties. It was introduced a family of q-Szász-Mirakjan-Kantorovich type positive linear operators that are generated by Dunkl's generalization of the exponential function [28].

In this paper, we present a few approximation theorems concerning with generating functions for constructing q-analogues of some discrete type positive linear operators (e.g., q-Lupas, q-Bernstein, q-Meyer-König and Zeller and q-Bleimann-Butzer-Hahn operators etc.). Finally, through the use of specific generating functions, we are able to provide some relevant exemplary applications of general operators.

2. Construction of operators

For $f \in C(I)$ $(I = [0,1]$ or $[0,\infty))$, $q > 0$ and each positive integer n, the authors introduced the following operators on $C(I)$, in [8]:

$$L_{n,q}(f;x) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \left. \frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k} \right|_{u=0} f\left(\frac{[k]_q}{\alpha_{n,k,q}} \right), \tag{1}$$

where $\alpha_{n,k,q}$ are positive numbers and $\{\varphi_{n,q}(x,u)\}$ generating real functions defined on $I \times [0,\infty)$, have the following conditions:

(i) $\varphi_{n,q}(x,0) \neq 0$ and $\varphi_{n,q}(x,1) = 1$ for all $n \in \mathbb{N}$ and $x \in I$.

(ii) $\left. \frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k} \right|_{u=0}$ exist and are continuous functions of x for all $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

(iii) For all $k \in \mathbb{N}_0$, $x,u \geqslant 0$,

$$\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k} \geqslant 0, \qquad n \in \mathbb{N}.$$

It is clear that the operators are linear and positive in view of (iii) on the space of bounded functions on I, $B(I)$.

The test functions $e_{r,i}$ are given by

$$e_{r,i}(t) = \left(\frac{t}{1 + (1-i)t} \right)^r, \quad r \in \mathbb{N}_0, \ \ i = 0,1,2.$$

The functions of $e_{r,0}$ are used as test functions for q-Butzer-Bleimann-Hahn operators, the functions $e_{r,1}$ are used as test functions for q-Bernstein, q-Szasz-Mirakyan, q-Lupas and q-Baskakov operators and the functions of $e_{r,2}$ for q-Meyer-König and Zeller operators.

In continuation of the relation for the numbers $\alpha_{n,k,q}$ indicated in (1), we assume the following:

$$e_{r,i}\left(\frac{[k]_q}{\alpha_{n,k,q}}\right)=\frac{[k]_q^r}{\alpha_{n,q}^r},\qquad r\in\mathbb{N}_0,$$

where $\alpha_{n,q}$ are positive numbers independent of k.

THEOREM 1. ([8]) *If the sequence $\left\{\varphi_{n,q}(x,u)\right\}$ satisfies the conditions (i)–(iii) for all $r\in\mathbb{N}_0$ and $n\in\mathbb{N}$, then the following relation is true:*

$$L_{n,q}(e_{r,i};x)=\frac{1}{\alpha_{n,q}^r}\sum_{m=0}^{r}q^{\binom{m}{2}}\mathbb{S}_q(r,m)\left.\frac{\partial_q^m\varphi_{n,q}(x,u)}{\partial_q u^m}\right|_{u=1},\tag{2}$$

$\mathbb{S}_q(r,m)$ appeared in are the q-Stirling numbers of the second kind for detail, see [13].

COROLLARY 1. *By virtue of equality (2), we have*

$$\begin{aligned}
L_{n,q}(e_{0,i};x)&=1;\\
L_{n,q}(e_{1,i};x)&=\frac{1}{\alpha_{n,q}}\left.\frac{\partial_q\varphi_{n,q}(x,u)}{\partial_q u}\right|_{u=1};\\
L_{n,q}(e_{2,i};x)&=\frac{1}{\alpha_{n,q}^2}\left.\left\{q\frac{\partial_q^2\varphi_{n,q}(x,u)}{\partial_q u^2}+\frac{\partial_q\varphi_{n,q}(x,u)}{\partial_q u}\right\}\right|_{u=1}.
\end{aligned}$$

3. Korovkin type theorem

In this section we will give the theorems on uniform convergence of operators (1). Let ω be a modulus of continuity, that is

(1) ω is a non-negative increasing function on $[0,\infty)$.
(2) $\omega(\delta_1+\delta_2)\leqslant\omega(\delta_1)+\omega(\delta_2)$.
(3) $\lim_{\delta\to 0}\omega(\delta)=0$.

Let $I_A=[0,A]$, $A>0$, and $I_\infty=[0,\infty)$. Let us denote by $H_\omega^i(I_A)$ the space of continuous functions f on I_A and satisfying the following condition:

$$|f(t)-f(x)|\leqslant\omega\left(|e_{1,i}(t)-e_{1,i}(x)|\right)\tag{3}$$

for all $x,t\in I_A$.

Let (q_n) be a sequence of real numbers in $(0,1)$ such that $1-q_n=o(\frac{1}{n})$. In the sequel for $j\in\mathbb{N}_0$, $n\in\mathbb{N}$, we use notations:

$$\mu_{n,i,j}(x,q):=L_{n,q}((e_{1,i}(\cdot)-e_{1,i}(x))^j;x),\qquad i=0,1,2.$$

$$\mu_{n,i,j}^*(x,q):=L_{n,q}((e_{1,i}(\cdot)-e_{1,i}(x))_q^j;x),\qquad i=0,1,2.$$

LEMMA 1. *If $A<1$, then*

$$\left|L_{n,q}\left(e_{r,i};x\right)-e_{r,i}(x)\right| \leqslant re_{r-1,i}(A)\sqrt{\mu_{n,i,2}(x,q)}, \qquad r\in\mathbb{N}_0,\ \ i=0,1,2$$

for all $t,x\in I_A$, where $e_{-1,i}(A):=0$. If $A>0$ is arbitrary, this inequality holds only for $i=0,1$.

Proof. For the case $r=0$ the assertion is obvious. We assume that $r\in\mathbb{N}$. For $t,x\in I_A$ with $A<1$,

$$\begin{aligned}
|e_{r,i}(t)-e_{r,i}(x)| &= \left|\left(\frac{t}{1+(1-i)t}\right)^r-\left(\frac{x}{1+(1-i)x}\right)^r\right| \\
&= \left|\frac{t}{1+(1-i)t}-\frac{x}{1+(1-i)x}\right| \\
&\quad\cdot\left|\left(\frac{t}{1+(1-i)t}\right)^{r-1}+\cdots+\left(\frac{x}{1+(1-i)x}\right)^{r-1}\right| \\
&\leqslant |e_{1,i}(t)-e_{1,i}(x)| \\
&\quad\cdot\left|\left(\frac{A}{1+(1-i)A}\right)^{r-1}+\cdots+\left(\frac{A}{1+(1-i)A}\right)^{r-1}\right| \\
&= |e_{1,i}(t)-e_{1,i}(x)|\,re_{r-1,i}(A).
\end{aligned}$$

For arbitrary $A>0$, by similar arguments, we have same inequality for $i=0,1$. By monotonicity of the operators $L_{n,q}$ and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left|L_{n,q}\left(e_{r,i};x\right)-e_{r,i}(x)\right| &\leqslant re_{r-1,i}(A)L_{n,q}\left(\left|e_{1,i}-e_{1,i}(x)\right|;x\right) \\
&\leqslant re_{r-1,i}(A)\sqrt{L_{n,q}\left(\left(e_{1,i}-e_{1,i}(x)\right)^2;x\right)}.
\end{aligned}$$

for all $n\in\mathbb{N}_0$, thus we obtain

$$\left|L_{n,q}\left(e_{r,i};x\right)-e_{r,i}(x)\right| \leqslant re_{r-1,i}(A)\sqrt{\mu_{n,i,2}(x,q)}$$

what we wanted to prove. □

COROLLARY 2. *If $\lim\limits_{n\to\infty}\mu_{n,i,2}(x,q_n)=0$, then*

$$\lim_{n\to\infty}L_{n,q_n}\left(e_{r,i};x\right)=e_{r,i}(x), \qquad r=0,1,2$$

for all $x\in I_A$.

THEOREM 2. *If $\lim\limits_{n\to\infty}\mu_{n,i,2}(x,q_n)=0$, then*

$$\lim_{n\to\infty}L_{n,q_n}\left(f;x\right)=f(x), \qquad x\in I_A$$

holds for all $f\in H_\omega^i$.

Proof. For $f \in H_{\omega}^{i}$, using the property (3) in the definition of the modulus of continuity ω, we can write $|f(t)-f(x)|<\varepsilon$ for $|e_{1,i}(t)-e_{1,i}(x)|<\delta$. Since f is continuous on I_A, for some $M>0$ we have $|f(t)-f(x)|\leqslant 2M$, for all $x,t\in I_A$. If $|e_{r,i}(t)-e_{r,i}(x)|\geqslant\delta$ then $(e_{1,i}(t)-e_{1,i}(x))^2\geqslant\delta^2$. Thus for all $t,x\in I_A$, we have

$$|f(t)-f(x)|\leqslant\varepsilon+\frac{2M}{\delta^2}\left(e_{1,i}(t)-e_{1,i}(x)\right)^2.$$

Therefore, we obtain

$$\begin{aligned}\left|L_{n,q_n}(f;x)-f(x)\right| &\leqslant L_{n,q_n}\left(|f(\cdot)-f(x)|;x\right)\\ &\leqslant \varepsilon+\frac{2M}{\delta^2}L_{n,q_n}\left(\left(e_{1,i}(\cdot)-e_{1,i}(x)\right)^2;x\right)\\ &=\varepsilon+\frac{2M}{\delta^2}\mu_{n,i,2}(x;q_n).\end{aligned}$$

Thus by the condition $\lim_{n\to\infty}\mu_{n,i,2}(x,q_n)=0$, the proof is completed. □

4. Rate of convergence of $L_{n,q}$ operators

For $f\in C(I_A)$ and $\delta>0$, we define the modulus of continuity of f of with step $\delta>0$ by

$$\omega(f;\delta):=\sup_{t,x\in I_A,\ |t-x|\leqslant\delta}|f(t)-f(x)|.$$

For any $\delta>0$, we have

$$|f(t)-f(x)|\leqslant\left(1+\frac{|t-x|}{\delta}\right)\omega(f,\delta). \tag{4}$$

THEOREM 3. *Let $L_{n,q}$ be defined by (1).*

(i) For $i=0,1,2$ and for $f\in H_{\omega}^{i}(I_A)$, we have

$$\left|L_{n,q}(f;x)-f(x)\right|\leqslant\left(1+\frac{1}{\delta}\sqrt{\mu_{n,i,2}(x,q)}\right)\omega(\delta),\qquad \delta>0.$$

(ii) For $i=1,2$ and for $f\in C(I_A)$, we have

$$\left|L_{n,q}(f;x)-f(x)\right|\leqslant\left(1+\frac{1}{\delta}\sqrt{\mu_{n,i,2}(x,q)}\right)\omega(f,\delta),\qquad \delta>0.$$

Proof. Since $L_{n,q}(e_{0,i})=1$ $(i=0,1,2)$, we can write

$$\left|L_{n,q}(f;x)-f(x)\right|\leqslant L_{n,q}\left(|f(\cdot)-f(x)|;x\right) \tag{5}$$

for all $n\in\mathbb{N}$. Now using (3) in inequality (5) we obtain

$$|f(t)-f(x)|\leqslant\omega\left(|e_{1,i}(t)-e_{1,i}(x)|\right)\leqslant\left(1+\frac{|e_{1,i}(t)-e_{1,i}(x)|}{\delta}\right)\omega(\delta), \tag{6}$$

for all $\delta > 0$. Applying the Cauchy-Schwartz Inequality it follows from (6) that

$$\left|L_{n,q}(f;x) - f(x)\right| \leqslant \left(1 + \frac{1}{\delta}\sqrt{L_{n,q}\left((e_{1,i}(\cdot) - e_{1,i}(x))^2;x\right)}\right)\omega(\delta).$$

thus we obtain the statement *(i)*. For the proof *(ii)* we use the fact that the modulus of continuity of $f \in C(I_A)$ has the property (3). So that by similar arguments in the proof of *(i)* we get *(ii)*. □

As usual, a function $f \in Lip_{M,i}\alpha$, $(M > 0,\ i = 0,1,2$ and $0 < \alpha \leqslant 1)$, if the inequality

$$|f(t) - f(x)| \leqslant M|e_{1,i}(t) - e_{1,i}(x)|^{\alpha} \tag{7}$$

holds for all $t,x \in I_A$. Note that, $Lip_{M,1}\alpha$ is usual $Lip_M\alpha$.

THEOREM 4. *For all $f \in Lip_{M,i}\alpha$ and $x \in I_A$, we have*

$$|L_{n,q}(f;x) - f(x)| \leqslant M(\mu_{n,i,2}(x,q))^{\alpha/2}.$$

Proof. Applying $L_{n,q}$ to the inequality (7), we have

$$\begin{aligned}\left|L_{n,q}(f;x) - f(x)\right| &\leqslant L_{n,q}\left(|f(\cdot) - f(e_{1,i}(x))|;x\right)\\ &\leqslant ML_{n,q}\left(|e_{1,i}(\cdot) - e_{1,i}(x)|^{\alpha};x\right).\end{aligned}$$

If we consider the Hölder Inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$ we have,

$$\begin{aligned}\left|L_{n,q}(f;x) - f(x)\right| &\leqslant M\left(L_{n,q}\left((e_{1,i}(\cdot) - e_{1,i}(x))^2;x\right)\right)^{\alpha/2}\\ &\leqslant M(\mu_{n,i,2}(x,q))^{\alpha/2}\end{aligned}$$

thus

$$|L_{n,q}(f;x) - f(x)| \leqslant M(\mu_{n,i,2}(x,q))^{\alpha/2}. \quad \square$$

5. A Quantitative Voronovskaya-type theorem

In this section, we obtain a Voronovskaya type estimate of the operators (1) for the test functions $e_{r,i}(x)$.

THEOREM 5. *Let $f \in C^2(I_A)$ such that $f(e_{r,i}(x)) \in B(I_A)$ $(i = 0,1,2)$. Then, we have*

$$\left|L_{n,q_n}(f(e_{1,i}(t));x) - \sum_{j=0}^{2}\frac{\left(D_{q_n}^j f\right)(e_{1,i}(x))}{[j]_{q_n}!}\mu_{n,i,j}^*(x,q_n)\right| \leqslant F_{n,i}(x)\frac{\omega\left(\left(D_{q_n}^2 f\right),\delta\right)}{[2]_{q_n}!}$$

for all $x \in I_A$ with $e_{r,i}(x) \in I_A$, where

$$F_{n,i}(x) = \mu_{n,i,2}^*(x,q_n) + \sqrt{\mu_{n,i,2}(x,q_n)\sum_{j=0}^{2}\binom{2}{j}e_{j,i}(x)(1-q_n)^j\mu_{n,i,4-j}(x,q_n)}.$$

Proof. Let $f \in C^2(I_A)$ and $x \in I_A$ with $e_{r,i}(x) \in I_A$ be fixed. By q-Taylor's formula [17] for $s, v \in I_A$, we write

$$f(s) = f(v) + (D_{q_n} f)(v)(s-v) + \frac{\left(D_{q_n}^2 f\right)(v)}{[2]_{q_n}!}(s-v)_{q_n}^2 + g_{q_n}(s;v)$$

where

$$g_{q_n}(s;v) = \frac{\left(D_{q_n}^2 f\right)(\xi_{s,v}) - \left(D_{q_n}^2 f\right)(s)}{[2]_{q_n}!}(s-v)_{q_n}^2$$

where $\xi_{s,v}$ is situated between s and v, therefore, $|\xi_{s,v} - v| < |s-v|$. By taking $s = e_{1,i}(t)$ and $v = e_{1,i}(x)$, we have

$$\begin{aligned} f(e_{1,i}(t)) &= f(e_{1,i}(x)) + (D_{q_n} f)(e_{1,i}(x))(e_{1,i}(t) - e_{1,i}(x)) \\ &\quad + \frac{\left(D_{q_n}^2 f\right)(e_{1,i}(x))}{[2]_{q_n}!}(e_{1,i}(t) - e_{1,i}(x))_{q_n}^2 + g_{q_n}(e_{1,i}(t); e_{1,i}(x)) \end{aligned} \tag{8}$$

Applying the operators (1) to the equality (8), we get

$$\begin{aligned} L_{n,q_n}(f(e_{1,i}(t));x) &= f(e_{1,i}(x)) + (D_{q_n} f)(e_{1,i}(x)) L_{n,q_n}((e_{1,i}(t) - e_{1,i}(x));x) \\ &\quad + \frac{\left(D_{q_n}^2 f\right)(e_{1,i}(x))}{[2]_{q_n}!} L_{n,q_n}\left((e_{1,i}(t) - e_{1,i}(x))_{q_n}^2;x\right) \\ &\quad + L_{n,q_n}(g_{q_n}(e_{1,i}(t); e_{1,i}(x));x). \end{aligned}$$

Consequently, we can write

$$\left| L_{n,q_n}(f(e_{1,i}(t));x) - \sum_{j=0}^{2} \frac{\left(D_{q_n}^j f\right)(e_{1,i}(x))}{[j]_{q_n}!} \mu_{n,i,j}^*(x,q_n) \right| \leqslant \left| L_{n,q_n}(g_{q_n}(e_{1,i}(t); e_{1,i}(x);x) \right|.$$

To estimate $L_{n,q_n}\left(\left|g_{q_n}(e_{1,i}(t); e_{1,i}(x)\right|;x\right)$, using the properties of modulus of continuity, we have

$$\begin{aligned} \left| \left(D_{q_n}^2 f\right)(\xi_{e_{1,i}(t),e_{1,i}(x)}) - \left(D_{q_n}^2 f\right)(e_{1,i}(x)) \right| &\leqslant \omega\left(\left(D_{q_n}^2 f\right), \left| \xi_{e_{1,i}(t),e_{1,i}(x)} - e_{1,i}(x) \right| \right) \\ &\leqslant \omega\left(\left(D_{q_n}^2 f\right), |e_{1,i}(t) - e_{1,i}(x)| \right) \\ &\leqslant \omega\left(\left(D_{q_n}^2 f\right), \delta \right) \left(1 + \frac{|e_{1,i}(t) - e_{1,i}(x)|}{\delta} \right), \end{aligned}$$

where $\delta > 0$. Hence, we can write

$$\begin{aligned} & L_{n,q_n}\left(\left|g_{q_n}(e_{1,i}(t); e_{1,i}(x)\right|;x\right) \\ \leqslant & \frac{\omega\left(\left(D_{q_n}^2 f\right), \delta\right)}{[2]_{q_n}!} \left[\mu_{n,i,2}^*(x,q_n) + L_{n,q_n}\left(|e_{1,i}(t) - e_{1,i}(x)| \left| (e_{1,i}(t) - e_{1,i}(x))_{q_n}^2 \right|;x \right) \right] \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$L_{n,q_n}\left(|e_{1,i}(t)-e_{1,i}(x)|\left|(e_{1,i}(t)-e_{1,i}(x))_{q_n}^2\right|;x\right)$$
$$\leqslant \sqrt{L_{n,q_n}\left((e_{1,i}(t)-e_{1,i}(x))^2;x\right)}\sqrt{L_{n,q_n}\left(\left[(e_{1,i}(t)-e_{1,i}(x))_{q_n}^2\right]^2;x\right)},$$

and using the equality

$$(e_{1,i}(t)-e_{1,i}(x))_{q_n}^2=(e_{1,i}(t)-e_{1,i}(x))^2+e_{1,i}(x)(1-q_n)\left(e_{1,i}(t)-e_{1,i}(x)\right)$$

and than by simple calculation, the equality

$$\begin{aligned}\left[(e_{1,i}(t)-e_{1,i}(x))_{q_n}^2\right]^2&=(e_{1,i}(t)-e_{1,i}(x))^4+2e_{1,i}(x)(1-q_n)\left(e_{1,i}(t)-e_{1,i}(x)\right)^3\\&\quad+(1-q_n)^2e_{2,i}(x)\left(e_{1,i}(t)-e_{1,i}(x)\right)^2\\&=\sum_{p=0}^{2}\binom{2}{p}e_{p,i}(x)(1-q_n)^p\left(e_{1,i}(t)-e_{1,i}(x)\right)^{4-p}.\end{aligned}$$

is obtained. Thus, we have

$$L_{n,q_n}\left(|e_{1,i}(t)-e_{1,i}(x)|\left|(e_{1,i}(t)-e_{1,i}(x))_{q_n}^2\right|;x\right)$$
$$\leqslant\sqrt{\mu_{n,i,2}(x,q_n)\sum_{j=0}^{2}\binom{2}{j}e_{j,i}(x)(1-q_n)^j\mu_{n,i,4-j}(x,q_n)}$$

Consequently, we get

$$\left|L_{n,q_n}\left(f\left(e_{1,i}(t)\right);x\right)-\sum_{j=0}^{2}\frac{\left(D_{q_n}^j f\right)\left(e_{1,i}(x)\right)}{[j]_{q_n}!}\mu_{n,i,j}^*(x,q_n)\right|\leqslant F_{n,i}(x)\frac{\omega\left(\left(D_{q_n}^2 f\right),\delta\right)}{[2]_{q_n}!}$$

where

$$F_{n,i}(x)=\mu_{n,i,2}^*(x,q_n)+\sqrt{\mu_{n,i,2}(x,q_n)\sum_{j=0}^{2}\binom{2}{j}e_{j,i}(x)(1-q_n)^j\mu_{n,i,4-j}(x,q_n)}.\qquad\square$$

6. Examples of $L_{n,q}$ operators

Firstly, we define the q-analogue of $(a+b+c)^n$, which we will use to construct the q-Bernstein operators.

DEFINITION 1. For $a,b,c\in\mathbb{R}$, we define

$$(a\boxplus b\boxplus c)_q^n:=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(a\oplus b)_q^{n-k}c^k;\qquad(a\boxplus b\boxplus c)_q^0:=1\tag{9}$$

and

$$(a \boxminus b \boxplus c)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a \ominus b)_q^{n-k} c^k$$

where $(a \ominus b)_q^n := (a \oplus (-b))_q^n$.

EXAMPLE 1. (q-Bernstein operators) For $n \in \mathbb{N}$ and $q \in (0,1)$, we consider the function

$$\varphi_{n,q}(x,u) := (1 \boxminus x \boxplus xu)_q^n, \qquad x \in [0,1]$$

where

$$(1 \boxminus x \boxplus xu)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1 \ominus x)_q^{n-k} (xu)^k.$$

It is easily verified that this sequence has the properties (*i-iii*). The operators generated by this sequence of functions have the form

$$L_{n,q}(f;x) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k}\bigg|_{u=0} f\left(\frac{[k]_q}{\alpha_{n,k,q}}\right).$$

For $\alpha_{n,k,q} = [n]_q$, we have the well-known q-Bernstein operators $B_{n,q}$ defined by G. M. Phillips [11]: For $f \in C[0,1]$,

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) f\left(\frac{[k]_q}{[n]_q}\right).$$

From Corollary 1 we have

$$\begin{aligned} B_{n,q}(e_{0,1};x) &= 1; \\ B_{n,q}(e_{1,1};x) &= x; \\ B_{n,q}(e_{2,1};x) &= x^2 + \frac{x(1-x)}{[n]_q}; \\ \mu_{n,1,1}(x,q) &= 0; \\ \mu_{n,1,2}^*(x,q) = \mu_{n,1,2}(x,q) &= \frac{x(1-x)}{[n]_q}. \end{aligned}$$

From Theorem 3 (ii), we have

$$|B_{n,q}(f;x) - f(x)| \leqslant \frac{3}{2}\omega\left(f; \frac{1}{\sqrt{[n]_q}}\right).$$

From Theorem 4, we have

$$|B_{n,q}(f;x) - f(x)| \leqslant M\left(\frac{1}{\sqrt{[n]_q}}\right)^{\alpha/2}.$$

From Theorem 5, we have

$$\lim_{n\to\infty}[n]_{q_n}\left[B_{n,q_n}(f;x)-f(x)\right]=\frac{x(1-x)}{2}f''(x).$$

EXAMPLE 2. (q-Meyer-König and Zeller operators) Let $n\in\mathbb{N}$. If we consider $\alpha_{n,k,q}=q^{-n}[k+n]_q$ and

$$\varphi_{n,q}(x,u):=\frac{(1\ominus x)_q^{n+1}}{(1\ominus xu)_q^{n+1}},\quad x\in[0,1),\quad q\in[0,1)$$

in the operators $L_{n,q}$ as defined by (1), then $L_{n,q}$ become the q-Meyer-König and Zeller operators. $M_{n,q}$ constructed in [16] as follows: For $f\in B[0,1)$

$$M_{n,q}(f;x)=(1\ominus x)_q^{n+1}\sum_{k=0}^{\infty}\begin{bmatrix}k+n\\k\end{bmatrix}_q x^k f\left(\frac{[k]_q}{q^{-n}[k+n]_q}\right).$$

From Corollary 1 we have

$$\begin{aligned}
M_{n,q}(e_{0,2};x)&=1;\\
M_{n,q}(e_{1,2};x)&=\frac{[n+1]_q}{[n]_q}\frac{xq^n}{(1-xq^{n+1})};\\
M_{n,q}(e_{2,2};x)&=\frac{[n+1]_q[n+2]_q}{[n]_q^2}\frac{x^2q^{2n+1}}{(1-xq^{n+1})(1-xq^{n+2})}+\frac{[n+1]_q}{[n]_q^2}\frac{xq^{2n}}{(1-xq^{n+1})}.
\end{aligned}$$

From Theorem 3 (ii), we have

$$\left|M_{n,q}(f;x)-f(x)\right|\leqslant 2\omega(f,\delta).$$

where

$$\delta=\sup_{x\in[0,1)}\sqrt{\mu_{n,2,2}(x,q)}$$

From Theorem 4, we have

$$|M_{n,q}(f;x)-f(x)|\leqslant M(\mu_{n,2,2}(x,q))^{\alpha/2}.$$

From Theorem 5, we have

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}\left[M_{n,q_n}\left(f\left(\frac{t}{1-t}\right);x\right)-f\left(\frac{x}{1-x}\right)\right]\\
&=\frac{x}{1-x}f'\left(\frac{x}{1-x}\right)+\frac{x}{2(1-x)^2}f''\left(\frac{x}{1-x}\right).
\end{aligned}$$

REMARK 1. By suitable choice of the generating function, one may construct the q-Durrmeyer type and Kantorovich type of discrete operators which are defined, for example, in [32], [33], [34], [35] and may investigate their general approximation properties.

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