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BERRY–ESSEEN TYPE INEQUALITY FOR A POISSON RANDOMLY INDEXED BRANCHING PROCESS VIA STEIN'S METHOD

ZHENLONG GAO

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Abstract. A Berry-Esseen type inequality is proved via Stein's method for the logarithm of a Poisson randomly indexed branching process $\{Z_{N_t}\}$, where $\{Z_n\}$ is a supercritical Galton–Watson process and $\{N_t\}$ is a Poisson process which is independent of $\{Z_n\}$.

1. Introduction

Consider a classical supercritical Galton-Watson process $\{Z_n, n \geqslant 0\}$ with offspring distribution $\{p_i, i \geqslant 0\}$ and an independent Poisson process with parameter $\lambda > 0$. In this paper, we deal with the continuous process $\{Z_{N_t}, t \geqslant 0\}$ which is called a Poisson randomly indexed branching process (PRIBP). For a PRIBP, we distinguish between the Shröder case and the Böttcher case depending on whether $p_0 + p_1 > 0$ or $p_0 + p_1 = 0$.

The model of PRIBP was introduced by [4] to study the evolution of stock prices and its statistical investigation was done in [3]. Recently, PRIBP has been brought to attention in the following three directions.

In applied direction, a formula for the fair price of an European call option was derived in [12]. Later on, [16] obtained a formula for the fair price of an up-and-out call option.

On more theoretical side, [14] and [15] considered a critical branching process subordinated by a general renewal process. They investigated the probability of non-extinction, the asymptotic behavior of the moments, and also limiting distributions under normalization. Results on subcritical case were done in [13].

For statistical inference, on the one hand, [17] indicated that $R_t := Z_{N_t+1} Z_{N_t}^{-1}$ is a reasonable estimator of the offspring mean m. They consider the supercritical PRIBP (Shröder case) and obtained the exponential rate of decay for the large deviation probability $P(|R_t - m| \geqslant x)$ under the condition that the offspring distribution has finite exponential moments. On the other hand, [8] showed that $(\lambda t)^{-1} \log Y_t$ is an estimator of $\log m$ and derived the consistency, asymptotic normality, large deviation and moderate deviation of the estimator when the PRIBP belongs to the Böttcher case. The large

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deviations in the Shröder case were given in [5], where the rate function $I(x)$ is deferent from the Böttcher case for small positive x. Similar results for branching process indexed by a renewal process were done in [6] and [7].

The asymptotic normality for a PRIBP was proved in [8]. Precisely, for any $t \geqslant 0$, define $Y_t = Z_{N_t}$. Then

$$\lim_{t\to\infty} P\left(\frac{\log Y_t - \lambda t \log m}{\sqrt{\lambda t}\log m} \leqslant x\right) = \Phi(x), \quad x \in R,$$

where $\Phi(x)$ is the distribution function of standard normal law. In this paper, we consider a Berry-Esseen type inequality for this asymptotic normality based on Stein's method.

Throughout the paper, we assume the following condition:

A1: $p_0 = 0, \;\; m \in (1,\infty), \;\; \sigma^2 = E(Z_1 - m)^2 \in (0,\infty)$.

THEOREM 1. Under the condition **A1**, we have

$$\sup_{x\in R}\left| P\left(\frac{\log Y_t - \lambda t \log m}{\sqrt{\lambda t}\log m} \leqslant x\right) - \Phi(x)\right| \leqslant \frac{C}{\sqrt{t}}, \tag{1}$$

where C is a positive constant.

The rest of the paper is organized as follows. In Section 2, we prove a Berry-Esseen inequality for the normalized Poisson process. Section 3 is devoted to the proof of the main result of the paper. Basic facts on Stein's method are given in the Appendix.

In the rest of the paper, we denote by C an absolute and positive constant which may differ from line to line.

2. Berry-Esseen type inequality for a Poisson process

In this section, we establish a Berry-Esseen type inequality for the normalized Poisson process with parameter $\lambda > 0$ based on Stein's method. The idea comes from [10].

LEMMA 1. *Let $\{N_t\}_{t\geqslant 0}$ be a Poisson process with parameter $\lambda > 0$, we have*

$$\sup_{x\in R}\left| P\left(\frac{N_t - \lambda t}{\sqrt{\lambda t}} \leqslant x\right) - \Phi(x)\right| \leqslant \frac{C}{\sqrt{t}}. \tag{2}$$

Proof. For any $t \geqslant 2$, there exists an integer $n = n(t)$ such that $n+1 \leqslant t < n+2$. For simplicity, let

$$U_t = \frac{N_n - \lambda n}{\sqrt{\lambda t}}, \quad V_t = \frac{N_t - \lambda t}{\sqrt{\lambda t}}, \quad Q_t = V_t - U_t.$$

By (33), we should find a suitable bound of $|E(f_x'(V_t)-V_tf_x(V_t))|$, where the bound is independent of $x\in R$ and f_x is the unique bounded solution of Stein's equation (30). For simplicity, we write f for f_x. By the triangular inequality, we have

$$\begin{aligned}|E(f'(V_t)-V_tf(V_t))| &\leqslant |E(f'(V_t)-U_tf(U_t))| \\ &\quad +|E(U_tf(U_t)-U_tf(V_t))|+|E(Q_tf(V_t))| \\ &=: I_1+I_2+I_3. \end{aligned} \tag{3}$$

By (31) and Hölder's inequality, one has

$$\begin{aligned} I_3 &\leqslant E|Q_t| = E\left|\frac{N_t-\lambda t}{\sqrt{\lambda t}}-\frac{N_n-\lambda n}{\sqrt{\lambda t}}\right| \\ &\leqslant \frac{1}{\sqrt{\lambda t}}\left(E\left|(N_t-\lambda t)-(N_n-\lambda n)\right|^2\right)^{1/2} \\ &= \frac{1}{\sqrt{\lambda t}}(\lambda(t-n))^{1/2}\leqslant\frac{C}{\sqrt{t}}. \end{aligned} \tag{4}$$

Using (31) and Hölder's inequality again, we have

$$\begin{aligned} I_2 &\leqslant E|U_t(f(V_t)-f(U_t))|\leqslant E|U_tQ_t| \\ &= E\left(\left|\frac{N_n-\lambda n}{\sqrt{\lambda t}}\right|\left|\frac{N_t-\lambda t}{\sqrt{\lambda t}}-\frac{N_n-\lambda n}{\sqrt{\lambda t}}\right|\right) \\ &\leqslant \frac{1}{\lambda t}(E|N_n-\lambda n|^2)^{1/2}\left(E\left|(N_t-\lambda t)-(N_n-\lambda n)\right|^2\right)^{1/2} \\ &= \frac{1}{\lambda t}\cdot\sqrt{\lambda n}\cdot\sqrt{\lambda(t-n)}\leqslant\frac{C}{\sqrt{t}}. \end{aligned} \tag{5}$$

Again, by the triangular inequality, we have

$$I_1\leqslant|E(f'(U_t+Q_t)-f'(U_t))|+|E(f'(U_t)-U_tf(U_t))| =: I_4+I_5. \tag{6}$$

For $i=1,2,\cdots$, define $X_i=N_i-N_{i-1}$. It is obvious that $\{X_i\}$ is an i.i.d. sequence of Poisson random variables with parameter $\lambda>0$. Let μ_t be the distribution function of U_t. Using (30), (33), the triangular inequality and (34), one gets

$$\begin{aligned} I_5 &= |\mu_t(x)-\Phi(x)| = \left|P\left(\frac{N_n-\lambda n}{\sqrt{\lambda n}}\leqslant x\sqrt{\frac{t}{n}}\right)-\Phi(x)\right| \\ &\leqslant \left|P\left(\frac{N_n-\lambda n}{\sqrt{\lambda n}}\leqslant x\sqrt{\frac{t}{n}}\right)-\Phi\left(x\sqrt{\frac{t}{n}}\right)\right|+\left|\Phi\left(x\sqrt{\frac{t}{n}}\right)-\Phi(x)\right| \\ &\leqslant \frac{C}{\sqrt{t}}+\left|\Phi\left(x\sqrt{\frac{t}{n}}\right)-\Phi(x)\right|. \end{aligned} \tag{7}$$

Since $(1-x)^{-1/2}=1+x/2+o(x)\ (x\to 0)$, we have

$$\sqrt{\frac{t}{n}}=\left(1-\frac{t-n}{t}\right)^{-1/2} =: 1+R_t,$$

where $0 \leqslant R_t \leqslant C/\sqrt{t}$. By the mean value theorem and the fact that $|x\Phi'(x)| \leqslant C$, we have

$$\left|\Phi\left(x\sqrt{\frac{t}{n}}\right) - \Phi(x)\right| = |\Phi(x(1+R_t)) - \Phi(x)| \leqslant R_t|x\Phi'(x)| \leqslant \frac{C}{\sqrt{t}}. \tag{8}$$

Applying (32) for $u = U_t, s = Q_t$ and $t = 0$, we get

$$\begin{aligned} I_4 \leqslant\ & E(|U_t||Q_t|) + E(|Q_t|) + P(U_t + Q_t \leqslant x, U_t \geqslant x) \\ & + P(U_t + Q_t \geqslant x, U_t \leqslant x). \end{aligned} \tag{9}$$

As for (4) and (5), we have

$$E(|U_t||Q_t|) + E(|Q_t|) \leqslant C/\sqrt{t}. \tag{10}$$

For simplicity, let

$$H_t = \frac{N_{t-t^{-2}} - \lambda(t - t^{-2})}{\sqrt{\lambda t}} - \frac{N_n - \lambda n}{\sqrt{\lambda t}}, \ \ L_t = Q_t - H_t.$$

Consequently,

$$\begin{aligned} P(U_t + Q_t \leqslant x, U_t \geqslant x) &\leqslant P(U_t + H_t \leqslant x + t^{-1/2}, U_t \geqslant x) + P(|L_t| \geqslant t^{-1/2}) \\ &= I_6 + P(|L_t| \geqslant t^{-1/2}). \end{aligned} \tag{11}$$

Let δ_t be the distribution function of H_t. By conditioning and using the independence between U_t and H_t, we have

$$\begin{aligned} I_6 &= \int P(U_t + s \leqslant x + t^{-1/2}, U_t \geqslant x) d\delta_t(s) \\ &= \int I(s \leqslant 1/\sqrt{t})(\mu_t(x + t^{-1/2} - s) - \mu_t(x)) d\delta_t(s). \end{aligned} \tag{12}$$

Note that $|\Phi'(x)| \leqslant 1$. Thus, by the triangular inequality, (7) and (8), we obtain

$$\begin{aligned} |\mu_t(x + t^{-1/2} - s) - \mu_t(x)| &\leqslant |\mu_t(x + t^{-1/2} - s) - \Phi(x + t^{-1/2} - s)| \\ &\quad + |\Phi(x + t^{-1/2} - s) - \Phi(x)| + |\Phi(x) - \mu_t(x)| \\ &\leqslant C/\sqrt{t} + |s|. \end{aligned} \tag{13}$$

By Hölder's inequality, we have

$$\begin{aligned} E|H_t| &= \frac{1}{\sqrt{\lambda t}} E|(N_{t-t^{-2}} - \lambda(t - t^{-2})) - (N_n - \lambda n)| \\ &\leqslant \frac{1}{\sqrt{\lambda t}} \left(E|(N_{t-t^{-2}} - \lambda(t - t^{-2})) - (N_n - \lambda n)|^2\right)^{1/2} \\ &= \frac{(\lambda(t - t^{-2} - n))^{1/2}}{\sqrt{\lambda t}} \leqslant \frac{C}{\sqrt{t}}. \end{aligned} \tag{14}$$

Furthermore, by Markov's inequality and Hölder's inequality, one has

$$\begin{aligned} P(|L_t| \geqslant t^{-1/2}) &\leqslant \sqrt{t}E|L_t| = \sqrt{t}E\left|\frac{N_t-\lambda t}{\sqrt{\lambda t}} - \frac{N_{t-t^{-2}}-\lambda(t-t^{-2})}{\sqrt{\lambda t}}\right| \\ &\leqslant \sqrt{t}\left(E\left|\frac{N_t-\lambda t}{\sqrt{\lambda t}} - \frac{N_{t-t^{-2}}-\lambda(t-t^{-2})}{\sqrt{\lambda t}}\right|^2\right)^{1/2} \\ &= t^{-1} \leqslant C/\sqrt{t}. \end{aligned} \tag{15}$$

Thus, (11)–(15) imply

$$P(U_t+Q_t \leqslant x, U_t \geqslant x) \leqslant C/\sqrt{t}. \tag{16}$$

In the same way, we get $P(U_t+Q_t \geqslant x, U_t \leqslant x) \leqslant C/\sqrt{t}$. Therefore, (2) follows from (33), (3)–(16).

3. Proof of the main result

In this section, we use the same method as in Lemma 1 to prove the main result. For a Galton-Watson process $\{Z_n\}$, define $W_n = Z_n/m^n$. It is well known that there exists a nonnegative random variable W such that $W_n \xrightarrow{a.s.} W$. Our proof depends on the following lemma.

LEMMA 2. *Under condition A1, one has*

$$\sup_n E|\log W_n|^i < +\infty, \quad i = 1,2$$

and there exists a constant $r \in (0,1)$ such that

$$E|\log W_n - \log W| \leqslant Cr^n.$$

Proof. Since $p_0 = 0$, W is a positive random variable. Note that for any $a > 0$,

$$W^{-a} = \Gamma(a)^{-1}\int_0^\infty e^{-uW}u^{a-1}du,$$

where $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$. According to Lemma 10.7 of [1], we have

$$EW^{-a} = \Gamma(a)^{-1}\int_0^\infty \phi(u)u^{a-1}du < \infty$$

for all $a > 0$ such that $p_1 m^a < 1$, where $\phi(u) = E(e^{-uW})$. Therefore, using the fact that, for any positive x, $|\log x|^i \leqslant C(x + x^{-a})$ $(i = 1,2)$, one gets

$$E|\log W|^i \leqslant C(E(W) + E(W^{-a})) < \infty, \quad i = 1,2.$$

For any nonnegative and convex function f, by Jensen's inequality, one has

$$E(f(W)|\mathscr{F}_n) \geqslant f(E(W|\mathscr{F}_n)) = f(W_n)\ a.s.,$$

where $\mathscr{F}_n$ is the $\sigma-$ field generated by $\{Z_0, Z_1, \cdots, Z_n\}$. Thus, $\sup_n E(f(W_n)) \leqslant E(f(W))$. On the other hand, using Fatou's lemma, one gets $\sup_n E(f(W_n)) \geqslant E(f(W))$.

Note that $|\log x|^i I(0 < x \leqslant 1)$ $(i = 1,2)$ is a nonnegative and convex function, we have

$$\sup_n E|\log W_n|^i I(W_n \leqslant 1) = \sup_n E|\log W|^i I(W \leqslant 1) < +\infty, \ \ i = 1,2.$$

By a standard truncation we obtain $\sup_n E|\log W_n|^i < \infty$, $i = 1,2$.

Next, define

$$U_n = \frac{W_{n+1}}{W_n} - 1 = \frac{1}{Z_n}\sum_{i=1}^{Z_n}\left(\frac{X_i}{m} - 1\right),$$

where $\{X_i\}$ is an i.i.d. sequence and independent of Z_n. Then

$$\log W_{n+1} - \log W_n = \log(1 + U_n).$$

We firstly show that there exists a constant $r \in (0,1)$ such that $[E(U_n^2)]^{1/2} \leqslant Cr^n$. In fact, by a standard moment inequality (see page 97 of [9]), one gets $E(U_n^2) \leqslant CE(Z_n^{-1})$. Note that $f(x) = x^{-1}, x > 0$ is a convex function, by Jensen's inequality,

$$E(Z_{n+1}^{-1}) = E\left(\sum_{i=1}^{Z_n} X_i\right)^{-1} \leqslant E\left(Z_n^{-1} Z_n^{-1} \sum_{i=1}^{Z_n} X_i^{-1}\right)$$

Since $\{X_i\}$ is independent of Z_n,

$$E(Z_{n+1}^{-1}) \leqslant E\left(E\left(Z_n^{-1} Z_n^{-1} \sum_{i=1}^{Z_n} X_i^{-1} | Z_n\right)\right) = E(Z_n^{-1})E(Z_1^{-1}).$$

By induction, we obtain

$$E(Z_n^{-1}) \leqslant \left(E(Z_1^{-1})\right)^n.$$

Let $r = \sqrt{E(Z_1^{-1})} < 1$, one gets $[E(U_n^2)]^{1/2} \leqslant Cr^n$.

Next, for any $b \in (0,1)$,

$$\begin{aligned} E|\log W_{n+1} - \log W_n| &= E|\log(1+U_n)I(U_n \geqslant -b)| + E|\log(1+U_n)I(U_n < -b)| \\ &= A_n + B_n. \end{aligned} \tag{17}$$

Using the fact that $|\log(1+x)| \leqslant C|x|$ for $x \geqslant -b$ and Hölder's inequality, one has

$$A_n \leqslant CE|U_n| \leqslant C(E(U_n^2))^{1/2} \leqslant Cr^n. \tag{18}$$

Note that $E(\log(1+U_n))^2 = E(\log W_{n+1} - \log W_n)^2 < \infty$, using Hölder's inequality and Chebyshev's inequality, we have

$$B_n \leqslant C(E(\log(1+U_n))^2)^{1/2}(P(U_n < -b))^{1/2} \leqslant C[E(U_n^2)]^{1/2} \leqslant Cr^n. \tag{19}$$

Thus by (17)–(19), one gets

$$E|\log W_{n+1} - \log W_n| \leqslant Cr^n.$$

Consequently, for any integer $k \geqslant 1$,

$$E|\log W_{n+k} - \log W_n| \leqslant C(r^n + r^{n+1} + \cdots + r^{n+k-1}) \leqslant Cr^n/(1-r).$$

Letting $k \to \infty$, we complete the proof of Lemma 2.

Proof of Theorem 1. For any $t \geqslant 1$, let

$$S_t = \frac{\log Y_t - \lambda t \log m}{\sqrt{\lambda t}\log m}, \quad V_t = \frac{N_t - \lambda t}{\sqrt{\lambda t}}, \quad T_t = S_t - V_t.$$

As for (3), we have

$$\begin{aligned}|E(f'(S_t) - S_t f(S_t))| \leqslant\ & |E(f'(S_t) - V_t f(V_t))| \\ & + |E(V_t f(S_t) - V_t f(V_t))| + |E(T_t f(S_t))| \\ =:\ & K_1 + K_2 + K_3. \end{aligned} \tag{20}$$

It is enough to prove that $K_1 + K_2 + K_3 \leqslant C/\sqrt{t}$. For K_3, let $W_n = Z_n/m^n$, by (31) and Lemma 2, one has

$$K_3 \leqslant E|T_t| = E\left|\frac{\log W_{N_t}}{\sqrt{\lambda t}\log m}\right| \leqslant \frac{C}{\sqrt{t}}. \tag{21}$$

Furthermore, using (31), Lemma 2 and Hölder's inequality, we have

$$\begin{aligned} K_2 &\leqslant E|V_t(f(S_t) - f(V_t))| \leqslant E|V_t T_t| \\ &\leqslant \frac{1}{\lambda t \log m}(E|N_t - \lambda t|^2)^{1/2}\left(E\,|\log W_{N_t}|^2\right)^{1/2} \leqslant \frac{C}{\sqrt{t}}. \end{aligned} \tag{22}$$

Finally, by the triangular inequality, we have

$$K_1 \leqslant |E(f'(V_t + T_t) - f'(V_t))| + |E(f'(V_t) - V_t f(V_t))| =: K_4 + K_5. \tag{23}$$

Lemma 1 implies $K_5 \leqslant C/\sqrt{t}$, then it is enough to show that $K_4 \leqslant C/\sqrt{t}$. Applying (32) for $u = V_t, s = T_t$ and $t = 0$, we get

$$\begin{aligned} I_4 \leqslant\ & E(|V_t||T_t|) + E(|T_t|) + P(V_t + T_t \leqslant x, V_t \geqslant x) \\ & + P(V_t + T_t \geqslant x, V_t \leqslant x). \end{aligned} \tag{24}$$

As for (21) and (22), we have $E(|V_t||T_t|) + E(|T_t|) \leqslant C/\sqrt{t}$. Firstly, we show that

$$P(V_t + T_t \leqslant x, V_t \geqslant x) \leqslant C/\sqrt{t}. \tag{25}$$

For simplicity, let

$$B_t = \frac{(N_t - N_{\sqrt{t}}) - \lambda(t - \sqrt{t})}{\sqrt{\lambda t}}, \quad C_t = V_t - B_t, \quad \rho_t(x) = P(B_t \leqslant x),$$

$$D_t = \frac{\log W_{N_{\sqrt{t}}}}{\sqrt{\lambda t}\log m}, \quad E_t = T_t - D_t.$$

Consequently,

$$\begin{aligned} P(V_t + T_t \leqslant x, V_t \geqslant x) &\leqslant P(V_t + D_t \leqslant x + t^{-1/2}, V_t \geqslant x) + P(|E_t| \geqslant t^{-1/2}) \\ &= K_6 + P(|E_t| \geqslant t^{-1/2}). \end{aligned} \tag{26}$$

Let ν_t be the joint distribution of (C_t, D_t). By conditioning and using the independence between B_t and (C_t, D_t), we have

$$\begin{aligned} K_6 &= \int P(B_t + s + v \leqslant x + t^{-1/2}, B_t + s \geqslant x)\nu_t(ds, dv) \\ &= \int I(s \leqslant 1/\sqrt{t})(\rho_t(x + t^{-1/2} - s - v) - \rho_t(x - s))\nu_t(ds, dv). \end{aligned} \tag{27}$$

As for (8) and (13), we obtain

$$|\rho_t(x + t^{-1/2} - s - v) - \rho_t(x - s)| \leqslant C/\sqrt{t} + |v| \tag{28}$$

By Lemma 2, one has

$$E|D_t| \leqslant C/\sqrt{t}, \quad P(|E_t| \geqslant t^{-1/2}) \leqslant C/\sqrt{t}. \tag{29}$$

(25) follows from (26)–(29).

One can obtain $P(V_t + T_t \geqslant x, V_t \leqslant x) \leqslant C/\sqrt{t}$ similarly. By (33), (20)–(25), we get Theorem 1.

A. Stein's method

Some basic facts on the Stein's method are given in this appendix. For more details, the reader can see the book [2].

Lemma 3 shows that the standard normal distribution can be characterized by the Stein's operator which is defined as following,

$$\mathscr{A} f(u) = f'(u) - u f(u),$$

where $f : R \mapsto R$ is an absolutely continuous function.

LEMMA 3. (Characterization of the normal law $N(0,1)$) *A random variable Z is of normal law $N(0,1)$ if and only if $E\mathscr{A} f(Z) = 0$ for all absolutely continuous function f such that $E|f'(Z)| < \infty$ (see Lemma 2.1 of* [2]*).*

The next lemma gives some facts on the solution of Stein's equation defined as follow. For any $x \in R$, define

$$I(u \leqslant x) - \Phi(x) = f'(u) - u f(u), \ \forall u \in R. \tag{30}$$

LEMMA 4. (Solution of Stein's equation) *For each $x \in R$, Stein's equation (30) has a unique bounded solution f_x which satisfies*

$$\|f_x\| \leqslant 1, \quad \|f_x'\| \leqslant 1 \tag{31}$$

and for all real u, s and t,

$$\begin{aligned} |f_x'(u+s) - f_x'(u+t)| \leqslant (|t|+|s|)(|u|+1) + I(x-t \leqslant u \leqslant x-s)I(s \leqslant t) \\ + I(x-s \leqslant u \leqslant x-t)I(s > t), \end{aligned} \tag{32}$$

where $\|\cdot\|$ denotes the infinity norm (see [11] for details).

Substituting u by X in (30), taking expectation and the supremum over $x \in R$, we obtain

$$\sup_{x \in R} |P(X \leqslant x) - \Phi(x)| = \sup_{x \in R} |E(f_x'(X) - X f_x(X))| = \sup_{x \in R} E \mathscr{A} f_x(X). \tag{33}$$

The last result gives the Berry-Esseen bound of a sum of i.i.d. random variables via Stein's method.

LEMMA 5. (Berry-Esseen bound via Stein's method) *Consider i.i.d. random variables $X_1, \cdots, X_n$ with $\mu = E(X_1)$, $\sigma^2 = Var(X_1)$ and $\rho = E|X_1|^3 < \infty$. Define $Y_n = \sum_{i=1}^n (X_i - \mu)/(\sigma\sqrt{n})$. For each $x \in R$, the unique bounded solution f_x of Stein's equation satisfies*

$$|E(f_x'(Y_n) - Y_n f_x(Y_n))| \leqslant \frac{C\rho}{\sqrt{n}}. \tag{34}$$

The proof of Lemma 5 was given in [11].

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Zhenlong Gao
School of Statistics
Qufu Normal University
Qufu 273165, China
e-mail: gzlkygz@163.com