THE BÉZIER VARIANT OF LUPAS KANTOROVICH
OPERATORS BASED ON POLYA DISTRIBUTION

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Abstract. In this paper we introduce the Bézier variant of Lupas Kantorovich operators based on Polya distribution. We establish a direct approximation by means of the Ditzian-Totik modulus of smoothness and a global approximation theorem in terms of second order modulus of continuity. Furthermore, we give the rate of convergence for absolutely continuous functions having a derivative equivalent to a bounded function. Our results extend the work of Agrawal [P. N. Agrawal, N. Ispir and A. Kajla, Approximation properties of Lupas-Kantorovich operators based on polya distribution, Rendiconti del Circolo Matematico di Palermo Series 2, 2016, 65 (2): 185–208] and Ispir [N. Ispir, P. N. Agrawal and A. Kajla, Rate of convergence of Lupas Kantorovich operators based on Polya distribution, Appl. Math. Comput., 2015, 261: 323–329].

1. Introduction

In the year 1987, Lupas [1] introduced Polya-Bernstein operators

$$P_n^{(1/n)}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{(1/n)}(x), \tag{1}$$

where $f \in C[0,1]$, $p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!}\binom{n}{k}(nx)_k(n-nx)_{n-k}$, $(n)_k = n(n+1)\cdots(n+k-1)$.

In [2], Miclaus studied some approximation properties of Bernstein-Stancu type operators based on Polya distribution. Recently, Gupta and Rassias [3] introduced the Durrmeyer variant of the operators (1) as follows:

$$D_n^{(1/n)}(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_0^1 p_{n,k}(t)f(t)dt, \tag{2}$$

where $p_{n,k}(t) = \binom{n}{k}t^k(1-t)^{n-k}$. They established some direct results which include an asymptotic formula, local and global approximaiton results. To approximate Lebesgue

integrable functions, Agrawal et al. [4] introduced the following integral modification
of the operators (1):

$$D_n^{*(1/n)}(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{3}$$

In [4], Agrawal et al. studied the voronovskaja type theorem, local approximation,
pointwise estimates and global approximation results. Later, Ispir et al. [5] estimated
the rate of convergence for absolutely continuous functions having a derivative coincid-
ing a.e. with a function of bounded variation. For the related research work, we can see
[6–9].

It is well known that Bézier curves play an important role in computer aided de-
signs and computer graphics systems. Zeng and Piriou [10] pioneered the study of two
Bernstein-Bézier type operators for bounded variation functions. Then many scholars
[11–14] have done research work in related fields. Agrawal et al. [15] and Neer et
al. [16] introduced the Bézier variant of Lupas-Durrmeyer type operators and genuine-
Durrmeyer type operators based on Polya distribution. They obtained some direct ap-
proximaiton theorem and the rate of convergence for absolutely continuous functions
having a derivative equivalent to a function of bounded variation.

Inspired by the idea of Zeng and Agrawal, we propose the modified variant of the
operators (3) in the following way:

$$D_{n,\alpha}^{*(1/n)}(f,x) = (n+1) \sum_{k=0}^{n} Q_{nk}^{(\alpha)}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \tag{4}$$

where $\alpha \geqslant 1$ and

$$Q_{nk}^{(\alpha)}(x) = \left[J_{n,k}(x)\right]^{\alpha} - \left[J_{n,k+1}(x)\right]^{\alpha}, \quad J_{n,k}(x) = \sum_{j=k}^{n} p_{nj}^{(1/n)}(x).$$

Obviously for $\alpha = 1$, the operators $D_{n,\alpha}^{*(1/n)}(f,x)$ reduces to the operators $D_n^{*(1/n)}(f,x)$.
Let

$$K_{n,\alpha}(x,t) = \sum_{k=0}^{n} (n+1)Q_{n,k}^{(\alpha)}(x)\chi_k(t)$$

and

$$R_{n,\alpha}(x,t) = \int_0^t K_{n,\alpha}(x,s)ds,$$

where $\chi_k(t)$ is the characteristic function of the interval $[\frac{k}{n+1}, \frac{k+1}{n+1}]$ with respect to
$I = [0,1]$. By the Lebesgue-Stieltjes integral representations, we have

$$D_{n,\alpha}^{*(1/n)}(f,x) = \int_0^1 f(t)K_{n,\alpha}(x,t)dt = \int_0^1 f(t)d_t R_{n,\alpha}(x,t). \tag{5}$$

The aim of this paper is to establish a direct approximaiton by means of the
Ditzian-Totik modulus of smoothness and a global approximation theorem in terms
of second order modulus of continuity. Furthermore, the rate of convergence for some
absolutely continuous functions having a derivative equivalent to a bounded function is
obtained. The results extend the work of Agrawal and Ispir.

2. Some lemmas

The proof of our results are based on the following lemmas.

LEMMA 1. [4] *For* $e_i = t^i$, $i = 0, 1, 2$, *we have*

$$D_n^{*(1/n)}(e_0, x) = 1, \quad D_n^{*(1/n)}(e_1, x) = \frac{2nx + 1}{2(n+1)},$$

$$D_n^{*(1/n)}(e_2, x) = \frac{3n^3x^2 + 9n^2x - 3n^2x^2 + 3nx + n + 1}{3(n+1)^3}.$$

By simple applications of Lemma 1, we get

$$D_n^{*(1/n)}(t - x, x) = \frac{1 - 2x}{2(n+1)}, \tag{6}$$

$$D_n^{*(1/n)}((t - x)^2, x) \leqslant \frac{2}{n+1}. \tag{7}$$

LEMMA 2. [4] *For* $f \in C[0,1]$, $x \in [0,1]$, *we have*

$$\|D_n^{*(1/n)}(f, x)\| \leqslant \|f\|.$$

LEMMA 3. *For* $f \in C[0,1]$, $x \in [0,1]$, *we have*

$$\|D_{n,\alpha}^{*(1/n)}(f, x)\| \leqslant \alpha \|f\|.$$

Proof. Using the inequality $|x^\alpha - y^\alpha| \leqslant \alpha |x - y|$ with $0 \leqslant x, y \leqslant 1$ and $\alpha \geqslant 1$, we get

$$0 < [J_{n,k}(x)]^\alpha - [J_{n,k+1}(x)]^\alpha \leqslant \alpha \left(J_{n,k}(x) - J_{n,k+1}(x) \right) = \alpha p_{n,k}^{(1/n)}(x).$$

Hence from the definition $D_{n,\alpha}^{*(1/n)}(f, x)$ and Lemma 2, we obtain

$$\|D_{n,\alpha}^{*(1/n)}(f, x)\| \leqslant \alpha \|D_n^{*(1/n)}(f, x)\| \leqslant \alpha \|f\|. \quad \square$$

LEMMA 4. *(i) For* $0 \leqslant y < x < 1$, *there holds*

$$R_{n,\alpha}(x, y) = \int_0^y K_{n,\alpha}(x, t)dt \leqslant \frac{2\alpha}{(n+1)(x-y)^2}. \tag{8}$$

(ii) For $0 < x < z \leqslant 1$, *there holds*

$$1 - R_{n,\alpha}(x, z) = \int_z^1 K_{n,\alpha}(x, t)dt \leqslant \frac{2\alpha}{(n+1)(z-x)^2}. \tag{9}$$

Proof. (i) By (5) and (7), we get

$$\begin{aligned}
R_{n,\alpha}(x,y) &= \int_0^y K_{n,\alpha}(x,t)dt \leqslant \int_0^y \left(\frac{x-t}{x-y}\right)^2 K_{n,\alpha}(x,t)dt \\
&\leqslant \frac{1}{(x-y)^2} \int_0^1 (t-x)^2 K_{n,\alpha}(x,t)dt \\
&= \frac{1}{(x-y)^2} D_{n,\alpha}^{*(1/n)}((t-x)^2,x) \\
&\leqslant \frac{\alpha}{(x-y)^2} D_n^{*(1/n)}((t-x)^2,x) \\
&\leqslant \frac{2\alpha}{(n+1)(x-y)^2}.
\end{aligned}$$

(ii) Using a similar method we can get (9) easily. $\quad\square$

3. Main results

Let $f(x) \in C[0,1]$, $t > 0$ and $W^2[0,1] = \{g \in C[0,1] : g'' \in C[0,1]\}$, the appropriate Peetre's K-functional is defined by

$$K_2(f,t) = \inf_{g \in W^2[0,1]} \{\|f-g\| + t\|g'\| + t^2\|g''\|\}.$$

From [17], there exists an absolute constant $C > 0$, such that

$$K_2(f,t) \leqslant C\omega_2(f,\sqrt{t}), \tag{10}$$

where ω_2 is the second order modulus of continuity of $f \in C[0,1]$, defined as

$$\omega_2(f,\sqrt{t}) = \sup_{0 < h \leqslant \sqrt{t}} \sup_{x,x+h,x+2h \in [0,1]} |f(x+2h) - 2f(x+h) + f(x)|.$$

THEOREM 1. *For $f \in C[0,1]$ and $x \in [0,1]$. Then there exists an absolute constant $C > 0$, such that*

$$|D_{n,\alpha}^{*(1/n)}(f,x) - f(x)| \leqslant C\omega_2\left(f, \sqrt{\frac{\alpha}{2(n+1)}}\right). \tag{11}$$

Proof. Let $g \in W^2$. By Taylor's expansion, we get

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

Applying $D_{n,\alpha}^{*(1/n)}(\cdot,x)$ to both sides of the above equation, we have

$$D_{n,\alpha}^{*(1/n)}(g,x) = g(x) + g'(x)D_{n,\alpha}^{*(1/n)}(t-x,x) + D_{n,\alpha}^{*(1/n)}\left(\int_x^t (t-u)g''(u)du,x\right).$$

By Cauchy Schwarz inequality, (7) and Lemma 3, we have

$$|D_{n,\alpha}^{*(1/n)}(g,x) - g(x)| \leqslant |g'(x)||D_{n,\alpha}^{*(1/n)}(|t-x|,x)| + \left|D_{n,\alpha}^{*(1/n)}\left(\int_x^t (t-u)g''(u)du,x\right)\right|$$

$$\leqslant \|g'\|D_{n,\alpha}^{*(1/n)}(|t-x|,x) + \frac{\|g''\|}{2}D_{n,\alpha}^{*(1/n)}((t-x)^2,x)$$

$$\leqslant \|g'\|D_{n,\alpha}^{*(1/n)}\left((t-x)^2,x\right)^{1/2} + \frac{\|g''\|}{2}D_{n,\alpha}^{*(1/n)}((t-x)^2,x)$$

$$\leqslant \sqrt{\alpha}\|g'\|D_n^{*(1/n)}\left((t-x)^2,x\right)^{1/2} + \alpha\frac{\|g''\|}{2}D_n^{*(1/n)}((t-x)^2,x)$$

$$\leqslant \sqrt{\alpha}\|g'\|\sqrt{\frac{2}{n+1}} + \alpha\frac{\|g''\|}{2}\frac{2}{n+1}.$$

Thus

$$|D_{n,\alpha}^{*(1/n)}(f,x) - f(x)| \leqslant |D_{n,\alpha}^{*(1/n)}(f-g,x)| + |f-g| + |D_{n,\alpha}^{*(1/n)}(g,x) - g(x)|$$

$$\leqslant 2\|f-g\| + \sqrt{\alpha}\|g'\|\sqrt{\frac{2}{n+1}} + \alpha\frac{\|g''\|}{2}\frac{2}{n+1}.$$

Taking the infimum on the right hand side over all $g \in W^2$, we obtain

$$|D_{n,\alpha}^{*(1/n)}(f,x) - f(x)| \leqslant 2K_2\left(f, \frac{\alpha}{2(n+1)}\right).$$

By (10), we get (11) immediately. This completes the proof. □

REMARK 1. When $\alpha = 1$, we have

$$|D_n^{*(1/n)}(f,x) - f(x)| \leqslant C\omega_2\left(f, \sqrt{\frac{1}{2(n+1)}}\right),$$

which extend the work of Agrawal et al. [4].

To describe our next result, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [18]. Let $\phi(x) = \sqrt{x(1-x)}$ and $f \in C[0,1]$, the first order modulus of smoothness is given by

$$\omega_\phi(f,t) = \sup_{0<h\leqslant t}\left|f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right)\right|, \quad x\pm\frac{h\phi(x)}{2} \in [0,1].$$

Further, the corresponding K-functional to $\omega_\phi(f,t)$ is defined by

$$K_\phi(f,t) = \inf_{g \in W_\phi[0,1]}\{\|f-g\| + t\|\phi g'\|\}(t>0),$$

where $W_\phi[0,1] = \{g : g \in AC[0,1], \|\phi g'\| < \infty\}$ and $AC[0,1]$ is the class of all absolutely continuous functions on $[0,1]$. From [18], there exists a constant $C > 0$ such that

$$K_\phi(f,t) \leqslant C\omega_\phi(f,t). \tag{12}$$

THEOREM 2. *For $f \in C[0,1]$, $x \in (0,1)$ and $\phi(x) = \sqrt{x(1-x)}$. Then there exists an absolute constant $C > 0$, such that*

$$|D_{n,\alpha}^{*(1/n)}(f,x) - f(x)| \leqslant C\omega_\phi\left(f, \sqrt{\frac{4\alpha}{(n+1)x(1-x)}}\right). \tag{13}$$

Proof. Using the representation

$$g(t) = g(x) + \int_x^t g'(u)du,$$

we get

$$D_{n,\alpha}^{*(1/n)}(g,x) = g(x) + D_{n,\alpha}^{*(1/n)}\left(\int_x^t g'(u)du, x\right).$$

For any $x, t \in (0,1)$, we find that

$$\left|\int_x^t g'(u)du\right| \leqslant \|\phi g'\| \left|\int_x^t \frac{1}{\phi(u)}du\right|.$$

But

$$\begin{aligned}
\left|\int_x^t \frac{1}{\phi(u)}du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| \\
&\leqslant \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \\
&\leqslant 2\left(|\sqrt{t} - \sqrt{x}| + |\sqrt{1-t} - \sqrt{1-x}|\right) \\
&= 2|t - x|\left(\frac{1}{\sqrt{t} + \sqrt{x}} + \frac{1}{\sqrt{1-t} + \sqrt{1-x}}\right) \\
&\leqslant 2|t - x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \leqslant \frac{2\sqrt{2}|t - x|}{\phi(x)}.
\end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|D_{n,\alpha}^{*(1/n)}(g,x) - g(x)| &\leqslant 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)D_{n,\alpha}^{*(1/n)}(|t-x|,x) \\
&\leqslant 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)\left(D_{n,\alpha}^{*(1/n)}((t-x)^2,x)\right)^{1/2} \\
&\leqslant 2\sqrt{2}\sqrt{\alpha}\|\phi g'\|\phi^{-1}(x)\left(D_n^{*(1/n)}((t-x)^2,x)\right)^{1/2} \\
&\leqslant 4\sqrt{\alpha}\|\phi g'\|\phi^{-1}(x)\sqrt{\frac{1}{n+1}}.
\end{aligned}$$

Thus

$$\begin{aligned}
|D_{n,\alpha}^{*(1/n)}(f,x) - f(x)| &\leqslant |D_{n,\alpha}^{*(1/n)}(f-g,x)| + |f-g| + |D_{n,\alpha}^{*(1/n)}(g,x) - g(x)| \\
&\leqslant 2\|f-g\| + 4\sqrt{\alpha}\|\phi g'\|\phi^{-1}(x)\sqrt{\frac{1}{n+1}}.
\end{aligned}$$

Taking the infimum on the right hand side over all $g \in W_\phi[0,1]$, we obtain

$$|D_{n,\alpha}^{*(1/n)}(f,x) - f(x)| \leqslant 2K_\phi\left(f, \sqrt{\frac{4\alpha}{(n+1)x(1-x)}}\right).$$

By (12), we get (13) immediately. This completes the proof. □

REMARK 2. When $\alpha = 1$, we have

$$|D_n^{*(1/n)}(f,x) - f(x)| \leqslant C\omega_\phi\left(f, \sqrt{\frac{4}{(n+1)x(1-x)}}\right),$$

which extend the work of Agrawal et al. [4].

Lastly, we study the approximation properties of $D_{n,\alpha}^{*(1/n)}(f,x)$ for some absolutely continuous functions $f \in \Phi_{DB}$, which is defined by

$$\Phi_{DB} = \left\{ f \mid f(x) = f(0) + \int_0^x h(t)dt; \ x \geqslant 0, \ h \text{ is bounded on } [0,1] \right\}.$$

For a bounded function f on $[0,1]$, we also introduce the following metric form

$$\Omega_x(f,\lambda) = \sup_{t \in [x-\lambda, x+\lambda]} |f(t) - f(x)|, \ \text{ where } x \in [0,1] \text{ is fixed}, \ \lambda \geqslant 0.$$

It is clear that
(i) $\Omega_x(f,\lambda)$ is monotone non-decreasing with respect to λ.
(ii) $\lim_{\lambda \to 0} \Omega_x(f,\lambda) = 0$, if f is continuous at the point x.

THEOREM 3. *Let* $f \in \Phi_{DB}$. *If* $h(x+)$ *and* $h(x-)$ *exist at a fixed point* $x \in (0,1)$, *then we have*

$$\left|D_{n,\alpha}^{*(1/n)}(f,x) - f(x)\right| \leqslant \alpha(|h(x+)| + |h(x-)|)\sqrt{\frac{2}{n+1}}$$

$$+ \frac{4\alpha + 2x(1-x)}{nx(1-x)} \sum_{k=1}^{[\sqrt{n}]} \Omega_x(\varphi_x, 1/k),$$

where

$$\varphi_x(t) = \begin{cases} h(t) - h(x+), & x < t \leqslant 1; \\ 0, & t = x; \\ h(t) - h(x-), & 0 \leqslant t < x. \end{cases}$$

Proof. Let f satisfy the conditions of Theorem 3, by using Bojanic-Cheng's method [19], we have

$$f(t) - f(x) = \int_x^t h(u)du \tag{14}$$

and $h(u)$ can be expressed as

$$h(u) = \frac{h(x+) + h(x-)}{2} + \varphi_x(u) + \frac{h(x+) - h(x-)}{2} sign(u-x)$$
$$+ \delta_x(u) \left[h(x) - \frac{h(x+) + h(x-)}{2} \right], \tag{15}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x; \\ 0, & u \neq x. \end{cases}$$

$$sign(x) = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases}$$

From (14), (15), and noting $\int_x^t sign(u-x)du = |t-x|, \int_x^t \delta_x(u)du = 0$, we find that

$$\left| D_{n,\alpha}^{*(1/n)}(f,x) - f(x) \right| = \left| D_{n,\alpha}^{*(1/n)}(f(t) - f(x),x) \right| = \left| D_{n,\alpha}^{*(1/n)}(\int_x^t h(u)du,x) \right|$$
$$= \left| \frac{h(x+) + h(x-)}{2} D_{n,\alpha}^{*(1/n)}(t-x,x) \right.$$
$$\left. + \frac{h(x+) - h(x-)}{2} D_{n,\alpha}^{*(1/n)}(|t-x|,x) + D_{n,\alpha}^{*(1/n)}(\int_x^t \varphi_x(u)du,x) \right|$$
$$\leqslant \left(|h(x+)| + |h(x-)| \right) D_{n,\alpha}^{*(1/n)}(|t-x|,x)$$
$$+ \left| D_{n,\alpha}^{*(1/n)}(\int_x^t \varphi_x(u)du,x) \right|.$$

By Cauchy Schwarz inequality, (7) and Lemma 3, we have

$$D_{n,\alpha}^{*(1/n)}(|t-x|,x) \leqslant \alpha D_n^{*(1/n)}(|t-x|,x) \leqslant \alpha D_n^{*(1/n)}\left((t-x)^2,x \right)^{1/2} \leqslant \alpha \sqrt{\frac{2}{n+1}}.$$

Thus we have

$$\left| D_{n,\alpha}^{*(1/n)}(f,x) - f(x) \right| \leqslant \alpha \left(|h(x+)| + |h(x-)| \right) \sqrt{\frac{2}{n+1}} + \left| D_{n,\alpha}^{*(1/n)}(\int_x^t \varphi_x(u)du,x) \right|. \tag{16}$$

To complete the proof, we must estimate the term $D_{n,\alpha}^{*(1/n)}(\int_x^t \varphi_x(u)du,x)$.

From (5), the term $D_{n,\alpha}^{*(1/n)}(\int_x^t \varphi_x(u)du,x)$ can be stated as

$$D_{n,\alpha}^{*(1/n)}(\int_x^t \varphi_x(u)du,x) = \int_0^1 (\int_x^t \varphi_x(u)du)K_{n,\alpha}(x,t)dt = \int_0^1 (\int_x^t \varphi_x(u)du)d_tR_{n,\alpha}(x,t)$$
$$= \int_0^x (\int_x^t \varphi_x(u)du)d_tR_{n,\alpha}(x,t) + \int_x^1 (\int_x^t \varphi_x(u)du)d_tR_{n,\alpha}(x,t).$$

Let

$$\Delta_{1n}(f,x) = \int_0^x (\int_x^t \varphi_x(u)du)d_tR_{n,\alpha}(x,t),$$

$$\Delta_{2n}(f,x) = \int_x^1 (\int_x^t \varphi_x(u)du)d_t R_{n,\alpha}(x,t).$$

Then we have

$$D_{n,\alpha}^{*(1/n)}(\int_x^t \varphi_x(u)du,x) = \Delta_{1n}(f,x) + \Delta_{2n}(f,x). \tag{17}$$

Using partial integration and noticing $R_{n,\alpha}(x,0) = 0$, $\int_x^x \varphi_x(u)du = 0$, we get

$$\Delta_{1n}(f,x) = R_{n,\alpha}(x,t) \int_x^t \varphi_x(u)du\Big|_0^x - \int_0^x R_{n,\alpha}(x,t)\varphi_x(t)dt$$

$$= -\int_0^x R_{n,\alpha}(x,t)\varphi_x(t)dt = -(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^x)R_{n,\alpha}(x,t)\varphi_x(t)dt.$$

Thus, it follows that

$$\left|\Delta_{1n}(f,x)\right| \leqslant \int_0^{x-\frac{x}{\sqrt{n}}} R_{n,\alpha}(x,t)\Omega_x(\varphi_x, x-t)dt + \int_{x-\frac{x}{\sqrt{n}}}^x R_{n,\alpha}(x,t)\Omega_x(\varphi_x, x-t)dt$$

From Lemma 4 (i) and $0 \leqslant R_{n,\alpha}(x,t) \leqslant 1$, we get

$$\left|\Delta_{1n}(f,x)\right| \leqslant \frac{2\alpha}{n+1} \int_0^{x-\frac{x}{\sqrt{n}}} \frac{\Omega_x(\varphi_x, x-t)}{(x-t)^2}dt + \frac{x}{\sqrt{n}}\Omega_x(\varphi_x, x/\sqrt{n}). \tag{18}$$

Putting $t = x - \frac{x}{u}$ for the integral of (18), we get

$$\int_0^{x-\frac{x}{\sqrt{n}}} \frac{\Omega_x(\varphi_x, x-t)}{(x-t)^2}dt = \frac{1}{x}\int_1^{\sqrt{n}} \Omega_x(\varphi_x, x/u)du \leqslant \frac{2}{x}\sum_{k=1}^{[\sqrt{n}]} \Omega_x(\varphi_x, 1/k). \tag{19}$$

From (17), (18), it follows that

$$\left|\Delta_{1n}(f,x)\right| \leqslant \frac{4\alpha}{(n+1)x}\sum_{k=1}^{[\sqrt{n}]} \Omega_x(\varphi_x, 1/k) + \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]} \Omega_x(\varphi_x, 1/k)$$

$$\leqslant \frac{4\alpha + 2x^2}{nx}\sum_{k=1}^{[\sqrt{n}]} \Omega_x(\varphi_x, 1/k). \tag{20}$$

From Lemma 4 (ii), using the same method, we also get

$$\left|\Delta_{2n}(f,x)\right| \leqslant \frac{4\alpha + 2(1-x)^2}{n(1-x)}\sum_{k=1}^{[\sqrt{n}]} \Omega_x(\varphi_x, 1/k). \tag{21}$$

Theorem 3 now follows from (16), (17), (20) and (21). This completes the proof. $\square$

REMARK 3. If f is a function with derivative of bounded variation, then $f \in \Phi_{DB}$. Thus when $\alpha = 1$, the approximaiton of $D_n^{*(1/n)}(f,x)$ for functions with derivatives of bounded variation in [5] is a special case of Theorem 3.

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