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A PARAMETER–BASED OSTROWSKI TYPE INEQUALITY ON TIME SCALES WITH k POINTS FOR FUNCTIONS HAVING BOUNDED SECOND DERIVATIVES

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Abstract. In this paper, we introduce a parameter $\lambda \in [0,1]$, and obtain a generalization of the Ostrowski type inequality on time scales for k points for functions whose second derivatives are bounded. Our results generalize some results in the literature and thereby, injects into the mathematical community some inequalities which we hope can be used to approximate the integral of a function in applied mathematics and/or mathematical physics. In addition, we apply our main theorem to the continuous, discrete, and quantum calculus to derive more inequalities in this direction.

1. Introduction

In 1938, Ostrowski [4] proved the following inequality which approximates a function by its integral average.

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(x)|\leqslant M$ for all $x\in(a,b)$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leqslant\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M$$

for all $x\in[a,b]$. The constant $1/4$ is the best possible in the sense that it cannot be replaced by a smaller constant.

In order to unify the theory of integral and differential calculus with the calculus of finite difference, the German mathematician Stefan Hilger [5] in 1988 introduced the concept of time scales (see Section 2 for a brief overview). This subject area has generated great deal of interest among mathematicians. The question of interest has always been: can we extend a classical inequality to time scales? In answering this question, Bohner and Matthews [3], in 2008, obtained the following time scale version of Theorem 1. In particular, they proved:

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THEOREM 2. *Let $a,b,x,t \in \mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable. Then for all $x\in[a,b]$, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f^{\sigma}(t)\Delta t\right| \leqslant \frac{M}{b-a}\left(h_2(x,a)+h_2(x,b)\right), \tag{1}$$

where $h_2(\cdot,\cdot)$ is given in Definition 11 and $M=\sup_{a<t<b}\left|f^{\triangle}(t)\right|<\infty$. This inequality is sharp in the sense that the right-hand side of (1) cannot be replaced by a smaller one.

The above theorem has been improved upon by many researchers. For more on this and related results, we refer the interested reader to the papers [6, 9, 12, 13, 14, 16, 19, 20, 21] and the references therein.

For functions whose second derivatives are bounded, Liu and Ngô [11] obtained the following perturbed version of Theorem 2.

THEOREM 3. *Let $a,b,x,t\in\mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and $f^{\Delta\Delta}:(a,b)\to\mathbb{R}$ is bounded, that is, $M:=\sup_{a<t<b}|f^{\Delta\Delta}(t)|<\infty$. Then for all $x\in[a,b]$, we have*

$$\left|\int_a^b f^{\sigma}(t)\Delta - f^{\sigma}(x)(b-a)+\big(h_2(x,a)-h_2(x,b)\big)f^{\Delta}(x)\right| \leqslant M\big(h_3(x,a)-h_3(x,b)\big).$$

Within the same period, Liu et al. [10] derived the following generalization of Theorem 3 for k points $x_1,x_2,\ldots,x_k$.

THEOREM 4. *Suppose that*

1. $a,b\in\mathbb{T}$, $I_k: a=x_0<x_1<\cdots<x_{k-1}<x_k=b$ *is a partition of the interval* $[a,b]$ *for* $x_0,x_1,\cdots,x_k\in\mathbb{T}$,
2. $\alpha_i\in\mathbb{T}$ $(i=0,1,\cdots,k+1)$ *is* $k+2$ *points so that* $\alpha_0=a$, $\alpha_i\in[x_{i-1},x_i]$ $(i=1,\cdots,k)$ *and* $\alpha_{k+1}=b$,
3. $f:[a,b]\to\mathbb{R}$ *is a twice differentiable function on* (a,b) *and* $f^{\Delta\Delta}:(a,b)\to\mathbb{R}$ *is bounded, that is,* $M:=\sup_{a<t<b}|f^{\Delta\Delta}(t)|<\infty$.

Then, we have the following inequality

$$\left|\int_a^b f^{\sigma}(t)\Delta t-\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f^{\sigma}(x_i)+\sum_{i=0}^{k-1}\left[h_2(x_{i+1},\alpha_{i+1})f^{\Delta}(x_{i+1})-h_2(x_i,\alpha_{i+1})f^{\Delta}(x_i)\right]\right|$$
$$\leqslant M\sum_{i=0}^{k-1}\Big[h_3(x_{i+1},\alpha_{i+1})-h_3(x_i,\alpha_{i+1})\Big].$$

The main objective of this work is the following: by introducing a parameter $\lambda\in[0,1]$, we further generalize Theorem 4 so that when $\lambda=0$, we recapture Theorem 4 and for $\lambda\in(0,1]$, we obtain completely new results. In addition, we also derive many interesting inequalities by applying our theorem to the continuous, discrete, and quantum time scales.

This present paper is organized as follows: Section 2 lays a brief background of the theory of time scales. In Section 3, we formulate and prove our main result. Thereafter, we conclude by applying our theorem to different time scales in Section 4.

2. Time scale essentials

We now present a brief overview of the theory of time scales. For an indepth study, we invite the interested reader to see references [1, 2].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. Since a time scale may not be connected, we need the following concept of jump operators.

The forward *jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward *jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set. The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ in this manner: if $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called *dense*. We also introduce the sets $\mathbb{T}^\kappa$, $\mathbb{T}_\kappa$, and $\mathbb{T}^\kappa_\kappa$, which are derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_\kappa = \mathbb{T} \setminus \{t_2\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Finally, we define $\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

For $a, b \in \mathbb{T}$ with $a \leqslant b$, we define the interval $[a,b]$ in $\mathbb{T}$ by $[a,b] = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. Open intervals and half-open intervals are defined in the same manner.

DEFINITION 5. The function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined as $f^\sigma(t) = f(\sigma(t))$.

DEFINITION 6. (Delta derivative) Assume $f : \mathbb{T} \to \mathbb{R}$ is a function. Then the delta derivative $f^\Delta(t) \in \mathbb{R}$ at $t \in \mathbb{T}^\kappa$ is defined to be the number (provided it exists) with property that for any $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\left| f^\sigma(t) - f(s) - f^\Delta(t)\,[\sigma(t) - s] \right| \leqslant \varepsilon\,|\sigma(t) - s|, \quad \forall s \in U.$$

We call $f^\Delta(t)$ the delta derivative of f at t. Moreover, we say that f is delta differentiable (or in short: differentiable) on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^\kappa$.

If $\mathbb{T}=\mathbb{R}$, then $f^{\Delta}(t)=\dfrac{df(t)}{dt}$, and if $\mathbb{T}=\mathbb{Z}$, then $f^{\Delta}(t)=f(t+1)-f(t)$.

THEOREM 7. *Let $f,g:\mathbb{T}\to\mathbb{R}$ be two differentiable functions at $t\in\mathbb{T}^{\kappa}$. Then the product $fg:\mathbb{T}\to\mathbb{R}$ is also differentiable at t with*

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f^{\sigma}(t)g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g^{\sigma}(t).$$

DEFINITION 8. The function $f:\mathbb{T}\to\mathbb{R}$ is said to be rd$-$continuous if it is continuous at all right-dense points $t\in\mathbb{T}$ and its left-sided limits exist at all left-dense points $t\in\mathbb{T}$.

DEFINITION 9. Let f be a rd$-$continuous function. Then $g:\mathbb{T}\to\mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\Delta}(t)=f(t)$ for any $t\in\mathbb{T}^{\kappa}$. In this case, we have

$$\int_a^b f(s)\Delta s=g(b)-g(a).$$

THEOREM 10. *If $a,b,c\in\mathbb{T}$ with $a<c<b$, $\alpha\in\mathbb{R}$ and f,g are* rd*-continuous, then*

(i) $\int_a^b[f(t)+g(t)]\Delta t=\int_a^b f(t)\Delta t+\int_a^b g(t)\Delta t$.

(ii) $\int_a^b \alpha f(t)\Delta t=\alpha\int_a^b f(t)\Delta t$.

(iii) $\int_a^b f(t)\Delta t=-\int_b^a f(t)\Delta t$.

(iv) $\int_a^b f(t)\Delta t=\int_a^c f(t)\Delta t+\int_c^b f(t)\Delta t$.

(v) $\left|\int_a^b f(t)\Delta t\right|\leqslant\int_a^b|f(t)|\Delta t$.

(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t=(fg)(b)-(fg)(a)-\int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t$.

(vii) *If $f(t)\geqslant 0$ for all $a\leqslant t\leqslant b$, then $\int_a^b f(t)\Delta t\geqslant 0$.*

DEFINITION 11. Let $h_k:\mathbb{T}^2\to\mathbb{R},\ k\in\mathbb{N}$ be functions that are recursively defined as

$$h_0(t,s)=1$$

and

$$h_{k+1}(t,s)=\int_s^t h_k(\tau,s)\Delta\tau \textit{ for all } s,t\in\mathbb{T}.$$

In view of the above definition and item (vii) of Theorem 10, we make the following important remarks that will be needed in the proof of our main result.

1. If $s\leqslant t$, then $h_{k+1}(t,s)\geqslant 0$ for all $t,s\in\mathbb{T}$.
2. If we let $h_k^{\Delta}(t,s)$ denote for each fixed s the derivative of $h_k(t,s)$ with respect to t, then
$$h_k^{\Delta}(t,s)=h_{k-1}(t,s),\ \text{for } k\in\mathbb{N},\ t\in\mathbb{T}^{\kappa}.$$

3. Main results

For the proof of our main result, we will need the following lemma.

LEMMA 12. (Generalized Montgomery identity with a parameter) Suppose that

1. $a,b \in \mathbb{T}$, $\lambda \in [0,1]$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ is a partition of the interval $[a,b]$ for $x_0, x_1, \cdots, x_k \in \mathbb{T}$,

2. $\alpha_i \in \mathbb{T}$ $(i = 0,1,\cdots,k+1)$ is $k+2$ points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1,\cdots,k)$ and $\alpha_{k+1} = b$,

3. $f : [a,b] \to \mathbb{R}$ is a twice differentiable function on (a,b).

Then we have the following equation

$$\begin{aligned}
&\int_a^b K(t,I_k) f^{\Delta\Delta}(t)\Delta t + (1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i) f^{\sigma}(x_i) + \lambda \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f^{\sigma}(\alpha_i)+f^{\sigma}(\alpha_{i+1})}{2} \\
&= \sum_{i=0}^{k-1}\left[h_2\left(\alpha_{i+1}, \alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right) - h_2\left(\alpha_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right] f^{\Delta}(\alpha_{i+1}) \\
&\quad + \sum_{i=0}^{k-1}\left[h_2\left(x_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right) f^{\Delta}(x_{i+1}) - h_2\left(x_i, \alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right) f^{\Delta}(x_i)\right] \\
&\quad + \int_a^b f^{\sigma}(t)\Delta t, \qquad (2)
\end{aligned}$$

where

$$K(t,I_k) = \begin{cases}
h_2\left(t, \alpha_1 - \lambda\frac{a-\alpha_1}{2}\right), & t \in [a,\alpha_1), \\
h_2\left(t, \alpha_1 + \lambda\frac{\alpha_2-\alpha_1}{2}\right), & t \in [\alpha_1, x_1), \\
h_2\left(t, \alpha_2 - \lambda\frac{\alpha_2-\alpha_1}{2}\right), & t \in [x_1, \alpha_2), \\
\vdots & \\
h_2\left(t, \alpha_{k-1} + \lambda\frac{\alpha_k-\alpha_{k-1}}{2}\right), & t \in [\alpha_{k-1}, x_{k-1}), \\
h_2\left(t, \alpha_k - \lambda\frac{\alpha_k-\alpha_{k-1}}{2}\right), & t \in [x_{k-1}, \alpha_k), \\
h_2\left(t, \alpha_k + \lambda\frac{\alpha_{k+1}-\alpha_k}{2}\right), & t \in [\alpha_k, b],
\end{cases}$$

provided for each $i \in \{0,1,2,\ldots,k-1\}$, $\alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}$ and $\alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}$ belong to $\mathbb{T}$.

Proof. In this proof, we will frequently switch between $f(\sigma(\cdot))$ and $f^{\sigma}(\cdot)$ as it deems convenient. Now, we apply twice item (vi) of Theorem 10 and the remarks given in Definition 11 to get

$$
\begin{aligned}
&\int_a^b K(t,I_k)f^{\Delta\Delta}(t)\Delta t\\
&=\sum_{i=0}^{k-1}\Bigg[\int_{x_i}^{\alpha_{i+1}} h_2\Big(t,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta\Delta}(t)\Delta t\\
&\quad+\int_{\alpha_{i+1}}^{x_{i+1}} h_2\Big(t,\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta\Delta}(t)\Delta t\Bigg]\\
&=\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(\alpha_{i+1})-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(x_i)\\
&\quad+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(x_{i+1})-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(\alpha_{i+1})\\
&\quad-\int_{x_i}^{\alpha_{i+1}} h_2^{\Delta}\Big(t,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(\sigma(t))\Delta t\\
&\quad-\int_{\alpha_{i+1}}^{x_{i+1}} h_2^{\Delta}\Big(t,\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(\sigma(t))\Delta t\Bigg]\\
&=\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(\alpha_{i+1})-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(x_i)\\
&\quad+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(x_{i+1})-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(\alpha_{i+1})\Bigg]\\
&\quad-\sum_{i=0}^{k-1}\Bigg[\int_{x_i}^{\alpha_{i+1}}\Big(t-\Big(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\Big)f^{\Delta}(\sigma(t))\Delta t\\
&\quad+\int_{\alpha_{i+1}}^{x_{i+1}}\Big(t-\Big(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big)f^{\Delta}(\sigma(t))\Delta t\Bigg]\\
&=\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(\alpha_{i+1})-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(x_i)\\
&\quad+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(x_{i+1})-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(\alpha_{i+1})\Bigg]\\
&\quad-\sum_{i=0}^{k-1}\Bigg[\Big(\alpha_{i+1}-\Big(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\Big)f^{\sigma}(\alpha_{i+1})-\Big(x_i-\Big(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\Big)f^{\sigma}(x_i)\\
&\quad-\int_{x_i}^{\alpha_{i+1}} f^{\sigma}(t)\Delta t+\Big(x_{i+1}-\Big(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big)f^{\sigma}(x_{i+1})\\
&\quad-\Big(\alpha_{i+1}-\Big(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big)f^{\sigma}(\alpha_{i+1})-\int_{\alpha_{i+1}}^{x_{i+1}} f^{\sigma}(t)\Delta t\Bigg]\\
&=\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(\alpha_{i+1})-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(x_i)\\
&\quad+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(x_{i+1})-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(\alpha_{i+1})\Bigg]
\end{aligned}
$$

$$-\Bigg[\sum_{i=0}^{k-1}\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f^{\sigma}(\alpha_{i+1})-x_0f^{\sigma}(x_0)+x_kf^{\sigma}(x_k)+\sum_{i=0}^{k-1}\alpha_{i+1}\Big(f^{\sigma}(x_i)-f^{\sigma}(x_{i+1})\Big)$$
$$+\sum_{i=0}^{k-1}-\frac{\lambda}{2}\Big((\alpha_{i+1}-\alpha_i)f^{\sigma}(x_i)+(\alpha_{i+2}-\alpha_{i+1})f^{\sigma}(x_{i+1})\Big)-\int_a^b f^{\sigma}(t)\Delta t\Bigg]$$
$$=\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(\alpha_{i+1})-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(x_i)$$
$$+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(x_{i+1})-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(\alpha_{i+1})\Bigg]$$
$$-\Bigg[\sum_{i=0}^{k-1}\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f^{\sigma}(\alpha_{i+1})+(\alpha_1-a)f^{\sigma}(a)+(b-\alpha_k)f^{\sigma}(b)+\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f^{\sigma}(x_i)$$
$$-\frac{\lambda}{2}\Big((\alpha_1-a)f^{\sigma}(a)+(b-\alpha_k)f^{\sigma}(b)+2\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f^{\sigma}(x_i)\Big)-\int_a^b f^{\sigma}(t)\Delta t\Bigg]$$
$$=\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(\alpha_{i+1})-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(x_i)$$
$$+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(x_{i+1})-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(\alpha_{i+1})\Bigg]$$
$$-\Bigg[\sum_{i=0}^{k-1}\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f^{\sigma}(\alpha_{i+1})+(1-\lambda)\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f^{\sigma}(x_i)$$
$$+\Big(1-\frac{\lambda}{2}\Big)\Big((\alpha_1-a)f^{\sigma}(a)+(b-\alpha_k)f^{\sigma}(b)\Big)-\int_a^b f^{\sigma}(t)\Delta t\Bigg].$$

Thus, we have that

$$\int_a^b K(t,I_k)f^{\Delta\Delta}(t)\Delta t$$
$$=\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(\alpha_{i+1})-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(x_i)$$
$$+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(x_{i+1})-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(\alpha_{i+1})\Bigg]$$
$$-\Bigg[\sum_{i=0}^{k-1}\frac{\lambda}{2}(\alpha_{i+2}-\alpha_i)f^{\sigma}(\alpha_{i+1})+(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f^{\sigma}(x_i)$$
$$+\frac{\lambda}{2}\Big((\alpha_1-a)f^{\sigma}(a)+(b-\alpha_k)f^{\sigma}(b)\Big)-\int_a^b f^{\sigma}(t)\Delta t\Bigg].$$

Here, we observe that

$$\sum_{i=0}^{k-1}(\alpha_{i+2}-\alpha_i)f^{\sigma}(\alpha_{i+1})=\sum_{i=0}^{k-1}(\alpha_{i+2}-\alpha_{i+1})f^{\sigma}(\alpha_{i+1})+\sum_{i=0}^{k-1}(\alpha_{i+1}-\alpha_i)f^{\sigma}(\alpha_{i+1})$$
$$=\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f^{\sigma}(\alpha_i)+f^{\sigma}(\alpha_{i+1})\Big)-\Big[(\alpha_1-\alpha_0)f^{\sigma}(\alpha_0)+(\alpha_{k+1}-\alpha_k)f^{\sigma}(\alpha_{k+1})\Big].$$

This implies that,

$$\sum_{i=0}^{k-1}\frac{\lambda}{2}(\alpha_{i+2}-\alpha_i)f^{\sigma}(\alpha_{i+1})$$
$$=\lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f^{\sigma}(\alpha_i)+f^{\sigma}(\alpha_{i+1})}{2}-\frac{\lambda}{2}\Big((\alpha_1-a)f^{\sigma}(a)+(b-\alpha_k)f^{\sigma}(b)\Big).$$

Hence, we have

$$\int_a^b K(t,I_k)f^{\Delta\Delta}(t)\Delta t$$
$$=\sum_{i=0}^{k-1}\Bigg[\bigg(h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\bigg)f^{\Delta}(\alpha_{i+1})$$
$$+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(x_{i+1})-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(x_i)\Bigg]$$
$$-\Bigg[(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f^{\sigma}(x_i)+\lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f^{\sigma}(\alpha_i)+f^{\sigma}(\alpha_{i+1})}{2}-\int_a^b f^{\sigma}(t)\Delta t\Bigg].$$

The identity follows by rearranging the terms. □

We are now ready to formulate and prove our main result.

THEOREM 13. (Ostrowski inequality on time scales with a parameter) *Suppose that*

1. $a,b\in\mathbb{T}$, $\lambda\in[0,1]$, $I_k: a=x_0<x_1<\cdots<x_{k-1}<x_k=b$ *is a partition of the interval* $[a,b]$ *for* $x_0,x_1,\cdots,x_k\in\mathbb{T}$,
2. $\alpha_i\in\mathbb{T}$ $(i=0,1,\cdots,k+1)$ *is* $k+2$ *points so that* $\alpha_0=a$, $\alpha_i\in[x_{i-1},x_i]$ $(i=1,\cdots,k)$ *and* $\alpha_{k+1}=b$,
3. $f:[a,b]\to\mathbb{R}$ *is a twice differentiable function on* (a,b) *and* $f^{\Delta\Delta}:(a,b)\to\mathbb{R}$ *is bounded, that is,* $M:=\sup\limits_{a<t<b}|f^{\Delta\Delta}(t)|<\infty$.

Then, we have the following inequality.

$$\Bigg|\int_a^b f^{\sigma}(t)\Delta t-(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f^{\sigma}(x_i)-\lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f^{\sigma}(\alpha_i)+f^{\sigma}(\alpha_{i+1})}{2}$$
$$+\sum_{i=0}^{k-1}\Bigg[\bigg(h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\bigg)f^{\Delta}(\alpha_{i+1})$$
$$+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^{\Delta}(x_{i+1})-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^{\Delta}(x_i)\Bigg]\Bigg|$$

$$\leqslant M\sum_{i=0}^{k-1}\Big[h_3\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_3\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$
$$+h_3\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)-h_3\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big],$$

provided for each $i\in\{0,1,2,\ldots,k-1\}$, $\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}$ *and* $\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}$ *belong to* $\mathbb{T}$.

Proof. Applying Lemma 12 and the items in Theorem 10, we get

$$\Bigg|\int_a^b f^\sigma(t)\Delta t-(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f^\sigma(x_i)-\lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f^\sigma(\alpha_i)+f^\sigma(\alpha_{i+1})}{2}$$
$$+\sum_{i=0}^{k-1}\Big[\Big(h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big)f^\Delta(\alpha_{i+1})$$
$$+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)f^\Delta(x_{i+1})-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)f^\Delta(x_i)\Big]\Bigg|$$
$$\leqslant\int_a^b\big|K(t,I_k)\big|\big|f^{\Delta\Delta}(t)\big|\Delta t$$
$$\leqslant M\sum_{i=0}^{k-1}\Big[\int_{x_i}^{\alpha_{i+1}}\Big|h_2\Big(t,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\Big|\Delta t+\int_{\alpha_{i+1}}^{x_{i+1}}\Big|h_2\Big(t,\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big|\Delta t\Big]$$
$$=M\sum_{i=0}^{k-1}\Big[\int_{\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}}^{\alpha_{i+1}}h_2\Big(t,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\Delta t$$
$$-\int_{\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}}^{x_i}h_2\Big(t,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\Delta t$$
$$+\int_{\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}}^{x_{i+1}}h_2\Big(t,\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Delta t$$
$$-\int_{\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}}^{\alpha_{i+1}}h_2\Big(t,\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Delta t\Big]$$
$$=M\sum_{i=0}^{k-1}\Big[h_3\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_3\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$
$$+h_3\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)-h_3\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big].$$

Hence, the proof is complete. □

REMARK 14. For $\lambda=0$, we recover Theorem 4.

COROLLARY 15. *Suppose that* $\alpha\in[a,b]\cap\mathbb{T}$. *Then we have that for all* $\lambda\in[0,1]$,

$$\Bigg|\int_a^b f^\sigma(t)\Delta t-(1-\lambda)\Big((\alpha-a)f^\sigma(a)+(b-\alpha)f^\sigma(b)\Big)$$
$$-\lambda\Big((\alpha-a)\frac{f^\sigma(a)+f^\sigma(\alpha)}{2}+(b-\alpha)\frac{f^\sigma(\alpha)+f^\sigma(b)}{2}\Big)$$
$$+\Big(h_2\Big(\alpha,\alpha-\lambda\frac{\alpha-a}{2}\Big)-h_2\Big(\alpha,\alpha+\lambda\frac{b-\alpha}{2}\Big)\Big)f^\Delta(\alpha)+h_2\Big(b,\alpha+\lambda\frac{b-\alpha}{2}\Big)f^\Delta(b)$$
$$-h_2\Big(a,\alpha-\lambda\frac{\alpha-a}{2}\Big)f^\Delta(a)\Bigg|$$
$$\leqslant M\Big[h_3\Big(\alpha,\alpha-\lambda\frac{\alpha-a}{2}\Big)-h_3\Big(a,\alpha-\lambda\frac{\alpha-a}{2}\Big)+h_3\Big(b,\alpha+\lambda\frac{b-\alpha}{2}\Big)$$
$$-h_3\Big(\alpha,\alpha+\lambda\frac{b-\alpha}{2}\Big)\Big],$$

provided $\alpha-\lambda\frac{\alpha-a}{2}$ *and* $\alpha+\lambda\frac{b-\alpha}{2}$ *belong to* $\mathbb{T}$ *and* $M=\sup_{a<t<b}\left|f^{\Delta\Delta}(t)\right|<\infty$.

Proof. The result follows directly from Theorem 13 by taking $k=1$ and choosing $x_0=a,\ x_1=b,\ \alpha_0=a,\ \alpha_1=\alpha$ and $\alpha_2=b$. □

REMARK 16. If we choose $\lambda=0$ in Corollary 15, then we obtain the inequality in Proposition 4.1 in [10]. To make this paper self content, we present the inequality here:

$$\left|\int_a^b f^\sigma(t)\Delta t-\Big((\alpha-a)f^\sigma(a)+(b-\alpha)f^\sigma(b)\Big)+h_2(b,\alpha)f^\Delta(b)-h_2(a,\alpha)f^\Delta(a)\right|$$
$$\leqslant M\Big(h_3(b,\alpha)-h_3(a,\alpha)\Big).$$

COROLLARY 17. *Suppose that* $x\in[a,b]\cap\mathbb{T}$, $\alpha_1\in[a,x]\cap\mathbb{T}$ *and* $\alpha_2\in[x,b]\cap\mathbb{T}$. *Then we have the inequality*

$$\Bigg|\int_a^b f^\sigma(t)\Delta t-(1-\lambda)\Big((\alpha_1-a)f^\sigma(a)+(\alpha_2-\alpha_1)f^\sigma(x)+(b-\alpha_2)f^\sigma(b)\Big)$$
$$-\lambda\Big((\alpha_1-a)\frac{f^\sigma(a)+f^\sigma(\alpha_1)}{2}+(\alpha_2-\alpha_1)\frac{f^\sigma(\alpha_1)+f^\sigma(\alpha_2)}{2}+(b-\alpha_2)\frac{f^\sigma(\alpha_2)+f^\sigma(b)}{2}\Big)$$
$$+\Big(h_2\Big(\alpha_1,\alpha_1-\lambda\frac{\alpha_1-a}{2}\Big)-h_2\Big(\alpha_1,\alpha_1+\lambda\frac{\alpha_2-\alpha_1}{2}\Big)\Big)f^\Delta(\alpha_1)$$
$$+\Big(h_2\Big(\alpha_2,\alpha_2-\lambda\frac{\alpha_2-\alpha_1}{2}\Big)-h_2\Big(\alpha_2,\alpha_2+\lambda\frac{b-\alpha_2}{2}\Big)\Big)f^\Delta(\alpha_2)$$
$$+h_2\Big(x,\alpha_1+\lambda\frac{\alpha_2-\alpha_1}{2}\Big)f^\Delta(x)-h_2\Big(a,\alpha_1-\lambda\frac{\alpha_1-a}{2}\Big)f^\Delta(a)$$

$$
\begin{aligned}
&+h_2\Big(b,\alpha_2+\lambda\frac{b-\alpha_2}{2}\Big)f^\Delta(b)-h_2\Big(x,\alpha_2-\lambda\frac{\alpha_2-\alpha_1}{2}\Big)f^\Delta(x)\Bigg|\\
&\leqslant M\Big[h_3\Big(\alpha_1,\alpha_1-\lambda\frac{\alpha_1-a}{2}\Big)-h_3\Big(a,\alpha_1-\lambda\frac{\alpha_1-a}{2}\Big)\\
&+h_3\Big(\alpha_2,\alpha_2-\lambda\frac{\alpha_2-\alpha_1}{2}\Big)-h_3\Big(x,\alpha_2-\lambda\frac{\alpha_2-\alpha_1}{2}\Big)\\
&+h_3\Big(x,\alpha_1+\lambda\frac{\alpha_2-\alpha_1}{2}\Big)-h_3\Big(\alpha_1,\alpha_1+\lambda\frac{\alpha_2-\alpha_1}{2}\Big)\\
&+h_3\Big(b,\alpha_2+\lambda\frac{b-\alpha_2}{2}\Big)-h_3\Big(\alpha_2,\alpha_2+\lambda\frac{b-\alpha_2}{2}\Big)\Big],
\end{aligned}
$$

provided for all $\lambda\in[0,1]$ *and for each* $i\in\{0,1\}$, $\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}$ *and* $\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}$ *belong to* $\mathbb{T}$ *and* $M=\sup_{a<t<b}\left|f^{\Delta\Delta}(t)\right|<\infty$.

Proof. The result follows directly from Theorem 13 by taking $k=2$ and choosing $x_0=a$, $x_1=x$, $x_2=b$, $\alpha_0=a$ and $\alpha_3=b$. □

REMARK 18. If we choose $\lambda=0$ in Corollary 17, then we obtain the inequality in Proposition 4.3 in [10] which states as follows:

$$
\begin{aligned}
&\Bigg|\int_a^b f^\sigma(t)\Delta t-\Big((\alpha_1-a)f^\sigma(a)+(\alpha_2-\alpha_1)f^\sigma(x)+(b-\alpha_2)f^\sigma(b)\Big)\\
&\qquad+h_2(x,\alpha_1)f^\Delta(x)-h_2(a,\alpha_1)f^\Delta(a)+h_2(b,\alpha_2)f^\Delta(b)-h_2(x,\alpha_2)f^\Delta(x)\Bigg|\\
&\leqslant M\Big(h_3(x,\alpha_1)-h_3(a,\alpha_1)+h_3(b,\alpha_2)-h_3(x,\alpha_2)\Big).
\end{aligned}
$$

4. Application to different time scales

In this section, we apply our theorem to the continuous, discrete, and quantum calculus to obtain the following results.

COROLLARY 19. (Continuous case) *Let* $\mathbb{T}=\mathbb{R}$. *Then we have the inequality,*

$$
\begin{aligned}
&\Bigg|\int_a^b f(t)dt-(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)-\lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f(\alpha_i)+f(\alpha_{i+1})}{2}\\
&+\sum_{i=0}^{k-1}\Bigg[\frac{\lambda^2}{8}\Big(\Big(\alpha_{i+1}-\alpha_i\Big)^2-\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big)f'(\alpha_{i+1})\\
&+\frac{1}{8}\Big(\Big(2x_{i+1}-\lambda\alpha_{i+2}+(\lambda-2)\alpha_{i+1}\Big)^2f'(x_{i+1})-\Big(2x_i-\lambda\alpha_i+(\lambda-2)\alpha_{i+1}\Big)^2f'(x_i)\Big)\Bigg]\Bigg|\\
&\leqslant\frac{M}{48}\Bigg[\sum_{i=0}^{k-1}\Big(\Big(2x_{i+1}-\lambda\alpha_{i+2}+(\lambda-2)\alpha_{i+1}\Big)^3-\Big(2x_i-\lambda\alpha_i+(\lambda-2)\alpha_{i+1}\Big)^3\Big)
\end{aligned}
$$

$$+\lambda^3\Big(\Big(\alpha_1-a\Big)^3-\Big(b-\alpha_k\Big)^3\Big)\Big],$$

where

$$M=\sup_{a<t<b}|f''(t)|<\infty.$$

Proof. The proof follows directly from Theorem 13 and using the fact that $f^\sigma(t)=f(t)$ and $h_k(t,s)=\dfrac{(t-s)^k}{k!}$. $\square$

COROLLARY 20. (Discrete case) *Let* $\mathbb{T}=\mathbb{Z}, a=0, b=n$ *and suppose*

1) $\mathbb{I}_k:=\{j_0,j_1,\cdots,j_k\}\subset\mathbb{Z}$, *where* $a=j_0<j_1<\cdots<j_k=b$, *is a partition of the set* $[0,n]\cap\mathbb{Z}$

2) $\{\alpha_0,\alpha_1,\cdots,\alpha_{k+1}\}\subset\mathbb{Z}$ *is a set of* $k+2$ *points such that* $\alpha_0=0$, $\alpha_i\in[j_{i-1},j_i]$ *for* $i=1,2,\cdots,k$ *and* $\alpha_{k+1}=n$;

3) $f(k)=x_k$.

We have the inequality,

$$\Bigg|\sum_{j=1}^{n}x_j-(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)x_{j_i+1}-\lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{x_{\alpha_i+1}+x_{\alpha_{i+1}+1}}{2}$$
$$+\sum_{i=0}^{k-1}\Bigg[\Bigg(h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Bigg)\Delta x_{\alpha_{i+1}}$$
$$+h_2\Big(j_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Delta x_{j_{i+1}}-h_2\Big(j_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\Delta x_{j_i}\Bigg]\Bigg|$$
$$\leqslant M\sum_{i=0}^{k-1}\Bigg[h_3\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_3\Big(j_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$
$$+h_3\Big(j_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)-h_3\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Bigg],$$

where

$$\Delta x_j=x_{j+1}-x_j,\quad M=\sup_{1\leqslant i\leqslant n}\Delta^2x_i<\infty \text{ and } h_k(t,s)=\binom{t-s}{k},\ \forall t,s\in\mathbb{Z}.$$

COROLLARY 21. (Quantum case) *Let* $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, $a=q^m$, $b=q^n$ *with* $m,n\in\mathbb{N}$ *and* $m<n$. *Suppose that*

1) $\mathbb{I}_k: q^m=q^{j_0}<q^{j_1}<\cdots<q^{j_k}=q^n$, *is a partition of the set* $[q^m,q^n]\cap q^{\mathbb{N}_0}$ *for* $j_0,j_1,\cdots,j_k\in\mathbb{N}$;

2) $q^{\alpha_i}\in q^{\mathbb{N}_0}$ $(i=0,1,\cdots,k+1)$ *is a set of* $k+2$ *points such that* $q^{\alpha_0}=q^m$, $q^{\alpha_i}\in[q^{j_{i-1}},q^{j_i}]\cap q^{\mathbb{N}_0}$ $(i=1,2,\cdots,k)$ *and* $q^{\alpha_{k+1}}=q^n$;

3) $f:[q^m,q^n]\to\mathbb{R}$ *is twice differentiable.*

Then we have the inequality,

$$\Bigg|\int_{q^m}^{q^n} f(qt)d_qt-(1-\lambda)\sum_{i=0}^{k}(q^{\alpha_{i+1}}-q^{\alpha_i})f(q^{j_i+1})-\lambda\sum_{i=0}^{k}(q^{\alpha_{i+1}}-q^{\alpha_i})\frac{f(q^{\alpha_i+1})+f(q^{\alpha_{i+1}+1})}{2}$$

$$+\sum_{i=0}^{k-1}\Bigg[\bigg(h_2\Big(q^{\alpha_{i+1}},q^{\alpha_{i+1}}-\lambda\frac{q^{\alpha_{i+1}}-q^{\alpha_i}}{2}\Big)-h_2\Big(q^{\alpha_{i+1}},q^{\alpha_{i+1}}+\lambda\frac{q^{\alpha_{i+2}}-q^{\alpha_{i+1}}}{2}\Big)\bigg)f^{\Delta}(q^{\alpha_{i+1}})$$

$$+h_2\Big(q^{j_{i+1}},q^{\alpha_{i+1}}+\lambda\frac{q^{\alpha_{i+2}}-q^{\alpha_{i+1}}}{2}\Big)f^{\Delta}(q^{j_{i+1}})-h_2\Big(q^{j_i},q^{\alpha_{i+1}}-\lambda\frac{q^{\alpha_{i+1}}-q^{\alpha_i}}{2}\Big)f^{\Delta}(q^{j_i})\Bigg]\Bigg|$$

$$\leqslant M\sum_{i=0}^{k-1}\Big[h_3\Big(q^{\alpha_{i+1}},q^{\alpha_{i+1}}-\lambda\frac{q^{\alpha_{i+1}}-q^{\alpha_i}}{2}\Big)-h_3\Big(q^{j_i},q^{\alpha_{i+1}}-\lambda\frac{q^{\alpha_{i+1}}-q^{\alpha_i}}{2}\Big)$$

$$+h_3\Big(q^{j_{i+1}},q^{\alpha_{i+1}}+\lambda\frac{q^{\alpha_{i+2}}-q^{\alpha_{i+1}}}{2}\Big)-h_3\Big(q^{\alpha_{i+1}},q^{\alpha_{i+1}}+\lambda\frac{q^{\alpha_{i+2}}-q^{\alpha_{i+1}}}{2}\Big)\Big],$$

where

$$f^{\Delta}(t)=\frac{f(qt)-f(t)}{(q-1)t},\quad M=\sup_{q^m<t<q^n}\frac{f(q^2t)-(q+1)f(qt)+qf(t)}{q(q-1)^2t^2}<\infty$$

and

$$h_k(t,s)=\prod_{\nu=0}^{k-1}\frac{t-q^{\nu}s}{\sum_{\mu=0}^{\nu}q^{\nu}},\ \ \forall\ t,s\in q^{\mathbb{N}_0}.$$

5. Conclusion

A new Montgomery identity, for functions with second delta derivatives, is established. Using this identity, we obtain an inequality (on time scales) of the Ostrowski type involving a parameter. By taking the parameter $\lambda=0$, we recapture Theorem 4. Furthermore, Theorem 13 is applied to different time scales to get more results in this direction. For some recent results related to the Ostrowski type inequalities via a parameter, we refer the interested reader to the papers [7, 8, 15, 17, 18] and the references therein.

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