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AN INEQUALITY FOR DISTANCES AMONG FIVE POINTS AND DISTANCE PRESERVING MAPPINGS

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Abstract. Using properties of norm and inner product, we prove a new inequality for distances between five points arbitrarily given in an inner product space. Moreover, we investigate the Aleksandrov-Rassias problem by proving that if the distance 1 is contractive and the golden ratio is extensive by a mapping f, then f is a linear isometry up to translation.

1. Introduction

In this paper, assume that V is a real (or complex) inner product space with the inner product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ is the norm on V defined as $\|x\| = \sqrt{\langle x,x\rangle}$ for all $x \in V$. When $\mathbb{E}^2$ is the two dimensional Euclidean space and if three points (vectors) x,y,z are the vertices of an acute triangle or a right triangle in $\mathbb{E}^2$, then the inequality

$$\|x-z\|^2 \leqslant \|x-y\|^2 + \|y-z\|^2$$

is true. Especially, the equality sign in the last inequality holds if and only if x,y,z are the vertices of a right triangle and the vectors $x-y$ and $y-z$ are orthogonal to each other. This is called the Pythagorean theorem which is one of the most famous theorems in mathematics.

In regard to this subject, Jung [5] and Jung and Lee [6] proved the following theorems dealing with inequalities for the distances between every two points among the given $2n$ points.

THEOREM 1.1. (Jung [5]) *For any real* (*or complex*) *inner product space V, the following inequality is true for all six points x_1, x_2, x_3, x_4, x_5, x_6 in V:*

$$\begin{aligned}
&2\left(\|x_6-x_1\|^2 + \|x_4-x_2\|^2 + \|x_5-x_3\|^2\right)\\
&\leqslant \|x_2-x_1\|^2 + \|x_6-x_2\|^2 + \|x_4-x_6\|^2 + \|x_1-x_4\|^2\\
&\quad + \|x_3-x_2\|^2 + \|x_4-x_3\|^2 + \|x_5-x_4\|^2 + \|x_2-x_5\|^2\\
&\quad + \|x_3-x_1\|^2 + \|x_6-x_3\|^2 + \|x_5-x_6\|^2 + \|x_1-x_5\|^2
\end{aligned}$$

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In addition, the equality sign is true if and only if each $\{x_2-x_1, x_6-x_2, x_4-x_6, x_1-x_4\}$, $\{x_3-x_1, x_6-x_3, x_5-x_6, x_1-x_5\}$, $\{x_3-x_2, x_4-x_3, x_5-x_4, x_2-x_5\}$ *comprises the sides of an appropriate (possibly degenerate) parallelogram such that* x_6-x_1 *and* x_4-x_2, *resp.,* x_6-x_1 *and* x_5-x_3, *resp.,* x_4-x_2 *and* x_5-x_3 *are the diagonals of the corresponding parallelogram.*

THEOREM 1.2. (Jung and Lee [6]) *Assume that* $n \geqslant 2$ *is an integer and* V *is a real (or complex) inner product space. The following inequality is true for any distinct* $2n$ *points* $x_{11}, x_{21}, \ldots, x_{n1}, x_{12}, x_{22}, \ldots, x_{n2}$ *in* V:

$$\sum_{\substack{i,j\in\{1,\ldots,n\}\\ k,\ell\in\{1,2\}\\ i<j}} \|x_{ik}-x_{j\ell}\|^2 \geqslant (n-1) \sum_{i\in\{1,\ldots,n\}} \|x_{i1}-x_{i2}\|^2$$

In addition, the equality sign holds if and only if for all $i, j \in \{1,\ldots,n\}$ *with* $i<j$, *the pair of four points* $\{x_{i1}, x_{i2}, x_{j1}, x_{j2}\}$ *comprises the vertices of an appropriate (possibly degenerate) parallelogram such that* x_{i1} *and* x_{j1} *are the opposite vertices to* x_{i2} *and* x_{j2}, *respectively.*

Moreover, Jung and Lee [6] made use of these inequalities to solve the Aleksandrov-Rassias problem. (We refer the reader to Section 3 of this paper for the exact definition of the Aleksandrov-Rassias problem.)

In Section 2 of this paper, using basic properties of norm and inner product, we prove a new inequality for distances between five points which are arbitrarily given in an inner product space. We devote Section 3 to a study of the Aleksandrov-Rassias problem. Indeed, we prove that if the distance 1 is contractive and the golden ratio $(\frac{1+\sqrt{5}}{2})$ is extensive by a mapping f, then f is a linear isometry up to translation, while Xiang proved in his paper [11] that if either distances 1 and $\sqrt{2}$ are preserved by a mapping f or 1 and $\sqrt{3}$ are preserved by f, then f is a linear isometry up to translation.

2. Inequality for distances among five points

From now on, we denote by ϕ the golden ratio, *i.e.*, $\phi = \frac{1+\sqrt{5}}{2}$. Then $\phi^2-\phi-1=0$, and ϕ is the ratio of a diagonal to a side in a regular pentagon. It is somewhat surprising that the golden ratio appears in an inequality for the distances between every two points among five points.

THEOREM 2.1. *For a real (or complex) inner product space* $(V, \langle\cdot,\cdot\rangle)$, *the inequality*

$$\begin{aligned}\phi^2\{\|x_1-x_2\|^2+\|x_2-x_3\|^2+\|x_3-x_4\|^2+\|x_4-x_5\|^2+\|x_5-x_1\|^2\}\\ \geqslant \|x_1-x_3\|^2+\|x_2-x_4\|^2+\|x_3-x_5\|^2+\|x_4-x_1\|^2+\|x_5-x_2\|^2\end{aligned} \tag{2.1}$$

is true for any five points x_1, x_2, x_3, x_4, x_5 *in* V. *Moreover, the equality sign holds if and only if*

$$x_4 = x_1 - \phi x_2 + \phi x_3 \quad \text{and} \quad x_5 = \phi x_1 - \phi x_2 + x_3 \tag{2.2}$$

for any $x_1, x_2, x_3 \in V$.

Proof. We will prove this theorem when V is a complex inner product space. For notational convenience, we set $x_6 = x_1$, $x_7 = x_2$ and $x_8 = x_3$. Let $S_j = \sum_{i=1}^{5} \langle x_i, x_{i+j} \rangle$ for each $j \in \{0,1,2\}$. Then for any $j \in \{0,1,2\}$, we get

$$\begin{aligned}\sum_{i=1}^{5} \|x_i - x_{i+j}\|^2 &= \sum_{i=1}^{5} \langle x_i - x_{i+j}, x_i - x_{i+j} \rangle \\ &= \sum_{i=1}^{5} \left\{ \langle x_i, x_i \rangle - \langle x_i, x_{i+j} \rangle - \overline{\langle x_i, x_{i+j} \rangle} + \langle x_{i+j}, x_{i+j} \rangle \right\} \\ &= \sum_{i=1}^{5} \left\{ 2\langle x_i, x_i \rangle - \langle x_i, x_{i+j} \rangle - \overline{\langle x_i, x_{i+j} \rangle} \right\} \\ &= 2S_0 - S_j - \overline{S}_j,\end{aligned} \tag{2.3}$$

where $\overline{c}$ denotes the complex conjugation of a complex number c.

On account of (2.3), $\phi^2 = \phi + 1$ and $2\phi = 1 + \sqrt{5}$, inequality (2.1) becomes

$$\begin{aligned}0 &\leqslant \phi^2 \sum_{i=1}^{5} \|x_i - x_{i+1}\|^2 - \sum_{i=1}^{5} \|x_i - x_{i+2}\|^2 \\ &= \phi^2(2S_0 - S_1 - \overline{S}_1) - (2S_0 - S_2 - \overline{S}_2) \\ &= (2\phi^2 - 2)S_0 - \phi^2(S_1 + \overline{S}_1) + (S_2 + \overline{S}_2) \\ &= 2\phi S_0 - (\phi + 1)(S_1 + \overline{S}_1) + (S_2 + \overline{S}_2),\end{aligned}$$

i.e., it is to prove that

$$(1 + \sqrt{5})S_0 - \frac{3 + \sqrt{5}}{2}(S_1 + \overline{S}_1) + (S_2 + \overline{S}_2) \geqslant 0. \tag{2.4}$$

Since

$$\begin{aligned}&\sum_{i=1}^{5} \langle x_i - x_{i+3}, x_{i+1} - x_{i+2} \rangle \\ &= \sum_{i=1}^{5} \left\{ \langle x_i, x_{i+1} \rangle + \overline{\langle x_{i+2}, x_{i+3} \rangle} - \overline{\langle x_{i+1}, x_{i+3} \rangle} - \langle x_i, x_{i+2} \rangle \right\} \\ &= \sum_{i=1}^{5} \langle x_i, x_{i+1} \rangle + \sum_{i=1}^{5} \overline{\langle x_i, x_{i+1} \rangle} - \sum_{i=1}^{5} \langle x_i, x_{i+2} \rangle - \sum_{i=1}^{5} \overline{\langle x_i, x_{i+2} \rangle} \\ &= (S_1 + \overline{S}_1) - (S_2 + \overline{S}_2),\end{aligned}$$

the following inequality is obviously true:

$$
\begin{aligned}
0 &\leqslant \sum_{i=1}^{5} \|x_i - x_{i+3} - \phi(x_{i+1} - x_{i+2})\|^2 \\
&= \sum_{i=1}^{5} \big\langle (x_i - x_{i+3}) - \phi(x_{i+1} - x_{i+2}),\, (x_i - x_{i+3}) - \phi(x_{i+1} - x_{i+2}) \big\rangle \\
&= \sum_{i=1}^{5} \|x_{i+3} - x_i\|^2 - \phi \sum_{i=1}^{5} \langle x_i - x_{i+3}, x_{i+1} - x_{i+2} \rangle \\
&\quad - \phi \sum_{i=1}^{5} \overline{\langle x_i - x_{i+3}, x_{i+1} - x_{i+2} \rangle} + \phi^2 \sum_{i=1}^{5} \|x_{i+1} - x_{i+2}\|^2 \\
&= \sum_{i=1}^{5} \|x_i - x_{i+2}\|^2 - 2\phi(S_1 + \overline{S}_1 - S_2 - \overline{S}_2) + \phi^2 \sum_{i=1}^{5} \|x_i - x_{i+1}\|^2 \\
&= (2\phi^2 + 2)S_0 - (\phi^2 + 2\phi)S_1 - (\phi^2 + 2\phi)\overline{S}_1 + (2\phi - 1)S_2 + (2\phi - 1)\overline{S}_2 \\
&= (5 + \sqrt{5})S_0 - \frac{5 + 3\sqrt{5}}{2} S_1 - \frac{5 + 3\sqrt{5}}{2} \overline{S}_1 + \sqrt{5} S_2 + \sqrt{5}\overline{S}_2
\end{aligned}
\tag{2.5}
$$

When we divide inequality (2.5) by $\sqrt{5}$, the resulting inequality is just the inequality (2.4), which is equivalent to our main inequality (2.1).

Equality condition. The right-hand side of (2.5) is 0 if and only if $x_i - x_{i+3} - \phi(x_{i+1} - x_{i+2}) = 0$ for all $i \in \{1, 2, \ldots, 5\}$, which is again equivalent to

$$
x_{i+3} = x_i - \phi x_{i+1} + \phi x_{i+2} \tag{2.6}
$$

for all $i \in \{1, 2, \ldots, 5\}$. Assume that the right-hand side of (2.5) is 0 and x_1, x_2, x_3 are given points in V. Then by (2.6), we have

$$
x_4 = x_1 - \phi x_2 + \phi x_3
$$

and

$$
\begin{aligned}
x_5 = x_2 - \phi x_3 + \phi x_4 &= x_2 - \phi x_3 + \phi(x_1 - \phi x_2 + \phi x_3) \\
&= \phi x_1 + (-\phi^2 + 1)x_2 + (\phi^2 - \phi)x_3 \\
&= \phi x_1 - \phi x_2 + x_3.
\end{aligned}
$$

Thus, the conditions in (2.2) are true.

On the other hand, we should check under the assumptions in (2.2) that the equation (2.6) is also true when $i \in \{1, 2, \ldots, 5\}$. Indeed, our assumption (2.2) implies that the equation (2.6) is true for $i \in \{1, 2\}$ as we see in the last paragraph. If $i = 3$, then

$$
\begin{aligned}
x_3 - \phi x_4 + \phi x_5 &= x_3 - \phi(x_1 - \phi x_2 + \phi x_3) + \phi(\phi x_1 - \phi x_2 + x_3) \\
&= (\phi^2 - \phi)x_1 + (\phi^2 - \phi^2)x_2 + (-\phi^2 + \phi + 1)x_3 \\
&= x_1,
\end{aligned}
$$

which is just the case of (2.6) for $i = 3$ (notice that $x_6 = x_1$, $x_7 = x_2$ and $x_8 = x_3$). For $i = 4$, we have

$$\begin{aligned} x_4 - \phi x_5 + \phi x_1 &= (x_1 - \phi x_2 + \phi x_3) - \phi(\phi x_1 - \phi x_2 + x_3) + \phi x_1 \\ &= (-\phi^2 + \phi + 1)x_1 + (\phi^2 - \phi)x_2 + (\phi - \phi)x_3 \\ &= x_2, \end{aligned}$$

which is just the case of (2.6) for $i = 4$. When $i = 5$, then we get

$$x_5 - \phi x_1 + \phi x_2 = (\phi x_1 - \phi x_2 + x_3) - \phi x_1 + \phi x_2 = x_3,$$

which is the case of (2.6) for $i = 5$.

Hence, equation (2.6) is true for every $i \in \{1, 2, \ldots, 5\}$. Since x_1, x_2, x_3 can be any element of V, the equality sign in (2.1) holds if and only if the conditions in (2.2) are true for any $x_1, x_2, x_3 \in V$. $\square$

3. Applications to Aleksandrov-Rassias problem

Assume that both V_1 and V_2 are either real normed spaces or complex normed spaces. A distance ρ is said to be contractive (or non-expanding) by a mapping $f : V_1 \to V_2$ if and only if $\|f(x) - f(y)\| \leqslant \rho$ for all $x, y \in V_1$ with $\|x - y\| = \rho$. In a similar way, we call a distance ρ extensive (or non-shrinking) by f if and only if $\|f(x) - f(y)\| \geqslant \rho$ for all $x, y \in V_1$ with $\|x - y\| = \rho$. Especially, ρ is said to be conservative (or preserved) provided ρ is contractive and extensive simultaneously.

Based on the fact that every distance ρ is conservative by an isometry, we may raise a question: *Is a mapping an isometry if the mapping preserves certain distances?* Indeed, Aleksandrov [1] raised a question whether a mapping $f : V_1 \to V_1$ is an isometry provided f preserves a distance ρ, which is known as the *Aleksandrov problem*. Without loss of generality, we may assume $\rho = 1$ when V_1 is a normed space (see [10]).

About twenty years earlier than Aleksandrov, Beckman and Quarles [2] investigated the Aleksandrov problem for the n-dimensional real Euclidean space $\mathbb{E}^n$.

THEOREM 3.1. (Beckman and Quarles [2]) *Assume that $n \geqslant 2$ is an integer and $\rho > 0$ is an arbitrary constant. Every mapping $f : \mathbb{E}^n \to \mathbb{E}^n$ preserving the distance ρ is a linear isometry up to translation.*

They could construct non-isometric mappings preserving unit distance for one-dimensional or for infinite-dimensional real Euclidean spaces (cf. [8]). Thereafter, Rassias [9] raised the question: *Is a mapping between normed spaces an isometry if it preserves two (or more) distances?* Such a problem is called the *Aleksandrov-Rassias problem*. For a strictly convex vector space, Benz gave an affirmative answer to this problem (see [3] and also [4]):

THEOREM 3.2. (Benz [3]) *Assume that V_1 is a real normed space with $\dim V_1 \geqslant 2$ and V_2 is a real normed space which is strictly convex. Suppose $N \geqslant 2$ is a fixed integer. If a distance ρ is contractive and $N\rho$ is extensive by a mapping $f : V_1 \to V_2$, then f is a linear isometry up to translation.*

Now, assume that V_1 is a real (or complex) inner product space and c_1, c_2, c_3, c_4, c_5, e_1, e_2, e_3, e_4, e_5 are positive numbers such that there exist points (vectors) x_1, x_2, x_3, x_4, x_5 of V_1 such that they satisfy the conditions in (2.2) as well as

$$\begin{aligned} &\|x_1 - x_2\| = c_1, \;\; \|x_2 - x_3\| = c_2, \;\; \|x_3 - x_4\| = c_3, \;\; \|x_4 - x_5\| = c_4, \;\; \|x_5 - x_1\| = c_5, \\ &\|x_1 - x_3\| = e_1, \;\; \|x_2 - x_4\| = e_2, \;\; \|x_3 - x_5\| = e_3, \;\; \|x_4 - x_1\| = e_4, \;\; \|x_5 - x_2\| = e_5, \end{aligned} \tag{3.1}$$

as we see in the following figure. (Obviously, due to (2.2), the five points x_1, x_2, x_3, x_4, x_5 lie on a two dimensional subspace of V_1.)

THEOREM 3.3. *Let V_1 and V_2 be either real inner product spaces or complex inner product spaces. Assume that the distances c_1, c_2, c_3, c_4, c_5 are contractive and the distances e_1, e_2, e_3, e_4, e_5 are extensive by a mapping $f : V_1 \to V_2$, where c_i's and e_i's are given by (3.1) and the corresponding x_i's satisfy the conditions in (2.2) (see Figure 3.1). Then f preserves all the distances c_i's and e_i's.*

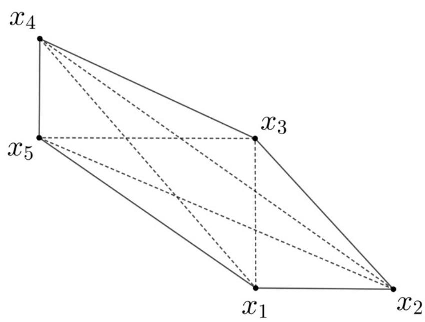

Figure 3.1: $x_4 = x_1 - \phi x_2 + \phi x_3$ *and* $x_5 = \phi x_1 - \phi x_2 + x_3$

Proof. First, we set $x_i' = f(x_i)$ for all $i \in \{1, 2, \ldots, 5\}$. Because the distances c_1, c_2, c_3, c_4, c_5 are contractive by f, we can use Theorem 2.1 to get

$$\begin{aligned} &\phi^2 \big\{ \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_4\|^2 + \|x_4 - x_5\|^2 + \|x_5 - x_1\|^2 \big\} \\ &\quad \geqslant \phi^2 \big\{ \|x_1' - x_2'\|^2 + \|x_2' - x_3'\|^2 + \|x_3' - x_4'\|^2 + \|x_4' - x_5'\|^2 + \|x_5' - x_1'\|^2 \big\} \\ &\quad \geqslant \|x_1' - x_3'\|^2 + \|x_2' - x_4'\|^2 + \|x_3' - x_5'\|^2 + \|x_4' - x_1'\|^2 + \|x_5' - x_2'\|^2. \end{aligned}$$

Moreover, since the distances e_1, e_2, e_3, e_4, e_5 are extensive, we consider (3.1) to obtain

$$\begin{aligned}
&\phi^2\{\|x_1-x_2\|^2+\|x_2-x_3\|^2+\|x_3-x_4\|^2+\|x_4-x_5\|^2+\|x_5-x_1\|^2\}\\
&\quad\geqslant \phi^2\{\|x_1'-x_2'\|^2+\|x_2'-x_3'\|^2+\|x_3'-x_4'\|^2+\|x_4'-x_5'\|^2+\|x_5'-x_1'\|^2\}\\
&\quad\geqslant \|x_1'-x_3'\|^2+\|x_2'-x_4'\|^2+\|x_3'-x_5'\|^2+\|x_4'-x_1'\|^2+\|x_5'-x_2'\|^2\\
&\quad\geqslant \|x_1-x_3\|^2+\|x_2-x_4\|^2+\|x_3-x_5\|^2+\|x_4-x_1\|^2+\|x_5-x_2\|^2.
\end{aligned}$$

Since x_4 and x_5 were given by (2.2) (see Figure 3.1), by the last two inequalities and Theorem 2.1, we conclude that

$$\begin{aligned}
&\phi^2\{\|x_1-x_2\|^2+\|x_2-x_3\|^2+\|x_3-x_4\|^2+\|x_4-x_5\|^2+\|x_5-x_1\|^2\}\\
&\quad\geqslant \phi^2\{\|x_1'-x_2'\|^2+\|x_2'-x_3'\|^2+\|x_3'-x_4'\|^2+\|x_4'-x_5'\|^2+\|x_5'-x_1'\|^2\}\\
&\quad\geqslant \|x_1'-x_3'\|^2+\|x_2'-x_4'\|^2+\|x_3'-x_5'\|^2+\|x_4'-x_1'\|^2+\|x_5'-x_2'\|^2\\
&\quad\geqslant \|x_1-x_3\|^2+\|x_2-x_4\|^2+\|x_3-x_5\|^2+\|x_4-x_1\|^2+\|x_5-x_2\|^2\\
&\quad= \phi^2\{\|x_1-x_2\|^2+\|x_2-x_3\|^2+\|x_3-x_4\|^2+\|x_4-x_5\|^2+\|x_5-x_1\|^2\}.
\end{aligned} \tag{3.2}$$

On the other hand, our hypotheses imply that

$$\begin{aligned}
&c_1=\|x_1-x_2\|\geqslant\|x_1'-x_2'\|, \quad c_2=\|x_2-x_3\|\geqslant\|x_2'-x_3'\|,\\
&c_3=\|x_3-x_4\|\geqslant\|x_3'-x_4'\|, \quad c_4=\|x_4-x_5\|\geqslant\|x_4'-x_5'\|,\\
&c_5=\|x_5-x_1\|\geqslant\|x_5'-x_1'\|, \quad e_1=\|x_1-x_3\|\leqslant\|x_1'-x_3'\|,\\
&e_2=\|x_2-x_4\|\leqslant\|x_2'-x_4'\|, \quad e_3=\|x_3-x_5\|\leqslant\|x_3'-x_5'\|,\\
&e_4=\|x_4-x_1\|\leqslant\|x_4'-x_1'\|, \quad e_5=\|x_5-x_2\|\leqslant\|x_5'-x_2'\|.
\end{aligned} \tag{3.3}$$

By combining (3.2) and (3.3), we conclude that

$$\begin{aligned}
&\|x_1-x_2\|=c_1=\|x_1'-x_2'\|, \quad \|x_2-x_3\|=c_2=\|x_2'-x_3'\|,\\
&\|x_3-x_4\|=c_3=\|x_3'-x_4'\|, \quad \|x_4-x_5\|=c_4=\|x_4'-x_5'\|,\\
&\|x_5-x_1\|=c_5=\|x_5'-x_1'\|, \quad \|x_1-x_3\|=e_1=\|x_1'-x_3'\|,\\
&\|x_2-x_4\|=e_2=\|x_2'-x_4'\|, \quad \|x_3-x_5\|=e_3=\|x_3'-x_5'\|,\\
&\|x_4-x_1\|=e_4=\|x_4'-x_1'\|, \quad \|x_5-x_2\|=e_5=\|x_5'-x_2'\|.
\end{aligned}$$

For arbitrarily given $x_1,x_2\in V_1$ with $\|x_1-x_2\|=c_1$, we can select three points (vectors) x_3,x_4,x_5 in V_1 such that x_1,x_2,x_3,x_4,x_5 determine a geometrical figure congruent to the one in Figure 3.1. In view of the above argument, we may conclude that $\|x_1'-x_2'\|=c_1$. For other distances such as c_2, c_3, c_4, c_5, e_1, e_2, e_3, e_4 and e_5, we can apply a similar argument. Therefore, f preserves the distances c_1, c_2, c_3, c_4, c_5, e_1, e_2, e_3, e_4 and e_5. □

REMARK 3.1. Assume that x_1, x_2, x_3, x_4, x_5 are the vertices of a regular pentagon with a unit side length as we see in Figure 3.2 below. If we let $c_1=c_2=c_3=c_4=c_5=1$ and $e_1=e_2=e_3=e_4=e_5=\phi$ in Theorem 3.3, then we see that f preserves the distances 1 and ϕ.

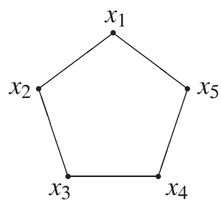

Figure 3.2: *regular pentagon*

THEOREM 3.4. *Assume that V_1 and V_2 are real Hilbert spaces with $\dim V_1 \geqslant 3$. If the distance 1 is contractive and the distance ϕ is extensive by a mapping $f : V_1 \to V_2$, then f is a linear isometry up to translation.*

Proof. By regarding Theorem 3.3 and Remark 3.1, f preserves both the distances 1 and ϕ. We will show that f preserves the distance $\sqrt{2}\phi$. Then because f preserves the distances ϕ and $\sqrt{2}\phi$, we can conclude that f is an isometry up to translation by [11, Theorem 2.8].

Assume that the distance between v_1 and v_3 of V_1 is $\sqrt{2}\phi$, *i.e.*, $\|v_1 - v_3\| = \sqrt{2}\phi$. Because $\dim V_1 \geqslant 3$, there exists a subspace U of V_1 containing v_1 and v_3 that is (inner product) isomorphic to 3-dimensional Euclidean space $\mathbb{E}^3$.

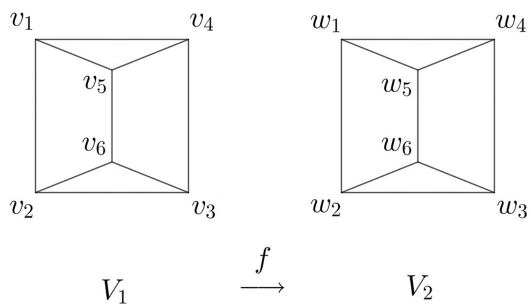

Figure 3.3.

In the first (roof-shaped) figure of Figure 3.3, $\{v_1, v_5, v_6, v_2\}$ is a part of vertices of a regular pentagon of unit side length in U and also $\{v_4, v_5, v_6, v_3\}$ is a part of vertices of another regular pentagon of unit side length in U such that $\|v_1 - v_4\| = \|v_2 - v_3\| = \phi$. In full detail, we have

$$\begin{aligned}
&\|v_1 - v_5\| = 1, \quad \|v_4 - v_5\| = 1, \quad \|v_5 - v_6\| = 1, \quad \|v_2 - v_6\| = 1, \quad \|v_3 - v_6\| = 1,\\
&\|v_1 - v_2\| = \phi, \quad \|v_2 - v_3\| = \phi, \quad \|v_3 - v_4\| = \phi, \quad \|v_4 - v_1\| = \phi,\\
&\|v_1 - v_6\| = \phi, \quad \|v_2 - v_5\| = \phi, \quad \|v_3 - v_5\| = \phi, \quad \|v_4 - v_6\| = \phi.
\end{aligned}$$

Denote $w_i = f(v_i)$ for $i \in \{1, 2, \ldots, 6\}$. Because f preserves the distances 1 and ϕ,

$$\|w_1-w_5\|=1,\quad \|w_4-w_5\|=1,\quad \|w_5-w_6\|=1,\quad \|w_2-w_6\|=1,\quad \|w_3-w_6\|=1,$$
$$\|w_1-w_2\|=\phi,\quad \|w_2-w_3\|=\phi,\quad \|w_3-w_4\|=\phi,\quad \|w_4-w_1\|=\phi,$$
$$\|w_1-w_6\|=\phi,\quad \|w_2-w_5\|=\phi,\quad \|w_3-w_5\|=\phi,\quad \|w_4-w_6\|=\phi. \tag{3.4}$$

Let $x_i = w_i - w_1$ for $i \in \{2,3,\dots,6\}$. Then

$$x_j - x_k = (w_j - w_1) - (w_k - w_1) = w_j - w_k \tag{3.5}$$

for any $j,k \in \{2,3,\dots,6\}$. The distances between 4 points w_1, w_2, w_5, and w_6 are given:

$$\begin{aligned} &\|x_2\| = \|w_2 - w_1\| = \phi,\ \|x_5 - x_2\| = \|w_5 - w_2\| = \phi,\\ &\|x_5\| = \|w_5 - w_1\| = 1,\ \|x_6 - x_2\| = \|w_6 - w_2\| = 1,\\ &\|x_6\| = \|w_6 - w_1\| = \phi,\ \|x_6 - x_5\| = \|w_6 - w_5\| = 1. \end{aligned} \tag{3.6}$$

Since

$$\|x_j - x_k\|^2 = \|x_j\|^2 - 2\langle x_j, x_k\rangle + \|x_k\|^2$$

for any $j,k \in \{2,3,\dots,6\}$, by using (3.6), we have

$$\langle x_2, x_5\rangle = \frac{1}{2}\left(\|x_2\|^2 + \|x_5\|^2 - \|x_2 - x_5\|^2\right) = \frac{1}{2}\left(\phi^2 + 1 - \phi^2\right) = \frac{1}{2}. \tag{3.7}$$

Similarly, we have $\langle x_2, x_6\rangle = \frac{1}{2}(2\phi^2 - 1)$ and $\langle x_5, x_6\rangle = \frac{1}{2}\phi^2$.

Hence, we can calculate $\|x_6 - x_5 - \frac{1}{\phi}x_2\|^2$:

$$\begin{aligned} \left\|x_6 - x_5 - \frac{1}{\phi}x_2\right\|^2 &= \left\langle x_6 - x_5 - \frac{1}{\phi}x_2,\ x_6 - x_5 - \frac{1}{\phi}x_2\right\rangle \\ &= \|x_6\|^2 + \|x_5\|^2 + \frac{1}{\phi^2}\|x_2\|^2 - 2\langle x_5, x_6\rangle - \frac{2}{\phi}\langle x_2, x_6\rangle + \frac{2}{\phi}\langle x_2, x_5\rangle \\ &= \phi^2 + 1 + \frac{1}{\phi^2}\phi^2 - \phi^2 - \frac{1}{\phi}\left(2\phi^2 - 1\right) + \frac{1}{\phi} \\ &= 0. \end{aligned}$$

Therefore,

$$x_6 = \frac{1}{\phi}x_2 + x_5. \tag{3.8}$$

Hence, we get

$$0 = x_6 - \frac{1}{\phi}x_2 - x_5 = w_6 - w_1 - \frac{1}{\phi}(w_2 - w_1) - (w_5 - w_1). \tag{3.9}$$

As we see in Figure 3.3, the structures of $\{w_1, w_2, w_5, w_6\}$ and $\{w_4, w_3, w_5, w_6\}$ are congruent. Thus, we can replace w_1, w_2, w_5, w_6 in (3.9) with w_4, w_3, w_5, w_6, respectively, and use (3.5) to get

$$0 = w_6 - w_4 - \frac{1}{\phi}(w_3 - w_4) - (w_5 - w_4) = x_6 - x_4 - \frac{1}{\phi}(x_3 - x_4) - (x_5 - x_4).$$

Therefore, $x_6 - x_4 = \frac{1}{\phi}(x_3 - x_4) + (x_5 - x_4)$. And it follows from (3.8) and the last equality that

$$x_3 = x_2 + x_4. \tag{3.10}$$

In view of (3.4) and (3.6), the distances between 3 points w_1, w_4, w_5 are given below.

$$\|x_4\| = \|w_4 - w_1\| = \phi, \quad \|x_5\| = \|w_5 - w_1\| = 1, \quad \|x_5 - x_4\| = \|w_5 - w_4\| = 1. \tag{3.11}$$

Hence

$$\langle x_4, x_5 \rangle = \frac{1}{2}\left(\|x_4\|^2 + \|x_5\|^2 - \|x_4 - x_5\|^2\right) = \frac{1}{2}\left(\phi^2 + 1 - 1\right) = \frac{1}{2}\phi^2. \tag{3.12}$$

By (3.4), (3.6), (3.7), (3.8), (3.9), (3.11), and by (3.12), we have

$$\begin{aligned}
\phi^2 = \|w_6 - w_4\|^2 = \|x_6 - x_4\|^2 &= \left\|\frac{1}{\phi}x_2 + x_5 - x_4\right\|^2 \\
&= \frac{1}{\phi^2}\|x_2\|^2 + \|x_4\|^2 + \|x_5\|^2 + \frac{2}{\phi}\langle x_2, x_5\rangle - \frac{2}{\phi}\langle x_2, x_4\rangle - 2\langle x_4, x_5\rangle \\
&= \frac{1}{\phi^2}\phi^2 + \phi^2 + 1 + \frac{1}{\phi} - \frac{2}{\phi}\langle x_2, x_4\rangle - \phi^2 \\
&= 2 + \frac{1}{\phi} - \frac{2}{\phi}\langle x_2, x_4\rangle
\end{aligned}$$

and hence

$$\langle x_2, x_4\rangle = \frac{\phi}{2}\left(2 + \frac{1}{\phi} - \phi^2\right) = \frac{\phi}{2}\left(2 + \frac{1}{\phi} - \phi - 1\right) = \frac{1}{2}\left(\phi + 1 - \phi^2\right) = 0.$$

Hence, by (3.4), (3.6), (3.10) and the last equality, we get

$$\|x_3\|^2 = \|x_2 + x_4\|^2 = \|x_2\|^2 + \|x_4\|^2 = \|x_2\|^2 + \|w_4 - w_1\|^2 = 2\phi^2,$$

i.e.,

$$\|x_3\| = \|w_3 - w_1\| = \sqrt{2}\phi.$$

Since f preserves distances ϕ and $\sqrt{2}\phi$, we conclude that f is an isometry up to translation by using [11, Theorem 2.8]. □

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REFERENCES

[1] A. D. ALEKSANDROV, *Mapping of families of sets*, Soviet Math. Dokl. **11** (1970), 116–120.
[2] F. S. BECKMAN AND D. A. QUARLES, *On isometries of Euclidean spaces*, Proc. Amer. Math. Soc. **4** (1953), 810–815.
[3] W. BENZ, *Isometrien in normierten Räumen*, Aequationes Math. **29** (1985), 204–209.
[4] W. BENZ AND H. BERENS, *A contribution to a theorem of Ulam and Mazur*, Aequationes Math. **34** (1987), 61–63.
[5] S.-M. JUNG, *Inequalities for distances between points and distance preserving mappings*, Nonlinear Anal. **62** (2005), no. 4, 675–681.
[6] S.-M. JUNG AND K.-S. LEE, *An inequality for distances between $2n$ points and the Aleksandrov-Rassias problem*, J. Math. Anal. Appl. **324** (2006), no. 2, 1363–1369.
[7] J. MATOUŠEK, *Lectures on Discrete Geometry*, Grad. Texts in Math., vol. 212, Springer, New York, 2002.
[8] TH. M. RASSIAS, *Is a distance one preserving mapping between metric spaces always an isometry?* Amer. Math. Monthly **90** (1983), 200.
[9] TH. M. RASSIAS, *Mappings that preserve unit distance*, Indian J. Math. **32** (1990), 275–278.
[10] TH. M. RASSIAS, *Properties of isometries and approximate isometries*, in Recent Progress in Inequalities (ed. G. V. Milovanovic), pp. 341–379, Kluwer, 1998.
[11] S. XIANG, *Mappings of conservative distances and the Mazur-Ulam theorem*, J. Math. Anal. Appl. **254** (2001), 262–274.

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