Journal of Mathematical Inequalities
Volume 13, Number 1 (2019), 1–16 doi:10.7153/jmi-2019-13-01

INTEGRAL OPERATORS WITH TWO VARIABLE INTEGRATION LIMITS ON THE CONE OF MONOTONE FUNCTIONS

AIGERIM KALYBAY, RYSKUL OINAROV AND AINUR TEMIRKHANOVA

(*Communicated by G. Sinnamon*)

Abstract. Weighted inequalities for the Hardy-Steklov operators with variable integration limits on the cone of monotone functions have been investigated with success. The similar problem for operators with kernels has remained unsolved up to now. In this paper we find characterizations for integral operators with a wide class of kernels.

1. Introduction

Let $I=(a,b)$, $-\infty\leqslant a<b\leqslant\infty$, $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Let ω and v be non-negative and measurable functions almost everywhere finite on I such that $\omega^{-q'}$, ω^{q}, $v^{p'}$ and $v^{-p'}$ are locally summable on I.

We denote by $L_{p,v}\equiv L_p(v,I)$ the set of all functions f measurable on I such that

$$\|f\|_{p,v}\equiv\|vf\|_p=\left(\int_a^b|vf|^p\right)^{\frac{1}{p}}<\infty.$$

Let $M\downarrow$ and $M\uparrow$ be sets of functions non-increasing and non-decreasing on I, respectively.

For the integral operators

$$K_-f(x)=\int_{\alpha(x)}^{\beta(x)}K(s,x)f(s)ds, \tag{1}$$

$$K_+f(x)=\int_{\alpha(x)}^{\beta(x)}K(x,s)f(s)ds, \tag{2}$$

we consider the inequalities

$$\|\omega K_-f\|_q\leqslant C\|vf\|_p,\quad f\in M\downarrow, \tag{3}$$

Mathematics subject classification (2010): 26D10, 47B38.

Keywords and phrases: Integral operator with variable integration limits, Hardy-Steklov operator, weighted inequality, non-increasing function, non-decreasing function.

$$\|\omega K_+ f\|_q \leqslant C\|vf\|_p, \quad f \in M\uparrow, \tag{4}$$

where the boundary functions α and β satisfy the following conditions:
(i) $\alpha(x)$ and $\beta(x)$ are functions differentiable and strictly increasing on I;
(ii) $\alpha(x) < \beta(x)$ for any $x \in I$; moreover, $\lim\limits_{x\to a^+} \alpha(x) = \lim\limits_{x\to a^+} \beta(x) = a$ and $\lim\limits_{x\to b^-} \alpha(x) = \lim\limits_{x\to b^-} \beta(x) = b$.

In the case when $K(x,s) \equiv 1$ operator (1) is denoted by

$$Hf(x) = \int\limits_{\alpha(x)}^{\beta(x)} f(s)ds \tag{5}$$

and called the Hardy-Steklov operator [18].

During the last decade the problem of boundedness and compactness of operators (5), (1) and (2) was intensively investigated in weighted Lebesgue spaces [3, 6, 10, 13, 18, 19, 20, 22, 23] and Banach spaces [4]. The main method of these investigations is the Batuev-Stepanov block-diagonal method [10] introduced in [2].

The operators of the forms (1) and (2) are important in different problems (see e.g. [11] and [12]).

From the early nineties of the last century in connection with characterizations of boundedness and estimates of norms of classical operators in weighted Lorentz spaces the investigation of inequalities of the forms (3) and (4) has got a rapid development on the cone of monotone functions [7, 14, 16, 17]. The main method to study weighted inequalities for operators on the cone of monotone functions is a "reduction method" that appeared almost at the moment of appearance of the problem. The idea of this method is to reduce the given inequality for monotone functions to some inequality for non-negative functions.

In work [14] E. Sawyer presents a reduction method that transfers the inequality of the form (3) for linear positive operators on the set of non-increasing functions to some inequality for non-negative functions. In the modern mathematical literature it is known as "the Sawyer duality principle" (for short "the Sawyer principle"). In works [16] and [7] the expansion of this principle for non-decreasing functions is given.

At the present time there are many works that establish the inequalities of the form (3) and (4) for different classes of operators by help of "the Sawyer principle" (see e.g. [1, 8, 9, 15]).

Recently as a development of "the Sawyer principle" A. Gogatishvili and V.D. Stepanov [5] present a reduction method that allows to reduce weighted inequalities for positive but not necessarily linear operators on the cone of monotone functions to some weighted inequalities on the set of non-negative functions.

As it is mentioned above the boundedness of operators (1) and (2) from $L_{p,v}$ to $L_{q,\omega}$ has been studied well enough, however the inequalities of the forms (3) and (4) have been characterized only for the Hardy-Steklov operators (see [20] and [21]); and when the function $K(\cdot,\cdot)$ depends on both variables the problem is still open even in the case when the function $K(\cdot,\cdot)$ satisfies the condition from works [3, 6, 18, 19, 20], where only the boundedness of operators (1) and (2) from $L_{p,v}$ in $L_{q,\omega}$ is found.

The aim of this paper is to get necessary and sufficient conditions for the validity of inequalities (3) and (4) when the function $K(\cdot,\cdot)$ satisfies a condition weaker from those in works [3, 6, 18, 19, 20].

In the paper the relation $A \ll B$ means $A \leqslant cB$, where a constant $c > 0$ depends only on unessential parameters. Moreover, we write $A \approx B$ instead of $A \ll B \ll A$.

2. Auxiliary notations, concepts and statements

"The Sawyer principle" consists in the following (see [5] and [14]):

Let $1 < p,q < \infty$ and $Tf(x) = \int_a^b G(x,s)f(s)ds$, $G(x,s) \geqslant 0$. Then the inequality

$$\|\omega Tf\|_q \leqslant C_-\|vf\|_p, \quad f \in M\downarrow \tag{6}$$

is equivalent to the inequality

$$\left(\int_a^b \left(\int_a^x T^*g(s)ds\right)^{p'} V_-^{-p'}(x)v^{p'}(x)dx\right)^{\frac{1}{p'}} + \frac{\left(\int_a^b T^*g(s)ds\right)}{(V_-(b))^{\frac{1}{p}}} \leqslant \tilde{C}_-\left(\int_a^b (g(x)\omega^{-1}(x))^{q'}dx\right)^{\frac{1}{q'}} \tag{7}$$

for $g \geqslant 0$, and the inequality

$$\|\omega Tf\|_q \leqslant C_+\|vf\|_p, \quad f \in M\uparrow \tag{8}$$

is equivalent to the inequality

$$\left(\int_a^b \left(\int_x^b T^*g(s)ds\right)^{p'} V_+^{-p'}(x)v^{p'}(x)dx\right)^{\frac{1}{p'}} + \frac{\left(\int_a^b T^*g(s)ds\right)}{(V_+(a))^{\frac{1}{p}}} \leqslant \tilde{C}_+\left(\int_a^b (g(x)\omega^{-1}(x))^{q'}dx\right)^{\frac{1}{q'}} \tag{9}$$

for $g \geqslant 0$, where

$$V_-(x) = \int_a^x v^{-p'}(t)dt, \quad V_+(x) = \int_x^b v^{-p'}(t)dt,$$

$$V_-(b) = \lim_{x\to b^-} V_-(x), \quad V_+(a) = \lim_{x\to a^+} V_+(x).$$

In addition, the least constants in (6) and (7) as in (8) and (9) are equivalent, i.e., $C_\mp \approx \tilde{C}_\mp$.

By Lemmas 2.1 and 2.2 from work [18] it follows

LEMMA A. *Let $1 < p \leqslant q < \infty$. Then the $L_{p,v} \to L_{q,\omega}$ norms of the operators*

$$H^+ f(x) = \int_a^{\alpha(x)} f(s)ds, \quad H^- g(s) = \int_{\beta(s)}^b g(x)dx$$

have the following relations

$$\|H^+\| \approx \sup_{t\in I} \left(\int_t^b \omega^q(x)dx\right)^{\frac{1}{q}} \left(\int_a^{\alpha(t)} v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$\|H^-\| \approx \sup_{t\in I} \left(\int_a^t \omega^q(x)dx\right)^{\frac{1}{q}} \left(\int_{\beta(t)}^b v^{-p'}(s)ds\right)^{\frac{1}{p'}}.$$

Let us present the required results and definitions of kernel classes for operators (1) and (2) given in [10].

Let the function $K^+(\cdot,\cdot) \geqslant 0$ be defined and measurable on the set $\Omega^+ \equiv \Omega^+_{\alpha,\beta} = \{(x,s) : a < x < b,\ \alpha(x) \leqslant s \leqslant \beta(x)\}$ and non-decrease in the first argument. For an integer $n \geqslant 0$ we define the classes $O^+_n(\alpha,\beta(\cdot),\Omega^+)$. The class $O^+_0(\alpha,\beta(\cdot),\Omega^+)$ consists of all functions of the type $K^+(x,s) \equiv K^+_0(x,s) = v(s)$ for all $(x,s) \in \Omega^+$. Let the classes $O^+_i(\alpha,\beta(\cdot),\Omega^+)$, $i = 0,1,\dots,n-1$, $n \geqslant 1$, be defined. The function $K^+(\cdot,\cdot)$ belongs to $O^+_n(\alpha,\beta(\cdot),\Omega^+)$ if and only if there exist functions $K^+_{n,i}(x,z) \geqslant 0$, $i = 0,1,\dots,n-1$, defined and measurable on $\Omega_{a,b} = \{(x,z) : a < z \leqslant x < b\}$ and functions $K^+_i(\cdot,\cdot) \in O^+_i(\alpha,\beta(\cdot),\Omega^+)$, $i = 0,1,\dots,n-1$, such that

$$K^+(x,s) \equiv K^+_n(x,s) \approx \sum_{i=0}^n K^+_{n,i}(x,z)K^+_i(z,s), \quad K^+_{n,n}(x,z) \equiv 1, \tag{10}$$

for

$$a < z \leqslant x < b, \quad \alpha(x) \leqslant s \leqslant \beta(z), \tag{11}$$

where the equivalence constants in (10) do not depend on x, z and s.

Similarly, let the function $K^-(\cdot,\cdot) \geqslant 0$ be define and measurable on the set $\Omega^- \equiv \Omega^-_{\alpha,\beta} = \{(x,s) : a < s < b,\ \alpha(s) \leqslant x \leqslant \beta(s)\}$ and non-increase in the second argument. We define the classes $\Omega^-_n(\alpha(\cdot),\beta,\Omega^-)$, $n \geqslant 0$. The class $\Omega^-_0(\alpha(\cdot),\beta,\Omega^-)$ consists of all functions of the type $K^-(x,s) \equiv K^-_0(x,s) = u(x)$ for all $(x,s) \in \Omega^-$. Let the classes $O^-_i(\alpha(\cdot),\beta,\Omega^-)$, $i = 0,1,\dots,n-1$, $n \geqslant 1$, be defined. Then the function $K^-(\cdot,\cdot)$ belongs to the class $O^-_n(\alpha(\cdot),\beta,\Omega^-)$ if and only if there exist functions $K^-_{i,n}(z,s)$, $i = 0,1,\dots,n-1$, defined and measurable on $\Omega_{a,b}$ and functions $K^-_i(x,z) \in O^-_i(\alpha(\cdot),\beta,\Omega^-)$, $i = 0,1,\dots,n-1$, such that

$$K^-(x,s) \equiv K^-_n(x,s) \approx \sum_{i=0}^n K^-_i(x,z)K^-_{i,n}(z,s), \quad K_{n,n}(\cdot,\cdot) \equiv 1 \tag{12}$$

for

$$a < s \leqslant z < b, \quad \alpha(z) \leqslant x \leqslant \beta(s), \tag{13}$$

where the equivalence constants in (8) do not depend on x, t and s.

Let us notice that $\Omega^+_{\alpha,\beta} = \Omega^-_{\beta^{-1},\alpha^{-1}}$.

Assume that

$$A_1^+ \equiv \sup_{z\in I} \sup_{y\in\Delta^-(z)} \left(\int_y^z \omega^q(x) \left(\int_{\alpha(z)}^{\beta(y)} K^{p'}(x,s) v^{-p'}(s) ds \right)^{\frac{q}{p'}} dx \right)^{\frac{1}{q}},$$

$$A_2^+ \equiv \sup_{z\in I} \sup_{y\in\Delta^-(z)} \left(\int_{\alpha(z)}^{\beta(y)} v^{-p'}(s) \left(\int_y^z K^q(x,s) \omega^q(x) dx \right)^{\frac{p'}{q}} ds \right)^{\frac{1}{p'}},$$

$$A_1^- \equiv \sup_{z\in I} \sup_{y\in\Delta^+(z)} \left(\int_z^y \omega^q(x) \left(\int_{\alpha(y)}^{\beta(z)} K^{p'}(s,x) v^{-p'}(s) ds \right)^{\frac{q}{p'}} dx \right)^{\frac{1}{q}},$$

$$A_2^- \equiv \sup_{z\in I} \sup_{y\in\Delta^+(z)} \left(\int_{\alpha(y)}^{\beta(z)} v^{-p'}(s) \left(\int_z^y K^q(s,x) \omega^q(x) dx \right)^{\frac{p'}{q}} ds \right)^{\frac{1}{p'}},$$

where $\Delta^+(z) = [z, \alpha^{-1}(\beta(z))]$, $\Delta^-(z) = [\beta^{-1}(\alpha(z)), z]$.

THEOREM A^-. [10] *Let $1 < p \leqslant q < \infty$. If the kernel of operator (1) belongs to $O_n^+(\alpha, \beta(\cdot), \Omega^+) \bigcup O_n^-(\beta^{-1}(\cdot), \alpha^{-1}, \Omega^+)$, $n \geqslant 0$, then operator (1) is bounded from $L_{p,v}$ to $L_{q,\omega}$ if and only if $A_1^- < \infty$ or $A_2^- < \infty$. Moreover, $\|K_-\| \approx A_1^- \approx A_2^-$, where $\|K_-\|$ is $L_{p,v} \to L_{q,\omega}$ norm of operator (1).*

THEOREM A^+. [10] *Let $1 < p \leqslant q < \infty$. If the kernel of operator (2) belongs to $O_n^+(\alpha(\cdot), \beta, \Omega^+) \bigcup O_n^-(\beta^{-1}, \alpha^{-1}(\cdot), \Omega^+)$, $n \geqslant 0$, then operator (2) is bounded from $L_{p,v}$ to $L_{q,\omega}$ if and only if $A_1^+ < \infty$ or $A_2^+ < \infty$. Moreover, $\|K_+\| \approx A_1^+ \approx A_2^+$, where $\|K_+\|$ is $L_{p,v} \to L_{q,\omega}$ norm of operator (2).*

3. Main results

Assume that

$$\mathbb{A}_0^- \equiv \sup_{z\in I} \left(\int_{\beta(z)}^b V_-^{-p'}(t) v^{p'}(t) dt \right)^{\frac{1}{p'}} \left(\int_a^z \omega^q(x) \left(\int_{\alpha(x)}^{\beta(x)} K(s,x) ds \right)^q dx \right)^{\frac{1}{q}},$$

$$\mathbb{A}_1^- \equiv \sup_{z\in I}\sup_{y\in\Delta^+(z)}\left(\int_z^y \omega^q(x)\left(\int_{\alpha(y)}^{\beta(z)}\left(\int_{\alpha(x)}^t K(s,x)ds\right)^{p'} V_-^{-p'}(t)v^{p'}(t)dt\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}},$$

$$\mathbb{A}_2^- \equiv \sup_{z\in I}\sup_{y\in\Delta^+(z)}\left(\int_{\alpha(y)}^{\beta(z)} V_-^{-p'}(t)v^{p'}(t)\left(\int_z^y \omega^q(x)\left(\int_{\alpha(x)}^t K(s,x)ds\right)^q dx\right)^{\frac{p'}{q}} dt\right)^{\frac{1}{p'}},$$

$$\mathbb{A}_0^+ \equiv \sup_{z\in I}\left(\int_a^{\alpha(z)} V_+^{-p'}(t)v^{p'}dt\right)^{\frac{1}{p'}}\left(\int_z^b \omega^q(x)\left(\int_{\alpha(x)}^{\beta(x)} K(x,s)ds\right)^q dx\right)^{\frac{1}{q}},$$

$$\mathbb{A}_1^+ \equiv \sup_{z\in I}\sup_{y\in\Delta^-(z)}\left(\int_y^z \omega^q(x)\left(\int_{\alpha(z)}^{\beta(y)}\left(\int_t^{\beta(x)} K(x,s)ds\right)^{p'} V_+^{-p'}(t)v^{p'}(t)dt\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}},$$

$$\mathbb{A}_2^+ \equiv \sup_{z\in I}\sup_{y\in\Delta^-(z)}\left(\int_{\alpha(z)}^{\beta(y)} V_+^{-p'}(t)v^{p'}(t)\left(\int_y^z\left(\int_t^{\beta(x)} K(x,s)ds\right)^q \omega^q(x)dx\right)^{\frac{p'}{q}} dt\right)^{\frac{1}{p'}},$$

$$\mathbb{A}_3^- \equiv (V_-(b))^{-\frac{1}{p}}\left(\int_a^b \omega^q(x)\left(\int_{\alpha(x)}^{\beta(x)} K(s,x)ds\right)^q dx\right)^{\frac{1}{q}},$$

$$\mathbb{A}_3^+ \equiv (V_+(a))^{-\frac{1}{p}}\left(\int_a^b \omega^q(x)\left(\int_{\alpha(x)}^{\beta(x)} K(x,s)ds\right)^q dx\right)^{\frac{1}{q}}.$$

Let us notice that $V_-^{1-p'}(b) = V_+^{1-p'}(a) = 0$ for $\int_a^b v^{p'}(t)dt = \infty$.

THEOREM 1. *Let $1 < p \leqslant q < \infty$. If the kernel of operator (1) belongs to $O_n^-(\alpha(\cdot), \beta, \Omega^-)\bigcup O_n^+(\beta^{-1}, \alpha^{-1}(\cdot), \Omega^-)$, $n \geqslant 0$, then inequality (3) for operator (1) holds if and only if $\mathbb{A}_0^- + \mathbb{A}_1^- + \mathbb{A}_3^- < \infty$ or $\mathbb{A}_0^- + \mathbb{A}_2^- + \mathbb{A}_3^- < \infty$. Moreover, $C \approx \mathbb{A}_0^- + \mathbb{A}_1^- + \mathbb{A}_3^- \approx \mathbb{A}_0^- + \mathbb{A}_2^- + \mathbb{A}_3^-$, where C is the least constant in (3).*

THEOREM 2. *Let $1 < p \leqslant q < \infty$. If the kernel of operator (2) belongs to $O_n^+(\alpha, \beta(\cdot), \Omega^+)\bigcup O_n^-(\beta^{-1}(\cdot), \alpha^{-1}, \Omega^+)$, $n \geqslant 0$, then inequality (4) for operator (2) holds if and only if $\mathbb{A}_0^+ + \mathbb{A}_1^+ + \mathbb{A}_3^+ < \infty$ or $\mathbb{A}_0^+ + \mathbb{A}_2^+ + \mathbb{A}_3^+$. Moreover, $C \approx \mathbb{A}_0^+ + \mathbb{A}_1^+ + \mathbb{A}_3^+ \approx \mathbb{A}_0^+ + \mathbb{A}_1^+ + \mathbb{A}_3^+$, where C is the least constant in (4).*

Proof of Theorem 1. By "the Sawyer principle" (7) inequality (3) is equivalent to the inequalities

$$\left(\int_a^b\left(\int_a^t K_-^* g(s)ds\right)^{p'} V_-^{-p'}(t)v^{p'}(t)dt\right)^{\frac{1}{p'}} \leqslant C_1\left(\int_a^b(\omega^{-1}(x)g(x))^{q'}dx\right)^{\frac{1}{q'}},\quad g\geqslant 0, \tag{14}$$

$$\left(\int_a^b K_-^* g(s)ds\right)(V_-(b))^{-\frac{1}{p}} \leqslant C_2\left(\int_a^b\left(\omega^{-1}(x)g(x)\right)^{q'}dx\right)^{\frac{1}{q'}},\quad g\geqslant 0. \tag{15}$$

Moreover, $C\approx C_1+C_2$, where C, C_1 and C_2 are the least constants in (1), (14) and (15), respectively.

Since

$$\int_a^b K_-^* g(s)ds = \int_a^b\int_{\beta^{-1}(s)}^{\alpha^{-1}(s)} K(s,x)g(x)dxds = \int_a^b g(x)\int_{\alpha(x)}^{\beta(x)} K(s,x)dsdx,$$

then by the duality principle in $L_{q',\omega^{-1}}$, inequality (15) holds if and only if

$$\mathbb{A}_3^- = C_2 = (V_-(b))^{-\frac{1}{p}}\left(\int_a^b \omega^q(x)\left(\int_{\alpha(x)}^{\beta(x)} K(s,x)ds\right)^q dx\right)^{\frac{1}{q}} < \infty.$$

Since the operator $\int_a^t K_-^* g(s)ds$ is the dual to the operator $K_-\left(\int_x^b f(t)dt\right)$ with respect to the form $\int_a^b g(t)f(t)dt$, then by transaction from inequality (14) to its dual inequality, we get

$$\left(\int_a^b\left(\int_{\alpha(x)}^{\beta(x)} K(s,x)\left(\int_s^b f(t)dt\right)ds\right)^q \omega^q(x)dx\right)^{\frac{1}{q}} \leqslant C_1\left(\int_a^b\left(V_-(t)v^{-1}(t)f(t)\right)^p dt\right)^{\frac{1}{p}},\quad f\geqslant 0. \tag{16}$$

We have that

$$\int_{\alpha(x)}^{\beta(x)} K(s,x)\left(\int_s^b f(t)dt\right)ds = \int_{\alpha(x)}^{\beta(x)} K(s,x)\int_s^{\beta(x)} f(t)dtds + \int_{\alpha(x)}^{\beta(x)} K(s,x)ds\int_{\beta(x)}^{b} f(t)dt$$
$$= \int_{\alpha(x)}^{\beta(x)} f(t)\int_{\alpha(x)}^{t} K(s,x)dsdt + \int_{\beta(x)}^{b} f(t)dt\int_{\alpha(x)}^{\beta(x)} K(s,x)ds.$$

Therefore, inequality (16) is equivalent to the validity of the inequalities

$$\left(\int_a^b\left(\int_{\beta(x)}^b f(t)dt\right)^q\left(\int_{\alpha(x)}^{\beta(x)} K(s,x)ds\right)^q \omega^q(x)dx\right)^{\frac{1}{q}}$$
$$\leqslant C_{1,0}\left(\int_a^b \left(V_-(t)v^{-1}(t)f(t)\right)^p dt\right)^{\frac{1}{p}}, \quad f \geqslant 0, \quad (17)$$

$$\left(\int_a^b\left(\int_{\alpha(x)}^{\beta(x)} f(t)\int_{\alpha(x)}^{t} K(s,x)dsdt\right)^q \omega^q(x)dx\right)^{\frac{1}{q}}$$
$$\leqslant C_{1,1}\left(\int_a^b \left(V_-(t)v^{-1}(t)f(t)\right)^p dt\right)^{\frac{1}{p}}, \quad f \geqslant 0. \quad (18)$$

Moreover, $C_1 \approx C_{1,0} + C_{1,1}$, where C_1, $C_{1,0}$ and $C_{1,1}$ are the least constants in (16), (17) and (18), respectively.

By Lemma A inequality (17) holds if and only if $\mathbb{A}_0^- < \infty$. Moreover, $C_{1,0} \approx \mathbb{A}_0^-$, where $C_{1,0}$ is the least constant in (17).

Now, to get necessary and sufficient conditions for the validity of inequality (18) we need the following properties of the function $\hat{K}(t,x) = \int_{\alpha(x)}^{t} K(s,x)ds$ that are of independent interest.

LEMMA 1. *If* $K(\cdot,\cdot) \in O_n^-(\alpha(\cdot),\beta,\Omega^-)$, $n \geqslant 0$, *then* $\hat{K}(\cdot,\cdot) \in O_{n+1}^-(\alpha(\cdot),\beta,\Omega^-)$.

LEMMA 2. *If* $K(\cdot,\cdot) \in O_n^+(\beta^{-1},\alpha^{-1}(\cdot),\Omega^-)$, $n \geqslant 0$, *then* $\hat{K}(\cdot,\cdot) \in O_{n+1}^+(\beta^{-1}, \alpha^{-1}(\cdot),\Omega^-)$.

Let us prove Lemmas 1 and 2 later and continue the proof of Theorem 1.

By the condition of Theorem 1 we have that

$$K(\cdot,\cdot) \in O_n^-(\alpha(\cdot),\beta,\Omega^-)\bigcup O_n^+(\beta^{-1},\alpha^{-1}(\cdot),\Omega^-).$$

By the Lemmas 1 and 2 it follows that

$$\hat{K}(\cdot,\cdot) \in O_{n+1}^-(\alpha(\cdot),\beta,\Omega^-)\bigcup O_{n+1}^+(\beta^{-1},\alpha^{-1}(\cdot),\Omega^-).$$

Then by Theorem A^- inequality (18) holds if and only if $\mathbb{A}_1^- < \infty$ or $\mathbb{A}_2^- < \infty$. Moreover, $C_{1,1} \approx \mathbb{A}_1^- \approx \mathbb{A}_2^-$.

From $C \approx C_1 + C_2$, $C_1 \approx C_{1,0} + C_{1,1}$, $C_{1,0} \approx \mathbb{A}_0^-$, $C_{1,1} \approx \mathbb{A}_1^- \approx \mathbb{A}_2^-$ and $C_2 = \mathbb{A}_3^-$ it follows that inequality (3) holds if and only if $\mathbb{A}_0^- + \mathbb{A}_1^- + \mathbb{A}_3^- < \infty$ or $\mathbb{A}_0^- + \mathbb{A}_2^- + \mathbb{A}_3^- < \infty$. Moreover, $C \approx \mathbb{A}_0^- + \mathbb{A}_1^- + \mathbb{A}_3^- \approx \mathbb{A}_0^- + +\mathbb{A}_2^- + \mathbb{A}_3^-$, where C is the least constant in (3).

Now, to complete the proof of Theorem 1 we prove Lemmas 1 and 2.

Proof of Lemma 1. Let

$$a < x < z < b, \quad \alpha(z) \leqslant t \leqslant \beta(x). \tag{19}$$

Then $\alpha(x) \leqslant \alpha(z) \leqslant t$ and

$$\hat{K}(t,x) = \int_{\alpha(z)}^{t} K(s,x)ds + \int_{\alpha(x)}^{\alpha(z)} K(s,x)ds \tag{20}$$

For $(s,z,x) : a < x \leqslant z < b, \ \alpha(z) \leqslant s \leqslant t \leqslant \beta(x)$ from the condition of Lemma $K(\cdot,\cdot) \in O_n^-(\alpha(\cdot),\beta,\Omega^-)$ we have

$$K(s,x) \equiv K_n^-(s,x) \approx \sum_{i=0}^{n} K_i^-(s,z)K_{i,n}^-(z,x).$$

This, together with (20), gives

$$\hat{K}(t,x) \approx \sum_{i=0}^{n} \int_{\alpha(z)}^{t} K_i^-(s,z)dsK_{i,n}^-(z,x) + \int_{\alpha(x)}^{\alpha(z)} K(s,x)ds.$$

Assume that $\hat{K}_{i+1}^-(t,z) = \int_{\alpha(z)}^{t} K_i^-(s,z)ds$, $\hat{K}_{i+1,n+1}^-(z,x) = K_{i,n}^-(z,x)$, $i = 0,1,\ldots,n$, $\hat{K}_0^-(t,z) \equiv 1$ and $\hat{K}_{0,n+1}^-(z,x) = \int_{\alpha(x)}^{\alpha(z)} K(s,x)ds$. Then the last relation have the form

$$\hat{K}(t,x) \equiv \hat{K}_{n+1}^-(t,x) \approx \sum_{i=0}^{n+1} \hat{K}_i^-(t,z)\hat{K}_{i,n+1}^-(z,x). \tag{21}$$

If we show that $\hat{K}_i^-(\cdot,\cdot) \in O_i^-(\alpha(\cdot),\beta,\Omega^-)$, $i = 0,1,\dots,n$, then from (21) we have that $\hat{K}(\cdot,\cdot) \in O_{n+1}^-(\alpha(\cdot),\beta,\Omega^-)$. The belonging $\hat{K}_0(\cdot,\cdot) \equiv 1 \in O_0^-(\alpha(\cdot),\beta,\Omega^-)$ is obvious.

Let $i = 1$, $K_0(s,x) = u_0(s)$ and

$$\hat{K}_1^-(t,x) = \int_{\alpha(x)}^{t} u_0(s)ds = \int_{\alpha(z)}^{t} u_0(s)ds + \int_{\alpha(x)}^{\alpha(z)} u_0(s)ds$$
$$= \hat{K}_0^-(t,z)\hat{K}_{0,1}^-(z,x) + \hat{K}_1^-(t,z)\hat{K}_{1,1}^-(z,x)$$

with condition (19). Therefore, $\hat{K}_1^-(\cdot,\cdot) \in O_1^-(\alpha(\cdot),\beta,\Omega^-)$.

Let $\hat{K}_j^-(\cdot,\cdot) \in O_j^-(\alpha(\cdot),\beta,\Omega^-)$, $j = 1,2,\dots,i-1$, $i \geqslant 2$. Then from the condition $K_{j-1}^-(\cdot,\cdot) \in O_{j-1}^-(\alpha(\cdot),\beta,\Omega^-)$, $j = 1,2,\dots,i-1$, as above for $\hat{K}_i^-(t,x)$ we get relation (21) for $n = i-1$, that means $\hat{K}_i^-(\cdot,\cdot) \in O_i^-(\alpha(\cdot),\beta,\Omega^-)$, $i = 0,1,\dots,n+1$.

The proof of Lemma 1 is complete. $\Box$

Proof of Lemma 2. Let

$$a < z \leqslant t < b, \quad \beta^{-1}(t) \leqslant x \leqslant \alpha^{-1}(z). \tag{22}$$

Then $\alpha(x) \leqslant z \leqslant t$ and

$$\hat{K}(t,x) = \int_{\alpha(x)}^{t} K(s,x)ds = \int_{z}^{t} K(s,x)ds + \int_{\alpha(x)}^{z} K(s,x)ds. \tag{23}$$

For $(s,z,x):\ a < z \leqslant s \leqslant t < b,\ \beta^{-1}(s) \leqslant \beta^{-1}(t) \leqslant x \leqslant \alpha^{-1}(z)$ from the condition $K(\cdot,\cdot) \in O_n^+(\beta^{-1},\alpha^{-1}(\cdot),\Omega^-)$ it follow that

$$K(s,x) \approx \sum_{i=0}^{n} K_{n,i}^+(s,z)K_i^+(z,x).$$

Then from (23) we have

$$\hat{K}(t,x) \approx \sum_{i=0}^{n} \int_{z}^{t} K_{n,i}^+(s,z)dsK_i^+(z,x) + \hat{K}(z,x).$$

That, assuming $\hat{K}_{n+1,i}^+(t,z) = \int_z^t K_{n,i}^+(s,z)ds$, $\hat{K}_i^+(z,x) = K_i^+(z,x) \in O_i^+(\beta^{-1},\alpha^{-1}(\cdot),\Omega^-)$, $i = 0,1,\dots,n$, $\hat{K}_{n+1,n+1}^+(t,z) \equiv 1$ and $\hat{K}(z,x) = \hat{K}_{n+1}(z,x)$, yields

$$\hat{K}(t,x) \equiv \hat{K}_{n+1}^+(t,x) \approx \sum_{i=0}^{n+1} \hat{K}_{n+1,i}^+(t,z)\hat{K}_i^+(z,x)$$

with condition (22). Hence, $\hat{K}(\cdot,\cdot) \in O_{n+1}^+(\beta^{-1},\alpha^{-1}(\cdot),\Omega^-)$.

The proof of Lemma 2 is complete. □

Thus, the proof of Theorem 1 is also complete. □

The proof of Theorem 2 is similar. Let us present only the scheme of the proof. First, we need to prove analogues of Lemmas 1 and 2 that have independent values.

Let $\tilde{K}(x,t) = \int\limits_t^{\beta(x)} K(x,s)ds$.

LEMMA 3. *If* $K(\cdot,\cdot) \in O_n^+(\alpha,\beta(\cdot),\Omega^+)$, $n \geqslant 0$, *then* $\tilde{K}(\cdot,\cdot) \in O_{n+1}^+(\alpha,\beta(\cdot),\Omega^+)$.

Proof of Lemma 3. Let

$$a < z \leqslant x < b, \ \ \alpha(x) \leqslant t \leqslant \beta(z). \tag{24}$$

Then

$$\tilde{K}(x,t) = \int\limits_t^{\beta(z)} K(x,s)ds + \int\limits_{\beta(z)}^{\beta(x)} K(x,s)ds.$$

From $a < z \leqslant x < b$, $\alpha(x) \leqslant t \leqslant s \leqslant \beta(z)$ and $K(\cdot,\cdot) \in O_n^+(\alpha,\beta(\cdot),\Omega^+)$, $n \geqslant 0$, we have

$$\tilde{K}(x,t) \equiv \tilde{K}_{n+1}^+(x,t) \approx \sum_{i=0}^{n} K_{n,i}^+(x,z) \int\limits_t^{\beta(z)} K_i^+(z,s)ds + \int\limits_{\beta(z)}^{\beta(x)} K(x,s)ds.$$

That, assuming $\tilde{K}_{n+1,i+1}^+(x,z) \equiv K_{n,i}^+(x,z)$, $\tilde{K}_{i+1}^+(z,t) \equiv \int\limits_t^{\beta(z)} K_i^+(z,s)ds$, $i = 0,1,\ldots,n$, $\tilde{K}_0(x,t) \equiv 1$, $\tilde{K}_{n+1,0}^+(x,z) \equiv \int\limits_{\beta(z)}^{\beta(x)} K(x,s)ds$, gives

$$\tilde{K}(x,t) \equiv \tilde{K}_{n+1}^+(x,t) \approx \sum_{i=0}^{n+1} \tilde{K}_{n+1,i}^+(x,z)\tilde{K}_i^+(z,t). \tag{25}$$

The belonging $\tilde{K}_i^+(\cdot,\cdot) \in O_i^+(\alpha,\beta(\cdot),\Omega^+)$, $i = 0,1,\ldots,n$, is proved by the induction as in Lemma 1. Therefore, from (25) and (24) it follows $\tilde{K}(\cdot,\cdot) \in O_{n+1}^+(\alpha,\beta(\cdot),\Omega^+)$.

The proof of Lemma 3 is complete. □

LEMMA 4. *If* $K(\cdot,\cdot) \in O_n^-(\beta^{-1}(\cdot),\alpha^{-1},\Omega^+)$, $n \geqslant 0$, *then* $\tilde{K}(x,t) \in O_{n+1}^-(\beta^{-1}(\cdot), \alpha^{-1},\Omega^+)$.

Proof of Lemma 4. Let

$$a < t \leqslant z < b, \ \ \beta^{-1}(z) \leqslant x \leqslant \alpha^{-1}(x). \tag{26}$$

Then $t \leqslant z \leqslant \beta(x)$ and

$$\tilde{K}(x,t) = \int_t^z K(x,s)ds + \int_z^{\beta(x)} K(x,s)ds.$$

That from $t \leqslant s \leqslant z$, $\beta^{-1}(z) \leqslant x \leqslant \alpha^{-1}(t)$ and $K(\cdot,\cdot) \in O_n^-(\beta^{-1}(\cdot),\alpha^{-1},\Omega^+)$ gives

$$\begin{aligned}\tilde{K}(x,t) \equiv \tilde{K}_{n+1}^-(x,t) &\approx \sum_{i=0}^{n} K_i^-(x,z)\int_t^z K_{i,n}^-(z,s)ds + \tilde{K}_{n+1}^-(x,z)\\ &= \sum_{i=0}^{n+1} \tilde{K}_i^-(x,z)\tilde{K}_{i,n+1}^-(z,t),\end{aligned}$$

with condition (26), where $\tilde{K}_i^-(x,z) = K_i^-(x,z) \in O_i^-(\beta^{-1}(\cdot),\alpha^{-1},\Omega^+)$, $\tilde{K}_{i,n+1}^-(z,t) = \int_t^z K_{i,n}^-(z,s)ds$, $i = 0,1,\ldots,n$, $\tilde{K}_{n+1,n+1}(z,t) \equiv 1$. Consequently, $\tilde{K}(\cdot,\cdot) \in O_{n+1}^-(\beta^{-1}(\cdot),$ $\alpha^{-1},\ \Omega^+)$.

The proof of Lemma 4 is complete. □

Using "the Sawyer principle" (9) to inequality (4), we get that inequality (4) is equivalent to the validity of inequalities

$$\left(\int_a^b \left(\int_a^{\alpha(x)} g(t)dt\right)^q \left(\int_{\alpha(x)}^{\beta(x)} K(x,s)ds\right)^q \omega^q(x)dx\right)^{\frac{1}{q}} \leqslant C_0 \left(\int_a^b (g(t)V_+(t)v^{-1}(t))^p dt\right)^{\frac{1}{p}}, \quad g \geqslant 0, \tag{27}$$

$$\left(\int_a^b \left(\int_{\alpha(x)}^{\beta(x)} g(t)\int_t^{\beta(x)} K(x,s)dsdt\right)^q \omega^q(x)dx\right)^{\frac{1}{q}} \leqslant C_1 \left(\int_a^b (g(t)V_+(t)v^{-1}(t))^p dt\right)^{\frac{1}{p}}, \quad g \geqslant 0, \tag{28}$$

$$V_+^{-\frac{1}{p}}(a)\int_a^b g(x)\int_{\alpha(x)}^{\beta(x)} K(x,s)dsdx \leqslant C_2 \left(\int_a^b (\omega^{-1}(t)g(t))^{q'}dt\right)^{\frac{1}{q'}}, \quad g \geqslant 0. \tag{29}$$

Moreover, $C \approx C_0 + C_1 + C_2$, where C, C_0, C_1 and C_2 are the least constants in (4), (27), (28) and (29), respectively. These inequalities hold if and only if $\mathbb{A}_0^+ < \infty$, $\mathbb{A}_1^+ < \infty$ or $\mathbb{A}_2^+ < \infty$, $\mathbb{A}_3^+ < \infty$, respectively. Moreover, $C_0 \approx \mathbb{A}_0^+$, $C_1 \approx \mathbb{A}_1^+ \approx \mathbb{A}_2^+$ and $C_2 = \mathbb{A}_3^+$. Therefore, inequality (4) holds if and only if $\mathbb{A}_0^+ + \mathbb{A}_1^+ + \mathbb{A}_3^+ < \infty$ or $\mathbb{A}_0^+ + \mathbb{A}_2^+ + \mathbb{A}_3^+ < \infty$. Moreover, $C \approx \mathbb{A}_0^+ + \mathbb{A}_1^+ + \mathbb{A}_3^+ \approx \mathbb{A}_0^+ + \mathbb{A}_2^+ + \mathbb{A}_3^+$, where C is the least constant in (4). The proof of Theorem 2 is complete. □

4. Some examples

In work [18] the boundedness of operator (1) from $L_{p,v}$ in $L_{q,\omega}$ is considered for the case when its kernel $K(\cdot,\cdot)$ satisfies the condition

$$K(s,x) \approx K(s,z) + K(\alpha(z),x), \quad a < x \leqslant z < b, \quad \alpha(z) \leqslant s \leqslant \beta(x). \tag{30}$$

Assuming that $K_1^-(s,z) \equiv K(s,z)$, $K_{0,1}^-(z,x) \equiv K(\alpha(z),x)$, $K_{1,1}^-(z,x) \equiv K_0^-(s,z) \equiv 1$, we have that $K(s,x) \equiv K_1^-(s,x) \approx K_0^-(s,z)K_{0,1}^-(z,x) + K_1^-(s,z)K_{1,1}^-(z,x)$, i.e., $K(\cdot,\cdot) \in O_1^-(\alpha(\cdot),\beta,\Omega^-)$.

Therefore, Theorem 1 is correct for operator (1) with condition (30) and $n = 1$. Using condition (30) we can find simpler values, the finiteness of which is equivalent to the finiteness of the values $\mathbb{A}_1^-$ or $\mathbb{A}_2^-$. In the expression $\mathbb{A}_1^-$ the variables z, y, x, s and t change within the following bounds: $a < z \leqslant x \leqslant y \leqslant b$, $\alpha(x) \leqslant \alpha(y) \leqslant t \leqslant \beta(z)$ and $\alpha(x) \leqslant s \leqslant t$. Hence,

$$\int_{\alpha(x)}^{t} K(s,x)ds = \int_{\alpha(y)}^{t} K(s,x)ds + \int_{\alpha(x)}^{\alpha(y)} K(s,x)ds. \tag{31}$$

Since in the function $\int_{\alpha(y)}^{t} K(s,x)ds$ the variables x, s and y satisfy the conditions $a < x \leqslant y < b$ and $\alpha(y) \leqslant s \leqslant \beta(z) \leqslant \beta(x)$, from (30) it follows that

$$\int_{\alpha(y)}^{t} K(s,x)ds \approx \int_{\alpha(y)}^{t} K(s,y)ds + (t - \alpha(y))K(\alpha(y),x).$$

If we replace the obtained expression in (31), we have

$$\int_{\alpha(x)}^{t} K(s,x)ds \approx \int_{\alpha(y)}^{t} K(s,y)ds + (t - \alpha(y))K(\alpha(y),x) + \int_{\alpha(x)}^{\alpha(y)} K(s,x)ds.$$

Therefore, for (30) the finiteness of the values $\mathbb{A}_1^-$ or $\mathbb{A}_2^-$ is equivalent to the finiteness of the values

$$\mathbb{A}_{1,1}^- = \sup_{z\in I} \sup_{y\in\Delta^+(z)} \left(\int_z^y \omega^q(x)dx \right)^{\frac{1}{q}} \left(\int_{\alpha(y)}^{\beta(z)} \left(\int_{\alpha(y)}^{t} K(s,y)ds \right)^{p'} V_-^{-p'}(t)v^{p'}(t)dt \right)^{\frac{1}{p'}},$$

$$\mathbb{A}^-_{1,2} = \sup_{z\in I}\sup_{y\in\Delta^+(z)}\left(\int_z^y \omega^q(x)K^q(\alpha(y),x)dx\right)^{\frac{1}{q}}\left(\int_{\alpha(y)}^{\beta(z)}(t-\alpha(y))^{p'}V_-^{-p'}(t)v^{p'}(t)dt\right)^{\frac{1}{p'}},$$

$$\mathbb{A}^-_{1,3} = \sup_{z\in I}\sup_{y\in\Delta^+(z)}\left(\int_z^y \omega^q(x)\left(\int_{\alpha(x)}^{\alpha(y)}K(s,x)ds\right)^q dx\right)^{\frac{1}{q}}\left(\int_{\alpha(y)}^{\beta(z)}V_-^{-p'}(t)v^{p'}(t)dt\right)^{\frac{1}{p}}.$$

From Theorem 1 we have

COROLLARY 1. *Let $1<p\leqslant q<\infty$. If the kernel of operator (1) satisfies condition (30), then inequality (3) holds if and only if $\mathbb{A}^- = \{\mathbb{A}^-_0, \mathbb{A}^-_{1,1}, \mathbb{A}^-_{1,2}, \mathbb{A}^-_{1,3}, \mathbb{A}^-_3\} < \infty$. Moreover, $C\approx\mathbb{A}^-$, where C is the least constant in (3).*

In works [3, 6, 18] the boundedness of operator (2) from $L_{p,v}$ to $L_{q,\omega}$ is studied in the case when its kernel satisfies the condition

$$K(x,s)\approx K(x,\beta(z))+K(z,s),\quad a<z\leqslant x<b,\quad \alpha(x)\leqslant s\leqslant\beta(z). \tag{32}$$

As above it is easy to see that $K(\cdot,\cdot)\in O_1^+(\alpha,\beta(\cdot),\Omega^+)$. Therefore, from the expression $\mathbb{A}_1^+$ and condition (32) we get

$$\int_t^{\beta(x)}K(x,s)ds\approx K(x,\beta(y))(\beta(y)-t)+\int_t^{\beta(y)}K(y,s)ds+\int_{\beta(y)}^{\beta(x)}K(x,s)ds.$$

Then the finiteness of the value $\mathbb{A}_1^+$ or $\mathbb{A}_2^+$ is equivalent to the finiteness of the following values

$$\mathbb{A}^+_{1,1} = \sup_{z\in I}\sup_{y\in\Delta^-(z)}\left(\int_y^z \omega^q(x)K^q(x,\beta(y))dx\right)^{\frac{1}{q}}\left(\int_{\alpha(z)}^{\beta(y)}(\beta(y)-t)^{p'}V_+^{-p'}(t)v^{p'}(t)dt\right)^{\frac{1}{p'}},$$

$$\mathbb{A}^+_{1,2} = \sup_{z\in I}\sup_{y\in\Delta^-(z)}\left(\int_y^z \omega^q(x)dx\right)^{\frac{1}{q}}\left(\int_{\alpha(z)}^{\beta(y)}\left(\int_t^{\beta(y)}K(y,s)ds\right)^{p'}V_+^{-p'}(t)v^{p'}(t)dt\right)^{\frac{1}{p'}},$$

$$\mathbb{A}^+_{1,3} = \sup_{z\in I}\sup_{y\in\Delta^-(z)}\left(\int_y^z \omega^q(x)\left(\int_{\beta(y)}^{\beta(x)}K(x,s)ds\right)^q dx\right)^{\frac{1}{q}}\left(\int_{\alpha(z)}^{\beta(y)}V_+^{-p'}(t)v^{p'}(t)dt\right)^{\frac{1}{p'}}.$$

From Theorem 2 we have

COROLLARY 2. *Let $1 < p \leqslant q < \infty$. If the kernel of operator* (2) *satisfies condition* (32)*, then inequality* (4) *holds if and only if* $\mathbb{A}^+ = \{\mathbb{A}_0^+, \mathbb{A}_{1,1}^+, \mathbb{A}_{1,2}^+, \mathbb{A}_{1,3}^+, \mathbb{A}_3^+\} < \infty$. *Moreover,* $C \approx \mathbb{A}^+$*, where* C *is the least constant in* (4)*.*

Acknowledgement. The paper was written under financial support by the Ministry of Education and Science of the Republic of Kazakhstan, Grant No. AP05130975 in area “Scientific research in the field of natural sciences”.

REFERENCES

[1] L. S. ARENDARENKO, R. OINAROV AND L.-E. PERSSON, *Some new Hardy-type integral inequalities on cones of monotone functions*, Advances in Harmonic Analysis and Operator Theory, edited by Alexandre Almeida, Luis Filipe Castro, Frank-Olme Speck; Springer, Basel, 2013, 77–89.

[2] E. N. BATUEV AND V. D. STEPANOV, *On weighted inequalities of Hardy type*, Siberian Math. J. **30**, (1989), 8–16.

[3] T. CHEN AND G. SINNAMON, *Generalized Hardy operators and normalizing measures*, J. Inequal. and Appl. **7**, 6 (2002), 829–866.

[4] A. GOGATISHVILI AND J. LANG, *The generalized Hardy operator with kernel and variable integral limits in Banach function spaces*, J. Inequal. Appl. **4**, 1 (1999), 1–16.

[5] A. GOGATISHVILI AND V. D. STEPANOV, *Reduction theorems for weighted integral inequalities on the cone of monotone functions*, Russian Math. Surveys **68**, 4 (2013), 597–664.

[6] H. P. HEINIG AND G. SINNAMON, *Mapping properties of integral overaging operators*, Stud. Math. **129**, (1998), 157–177.

[7] H. P. HEINIG AND V. D. STEPANOV, *Weighted Hardy inequalities for increasing functions*, Canad. J. Math. **45**, 1 (1993), 104–116.

[8] A. KUFNER, L. MALIGRANDA AND L.-E. PERSSON, *The Hardy inequality. About its history and some related results*, Vydavatelský servis, Pilsen, 2007.

[9] A. KUFNER AND L.-E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific, New Jersey/London/Singapore/Hong Kong, 2003.

[10] R. OINAROV, *Boundedness and compactness in weighted Lebesgue spaces of integral operators with variable integration limits*, Siberian Math. J. **52**, 6 (2011), 1042–1055.

[11] R. OINAROV, *Boundedness of integral operators from weighted Sobolev space to weighted Lebesgue space*, Complex Variables and Elliptic Equations **56**, 10–11 (2011), 1021–1038.

[12] R. OINAROV, *Boundedness of integral operators in weighted Sobolev spaces*, Izvestiya: Mathematics **78**, 4 (2014), 836–853.

[13] D. V. PROKHOROV, V. D. STEPANOV AND E. P. USHAKOVA, *Hardy-Steklov integral operators*, Part 1. Proc. Steklov Inst. Math. **300**, (Suppl. 2) (2018), 1–111; translated from Sovrem. Probl. Mat. **22**, Steklov Math. Institute of RAS, Moscow, 2016.

[14] E. SAWYER, *Boundedness of classical operators on classical Lorentz spaces*, Stud. Math. **96**, 2 (1990), 145–158.

[15] G. SINNAMON, *Hardy's Inequality and Monotonicity*, Proceedings Function Spaces, Differential Operators and Nonlinear Analysis, Mathematical Institute of the Academy of Sciences of the Czech Republic, Prague, 2005, 292–310.

[16] V. D. STEPANOV, *Integral operators on the cone of monotone functions*, J. London Math. Soc. **48**, 3 (1993), 465–487.

[17] V. D. STEPANOV, *The weighted Hardy's inequality for nonincreasing functions*, Trans. Amer. Math. Soc. **338**, 1 (1993), 173–186.

[18] V. D. STEPANOV AND E. P. USHAKOVA, *On integral operators with variable limits of integration*, Proc. Steklov Inst. Math. **232**, (2001), 290–309.

[19] V. D. STEPANOV AND E. P. USHAKOVA, *On the geometric mean operator with variable limits of integration*, Proc. Steklov Inst. Math. **260**, (2008), 254–278.

[20] V. D. STEPANOV AND E. P. USHAKOVA, *Kernel operators with variable intervals of integration in Lebesgue spaces and applications*, Math. Inequal. Appl. **13**, 3 (2010), 449–510.

[21] V. D. STEPANOV AND E. P. USHAKOVA, *Hardy operator with variable limits on monotone functions*, J. Funct. Spaces Appl. **1**, 1 (2003), 1–15.

[22] E. P. USHAKOVA, *Alternative boundedness characteristics for the Hardy-Steklov operator*, Eurasian Math. J. **8**, 2 (2017), 74–96.

[23] E. P. USHAKOVA, *On boundedness and compactness of a certain class of kernel operators*, J. Funct. Spaces Appl. **9**, (2011), 67–107.

(Received July 13, 2017)

Aigerim Kalybay
KIMEP University
4 Abai Ave., 050010 Almaty, Kazakhstan
e-mail: kalybay@kimep.kz

Ryskul Oinarov
L. N. Gumilyov Eurasian National University
5 Munaytpasov St., 010008 Astana, Kazakhstan
e-mail: o_ryskul@mail.ru

Ainur Temirkhanova
L. N. Gumilyov Eurasian National University
5 Munaytpasov St., 010008 Astana, Kazakhstan
e-mail: ainura-t@yandex.kz