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A REVERSE OF YOUNG INEQUALITY

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Abstract. In this paper, we prove several multi-term refinements of reverse of Young inequality with Kantorovich constant for both real numbers and operators. Among other results, for all $0 \leqslant v \leqslant 1$ and $N \in \mathbb{N}$, $(1-v)a+vb \leqslant (\sqrt{a}-\sqrt{b})^2 - S_N(v;a,b) + K(\sqrt[2^N]{h},2)^{-\beta_N(v)}a^{1-v}b^v$ for all real numbers a and b, where $S_N(v;a,b)$ is a certain function defined by Sababheh. Furthermore, we also improved some inequalities with Kantorovich constant.

1. Introduction

The simple inequality

$$a^v b^{1-v} \leqslant va+(1-v)b, \quad a,b>0 \text{ and } 0 \leqslant v \leqslant 1 \tag{1.1}$$

is the Young inequality. Even though this inequality looks very simple, it is of great interest in operator theory. Refining this inequality has taken the attention of many researchers in the field, where adding a positive term to the left side is possible.

Among the first refinements of this inequality is the squared version proved in [3]

$$(a^v b^{1-v})^2 + \min\{v,1-v\}^2(a-b)^2 \leqslant (va+(1-v)b)^2. \tag{1.2}$$

Later, the authors in [5] obtained the other interesting refinement

$$a^v b^{1-v} + \min\{v,1-v\}(\sqrt{a}-\sqrt{b})^2 \leqslant va+(1-b). \tag{1.3}$$

A common fact about the refinements (1.2) and (1.3) are having one refining term.

In recent paper [4], some reverses and refinements of Young's inequality were presented, it is proved that

$$\begin{cases} a^v b^{1-v} + v(\sqrt{a}-\sqrt{b})^2 + r_0(\sqrt[4]{ab}-\sqrt{b})^2 \leqslant va+(1-v)b, \quad 0 \leqslant v \leqslant \frac{1}{2} \\ a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + r_0(\sqrt[4]{ab}-\sqrt{a})^2 \\ \leqslant va+(1-v)b, \quad \frac{1}{2} \leqslant v \leqslant 1 \end{cases} \tag{1.4}$$

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where $r_0 = \min\{2r, 1-2r\}$ for $r = \min\{v, 1-v\}$. These inequalities adding a second refining term to the original Young's inequality. In the same paper, it is proved the following reversed versions

$$\begin{cases} va+(1-v)b+r_0(\sqrt[4]{ab}-\sqrt{a})^2 \leqslant a^v b^{1-v}+(1-v)(\sqrt{a}-\sqrt{b})^2, & 0 \leqslant v \leqslant \frac{1}{2} \\ va+(1-v)b+r_0(\sqrt[4]{ab}-\sqrt{b})^2 \leqslant a^v b^{1-v}+v(\sqrt{a}-\sqrt{b})^2, & \frac{1}{2} \leqslant v \leqslant 1 \end{cases} \tag{1.5}$$

where $r_0 = \min\{2r, 1-2r\}$ for $r = \min\{v, 1-v\}$.

To state our paper, we adopt the following notations. Like in [6], let $a, b \geqslant 0$ and $v \in [0,1]$ for $N \in \mathbb{N}$ and $j = 1, 2, \cdots, N$, let $k_j(v) = [2^{j-1}v], r_j(v) = [2^j v]$ and

$$s_j(v) = (-1)^{r_j(v)} 2^{j-1} v + (-1)^{r_j(v)+1} [\frac{r_j(v)+1}{2}]. \tag{1.6}$$

Then define the nonnegative function

$$S_N(v;a,b) = \sum_{j=1}^{N} s_j(v) \left(\sqrt[2^j]{b^{2^{j-1}-k_j(v)} a^{k_j(v)}} - \sqrt[2^j]{a^{k_j(v)+1} b^{2^{j-1}-k_j(v)-1}} \right)^2, \tag{1.7}$$

it is proved that

$$a^v b^{1-v} + S_N(v;a,b) \leqslant va + (1-v)b. \tag{1.8}$$

Also, in [1], authors proved that

$$S_N(v;a,b) = S_N(1-v;b,a) \tag{1.9}$$

and

$$K(\sqrt[2^N]{h}, 2)^{\beta_N(v)} a^v b^{1-v} + S_N(v;a,b) \leqslant va + (1-v)b, \tag{1.10}$$

where $K(t,2) = \frac{(t+1)^2}{4t}$, $t > 0$, $h = \frac{b}{a}$, $\alpha_N(v) = 1 + [2^N v] - 2^N v$ and $\beta_N(v) = \min\{\alpha_N(v), 1 - \alpha_N(v)\}$ for $N \in \mathbb{N}$.

Paper [7] pointed out that Specht's ratio and the Kantorovich constant have the relationship as follows:

$$S(t^r) \leqslant K(t,2)^r, \quad t > 0 \text{ and } 0 \leqslant r \leqslant \frac{1}{2}. \tag{1.11}$$

In a recent work, Kai [10] gave the following Young type inequalities

$$\begin{cases} (v^2 a)^v b^{1-v} + v^2(\sqrt{a}-\sqrt{b})^2 \leqslant v^2 a + (1-v)^2 b, & 0 \leqslant v \leqslant \frac{1}{2} \\ a^v((1-v)^2 b)^{1-v} + (1-v)^2(\sqrt{a}-\sqrt{b})^2 \leqslant v^2 a + (1-v)^2 b, & \frac{1}{2} \leqslant v \leqslant 1. \end{cases} \tag{1.12}$$

Later, Nasiri in [12] improved the inequalities (1.12) with Kantorovich constant

$$\begin{cases} v^2 a + (1-v)^2 b \geqslant v^2(\sqrt{a}-\sqrt{b})^2 + r(\sqrt{b} - \sqrt[4]{v^2 ab})^2 \\ \qquad\qquad + K(\sqrt[4]{v^2 h}, 2)^{r'} (v^2 a)^v b^{1-v}, \quad 0 \leqslant v \leqslant \frac{1}{2} \\ v^2 a + (1-v)^2 b \geqslant (1-v)^2(\sqrt{a}-\sqrt{b})^2 + r(\sqrt{a} - \sqrt[4]{(1-v)^2 ab})^2 \\ \qquad\qquad + K(\sqrt[4]{\frac{h}{(1-v)^2}}, 2)^{r'} a^v [(1-v)^2 b]^{1-v}, \quad \frac{1}{2} \leqslant v \leqslant 1. \end{cases} \tag{1.13}$$

where $h=\frac{b}{a}$, $r=\min\{2v,1-2v\}$ and $r'=\min\{2r,1-2r\}$.

Furthermore, authors in [11] proved that

$$\begin{cases} v^2a+(1-v)^2b \leqslant (1-v)^2(\sqrt{a}-\sqrt{b})^2+a^v[(1-v)^2b]^{1-v}, & 0\leqslant v\leqslant \frac{1}{2} \\ v^2a+(1-v)^2b \leqslant v^2(\sqrt{a}-\sqrt{b})^2+(v^2a)^v b^{1-v}, & \frac{1}{2}\leqslant v\leqslant 1. \end{cases} \tag{1.14}$$

which can be regarded as the reverse of Young type inequalities. In this paper, we will present a refinement that has as many terms as we wish.

Throughout the paper, M_n denotes the space of all $n\times n$ complex matrices. M_n^+ is the set of positive semidefinite matrices in M_n, $X\geqslant Y$ for $X,Y\in M_n$ means that X and Y are hermitian and $X-Y\in M_n^+$. M_n^{++} is the set of strictly positive definite matrices in M_n, and $|||\cdot|||$ is a unitarily invariant norm defined on M_n. We defined

$$A\nabla_v B=(1-v)A+vB, \quad v\in[0,1]$$

$$A\sharp_v B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}}, \quad v\in R$$

denoted by $A\nabla B$ and $A\sharp B$ respectively when $v=\frac{1}{2}$. Also, we defined that

$$H_v(A,B)=\frac{A\sharp_v B+B\sharp_v A}{2}. \quad v\in[0,1]$$

The paper is organized in the following way: In Section 2, we give the reversed version of (1.10) and some auxiliary results. In Section 3, reversed version of Young inequality with Kantorovich constant is presented. In the last Section, we establish the matrix version of inequality (2.2) for unitarily invariant norm based on Corollary 2.2.

2. Reverses of Young and Young type inequality with Kantorovich constant

First, we prove a reverse of Young inequality with Kantorovich constant.

THEOREM 2.1. *Let $a,b>0$, $0\leqslant v\leqslant 1$ and $N\in\mathbb{N}$, then*

$$(1-v)a+vb\leqslant(\sqrt{a}-\sqrt{b})^2-S_N(v;a,b)+K(\sqrt[2^N]{h},2)^{-\beta_N(v)}a^{1-v}b^v \tag{2.1}$$

Proof. By (1.10) we have

$$\begin{aligned} & K(\sqrt[2^N]{h},2)^{-\beta_N(v)}a^{1-v}b^v+(\sqrt{a}-\sqrt{b})^2-(1-v)a-vb \\ &= K(\sqrt[2^N]{h},2)^{-\beta_N(v)}a^{1-v}b^v+va+(1-v)b-2\sqrt{ab} \\ &\geqslant K(\sqrt[2^N]{h},2)^{-\beta_N(v)}a^{1-v}b^v+K(\sqrt[2^N]{h},2)^{\beta_N(v)}a^v b^{1-v}+S_N(v;a,b)-2\sqrt{ab} \\ &\geqslant 2\sqrt{ab}+S_N(v;a,b)-2\sqrt{ab} \\ &= S_N(v;a,b). \quad \square \end{aligned}$$

COROLLARY 2.2. *Replace a,b by a^2,b^2 in Theorem 2.1, we can get the following result.*

$$((1-v)a+vb)^2 \leqslant (v^2-v+1)(a-b)^2 - S_N(v;a^2,b^2) + K\left(\sqrt[2^{N-1}]{h},2\right)^{-\beta_N(v)}(a^{1-v}b^v)^2. \tag{2.2}$$

By the similar way in Theorem 2.1, we can get a reverse of Young type inequality with Kantorovich constant, which is the refinement of (1.14).

THEOREM 2.3. *Let $a,b>0$ and $N\in\mathbb{N}$.*
If $0\leqslant v\leqslant \frac{1}{2}$, then

$$v^2a+(1-v)^2b \leqslant (1-v)^2(\sqrt{a}-\sqrt{b})^2 - S_N(2v;(1-v)\sqrt{ab},a) + K\left(\sqrt[2^{N+1}]{(1-v)^2h},2\right)^{-\beta_N(2v)}(1-v)^{2(1-v)}a^vb^{1-v}. \tag{2.3}$$

If $\frac{1}{2}\leqslant v\leqslant 1$, then

$$v^2a+(1-v)^2b \leqslant v^2(\sqrt{a}-\sqrt{b})^2 - S_N(2v-1;b,v\sqrt{ab}) + K\left(\sqrt[2^{N+1}]{\frac{h}{v^2}},2\right)^{-\beta_N(2v-1)}v^{2v}a^vb^{1-v}. \tag{2.4}$$

Proof. Let $0\leqslant v\leqslant \frac{1}{2}$. Then we have

$$\begin{aligned}
&(1-v)^2(\sqrt{a}-\sqrt{b})^2 - v^2a-(1-v)^2b\\
&\quad +K\left(\sqrt[2^{N+1}]{(1-v)^2h},2\right)^{-\beta_N(2v)}(1-v)^{2(1-v)}a^vb^{1-v}\\
&= 2v(1-v)\sqrt{ab}+(1-2v)a+(2v-2)\sqrt{ab}\\
&\quad +K\left(\sqrt[2^{N+1}]{(1-v)^2h},2\right)^{-\beta_N(2v)}(1-v)^{2(1-v)}a^vb^{1-v}\\
&\geqslant K\left(\sqrt[2^{N+1}]{(1-v)^2h},2\right)^{\beta_N(2v)}(1-v)^{2v}a^{1-v}b^v + S_N(2v;(1-v)\sqrt{ab},a)\\
&\quad +(2v-2)\sqrt{ab}+K\left(\sqrt[2^{N+1}]{(1-v)^2h},2\right)^{-\beta_N(2v)}(1-v)^{2(1-v)}a^vb^{1-v}\\
&\geqslant 2(1-v)\sqrt{ab}+(2v-2)\sqrt{ab}+S_N(2v;(1-v)\sqrt{ab},a)\\
&= S_N(2v;(1-v)\sqrt{ab},a)
\end{aligned}$$

Thus, (2.3) holds.

For $\frac{1}{2} \leqslant v \leqslant 1$, compute:

$$
\begin{aligned}
& v^2(\sqrt{a}-\sqrt{b})^2 - v^2 a - (1-v)^2 b + K\left(\sqrt[2^{N+1}]{\frac{h}{v^2}}, 2\right)^{-\beta_N(2v-1)} v^{2v} a^v b^{1-v} \\
&= (2v-1)b + (2-2v)v\sqrt{ab} - 2v\sqrt{ab} + K\left(\sqrt[2^{N+1}]{\frac{h}{v^2}}, 2\right)^{-\beta_N(2v-1)} v^{2v} a^v b^{1-v} \\
&\geqslant K\left(\sqrt[2^{N+1}]{\frac{h}{v^2}}, 2\right)^{\beta_N(2v-1)} v^{2-2v} a^{1-v} b^v + S_N(2v-1; b, v\sqrt{ab}) - 2v\sqrt{ab} \\
&\quad + K\left(\sqrt[2^{N+1}]{\frac{h}{v^2}}, 2\right)^{-\beta_N(2v-1)} v^{2v} a^v b^{1-v} \\
&\geqslant S_N(2v-1; b, v\sqrt{ab}).
\end{aligned}
$$

Thus, (2.4) holds. So we completed the proof. □

LEMMA 2.4.

$$\beta_N(v) = \beta_N(1-v) \geqslant 0 \tag{2.5}$$

where $\beta_N(v) = \min\{\alpha_N(v), 1-\alpha_N(v)\}$ *for* $v \in [0,1]$ *and* $N \in \mathbb{N}$, $\alpha_N(v) = 1 + [2^N v] - 2^N v$.

Proof. *Case* i) when $[2^N v]$ is an integer, then $\alpha_N(v) = 1$ and $1 - \alpha_N(v) = 0$, which means that $\beta_N(v) = 0$. Meanwhile, $\alpha_N(1-v) = 1$ and $1 - \alpha_N(1-v) = 0$ means that $\beta_N(1-v) = 0$. So we can get $\beta_N(v) = \beta_N(1-v)$.

Case ii) when $[2^N v]$ is not an integer, then we have

$$1 + [2^N v] - 2^N v = -2^N v - [-2^N v]$$

and

$$2^N v - [2^N v] = 1 + [-2^N v] + 2^N v$$

which means that $\beta_N(v) = \beta_N(1-v)$. So we completed the proof. □

Next, we will get some of the Kantorovich constant of Heinz inequalities.

PROPOSITION 2.5.

$$\frac{1}{2}(S_N(v;a,b) + S_N(v;b,a)) \leqslant \frac{1}{2}(a+b) - K\left(\sqrt[2^N]{h}, 2\right)^{\beta_N(v)} H_v(a,b) \tag{2.6}$$

and

$$\frac{1}{2}(S_N(v;a,b) + S_N(v;b,a)) \leqslant \frac{1}{2}(a+b) - 2\sqrt{ab} + K\left(\sqrt[2^N]{h}, 2\right)^{-\beta_N(v)} H_v(a,b). \tag{2.7}$$

Proof. By (1.10) and (2.5) we have

$$K\left(\sqrt[2^N]{h}, 2\right)^{\beta_N(v)} a^v b^{1-v} + S_N(v;a,b) \leqslant va + (1-v)b$$

and

$$K\left(\sqrt[2^N]{h},2\right)^{\beta_N(1-v)} a^{1-v}b^v + S_N(1-v;a,b) \leqslant (1-v)a+vb.$$

So we have

$$\frac{1}{2}(S_N(v;a,b)+S_N(1-v;a,b)) \leqslant \frac{1}{2}(a+b) - K\left(\sqrt[2^N]{h},2\right)^{\beta_N(v)} H_v(a,b),$$

by (1.9), so we have the desired result (2.6).

Similarly, by (2.1) and (2.5) we have

$$(1-v)a+vb \leqslant (a+b)-2\sqrt{ab}-S_N(v;a,b)+K\left(\sqrt[2^N]{h},2\right)^{-\beta_N(v)} a^{1-v}b^v$$

and

$$va+(1-v)b \leqslant (a+b)-2\sqrt{ab}-S_N(1-v;,a,b)+K\left(\sqrt[2^N]{h},2\right)^{-\beta_N(1-v)} a^v b^{1-v}.$$

So we have

$$\frac{1}{2}(S_N(v;a,b)+S_N(1-v;a,b)) \leqslant \frac{1}{2}(a+b)-2\sqrt{ab}+K\left(\sqrt[2^N]{h},2\right)^{-\beta_N(v)} H_v(a,b)$$

by (1.9), so we have the desired result (2.7). □

To achieve our further results, we need a following lemma.

LEMMA 2.6. *Let ϕ be a strictly increasing convex function defined on an interval I. If x,y,z and w are points in I such that*

$$z-w \leqslant x-y,$$

where $w \leqslant z \leqslant x$ and $y \leqslant x$, then

$$(0\leqslant) \quad \phi(z)-\phi(w) \leqslant \phi(x)-\phi(y).$$

We can see [8] *for more details.*

PROPOSITION 2.7. *Let $\phi : [0,\infty) \longrightarrow R$ be a strictly increasing convex function, if $a,b>0$, then*

(1) $\phi(K(\sqrt[2^N]{h},2)^{\beta_N(v)}H_v(a,b)) - \phi(\sqrt{ab}) \leqslant \phi(\frac{a+b}{2}) - \phi(\frac{1}{2}(S_N(v;a,b)+S_N(v;b,a)) + \sqrt{ab})$

(2) $\phi(\frac{a+b}{2}) - \phi(K(\sqrt[2^N]{h},2)^{-\beta_N(v)}H_v(a,b)) \leqslant \phi(a+b) - \phi(2\sqrt{ab}+\frac{1}{2}(S_N(v;a,b)+S_N(v;b,a)))$

Proof. (1). For our convenience, we denote that $x=\frac{a+b}{2}$, $y=\frac{1}{2}(S_N(v;a,b)+S_N(v;b,a))+\sqrt{ab}$, $z=K(\sqrt[2^N]{h},2)^{\beta_N(v)}H_v(a,b)$ and $w=\sqrt{ab}$. It is clearly that $z \geqslant w$ by $H_v(a,b) \geqslant \sqrt{ab}$ and $K(\sqrt[2^N]{h},2)^{\beta_N(v)} \geqslant 1$. So we only need to prove i) $x \geqslant z$, ii) $x \geqslant y$ and iii) $x-y \geqslant z-w$. In fact, we have

i) by (1.7) and (2.6) we can easily get $x \geqslant z$.

ii) $\frac{a+b}{2} - \sqrt{ab} \geqslant \frac{a+b}{2} - K(\sqrt[2^N]{h},2)^{\beta_N(v)} H_v(a,b) \geqslant \frac{1}{2}(S_N(v;a,b) + S_N(v;b,a))$.

iii) $x-y \geqslant z-w$ imply $x-z \geqslant y-w$, which can be achieved by (2.6).

(2). Let $x = a+b$, $y = \frac{1}{2}(S_N(v;a,b)+S_N(v;b,a)) + 2\sqrt{ab}$, $z = \frac{a+b}{2}$ and $w = K(\sqrt[2^N]{h},2)^{-\beta_N(v)} H_v(a,b)$. It is easy to see that $x \geqslant z \geqslant w$ by $K(\sqrt[2^N]{h},2)^{-\beta_N(v)} \leqslant 1$ and $H_v(a,b) \leqslant \frac{a+b}{2}$. So we only prove $x \geqslant y$ and $x-y \geqslant z-w$.

By (2.7), we can get $x-y \geqslant z-w(\geqslant 0)$, which imply that $x \geqslant y$.

Then by lemma 2.6, so we completed the proof. $\square$

COROLLARY 2.8. *Let $\phi(x) = x^m$, $m \geqslant 1$, then we have the following results.*

$$\left(K\left(\sqrt[2^N]{h},2\right)^{\beta_N(v)} H_v(a,b)\right)^m + \left(\frac{1}{2}(S_N(v;a,b)+S_N(v;b,a)) + \sqrt{ab}\right)^m \leqslant \left(\frac{a+b}{2}\right)^m + (\sqrt{ab})^m$$

and

$$\left(\frac{a+b}{2}\right)^m + \left(\frac{1}{2}(S_N(v;a,b)+S_N(v;b,a)) + 2\sqrt{ab}\right)^m \leqslant (a+b)^m + \left(K\left(\sqrt[2^N]{h},2\right)^{-\beta_N(v)} H_v(a,b)\right)^m.$$

3. Reversed versions for operators

In this section, we will give some reversed versions of Young inequalities for operators by the monotonic property of operator functions. First, we give the basic of the following discussion.

LEMMA 3.1. *Let $X \in B(H)$ be self-adjoint and let f and g be continuous real functions such that $f(t) \geqslant g(t)$ for all $t \in Sp(X)$ (the Spectrum of X). Then $f(X) \geqslant g(X)$.*

THEOREM 3.2. *Let $A,B \in B(H)$ be positive invertible operators, I be the identity operator and $v \in [0,1]$. If all positive numbers m,m' and M,M' satisfy either of the following conditions*

i) $0 < mI \leqslant A \leqslant m'I < M'I \leqslant B \leqslant MI$;

ii) $0 < mI \leqslant B \leqslant m'I < M'I \leqslant A \leqslant MI$.

Then

$$A\nabla_v B \leqslant (A\nabla B - 2A\sharp B) - \sum_{j=1}^{N} s_j(v)(A\sharp_{1-\alpha_j(v)}B + A\sharp_{1-\alpha_j(v)-2^{1-j}}B - 2A\sharp_{1-\alpha_j(v)-2^{-j}}B) + K\left(\sqrt[2^N]{h'},2\right)^{-\beta_N(v)} A\sharp_v B \tag{3.1}$$

where $h' = \frac{M'}{m'}$, $\alpha_j(v) = \frac{k_j(v)}{2^{j-1}}$, $k_j(v) = [2^j v]$ *and* $\beta_N(v) = \min\{1 + [2^N v] - 2^N v, 2^N v - [2^N v]\}$.

Proof. Let $a = 1$ in (2.1) and expand the summand to get

$$(1-v) + vb \leqslant (1 + b - 2\sqrt{b}) - \sum_{j=1}^{N} s_j(v)(b^{1-\alpha_j(v)} + b^{1-\alpha_j(v)-2^{1-j}} - 2b^{1-\alpha_j(v)-2^{-j}}) + K\left(\sqrt[2^N]{b}, 2\right)^{-\beta_N(v)} b^v. \tag{3.2}$$

For $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, under the first condition, we can get $I \leqslant h'I = \frac{M'}{m'}I \leqslant X \leqslant hI = \frac{M}{m}I$, and then $Sp(X) \subseteq [h', h] \subseteq (1, +\infty)$. By lemma 3.1 and (3.2), we have

$$\begin{aligned}(1-v)I + vX \leqslant{}& (I + X - 2X^{\frac{1}{2}}) \\ &- \sum_{j=1}^{N} s_j(v)\left(X^{1-\alpha_j(v)} + X^{1-\alpha_j(v)-2^{1-j}} - 2X^{1-\alpha_j(v)-2^{-j}}\right) \\ &+ \max_{h' \leqslant x \leqslant h} K\left(\sqrt[2^N]{x}, 2\right)^{-\beta_N(v)} X^v. \end{aligned} \tag{3.3}$$

Since the Kantorovich constant $K(t,2) = \frac{(t+1)^2}{4t}$ is an increasing function on $(1, +\infty)$, then

$$\begin{aligned}(1-v)I + vA^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant{}& (I + A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}) \\ &- \sum_{j=1}^{N} s_j(v)\Big((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha_j(v)} + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha_j(v)-2^{1-j}} \\ &- 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha_j(v)-2^{-j}}\Big) \\ &+ K\left(\sqrt[2^N]{h'}, 2\right)^{-\beta_N(v)} (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v. \end{aligned}$$

In a similar way, under the second condition, we can get $I \leqslant \frac{1}{h}I = \frac{m}{M}I \leqslant X \leqslant \frac{1}{h'}I = \frac{m'}{M'}I$, and then $Sp(X) \subseteq [\frac{1}{h}, \frac{1}{h'}] \subseteq (0, 1)$. By lemma 3.1 and (3.2), we have

$$\begin{aligned}(1-v)I + vX \leqslant{}& (I + X - 2X^{\frac{1}{2}}) \\ &- \sum_{j=1}^{N} s_j(v)\left(X^{1-\alpha_j(v)} + X^{1-\alpha_j(v)-2^{1-j}} - 2X^{1-\alpha_j(v)-2^{-j}}\right) \\ &+ \max_{\frac{1}{h} \leqslant x \leqslant \frac{1}{h'}} K\left(\sqrt[2^N]{x}, 2\right)^{-\beta_N(v)} X^v. \end{aligned}$$

Since the Kantorovich constant $K(t,2)=\frac{(t+1)^2}{4t}$ is a decreasing function on $(0,1)$, then

$$\begin{aligned}
&(1-v)I+vA^{-\frac{1}{2}}BA^{-\frac{1}{2}}\\
&\leqslant (I+A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}})-\sum_{j=1}^{N}s_j(v)\Big((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha_j(v)}\\
&+(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha_j(v)-2^{1-j}}-2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha_j(v)-2^{-j}}\Big)\\
&+K\left(\sqrt[2^N]{h'},2\right)^{-\beta_N(v)}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{v}. \qquad (3.4)
\end{aligned}$$

Then multiplying inequalities (3.3) and (3.4) by $A^{\frac{1}{2}}$ on both sides, we can get the required inequality (3.1). □

4. Reverse version of Young inequality for the Hilbert-Schmidt norm

In this section, we will give the reverse version of Young inequality (2.2) for Hilbert-Schmidt norm.

Based on Kittaneh and Manasrah [2] showed that if $A,B,X\in M_n$ with A and B are positive semidefinite matrices and $v\in[0,1]$, then

$$||(1-v)AX+vXB||_2^2\leqslant ||A^{1-v}XB^v||_2^2+R^2||AX-XB||_2^2$$

and

$$||AX+XB||_2^2\leqslant ||A^{1-v}XB^v+A^vXB^{1-v}||_2^2+2R||AX-XB||_2^2$$

where $R=\max\{v,1-v\}$.

Liao and Wu [9] proved that if $A,B,X\in M_n$ such that A and B are two positive definite matrices and satisfy $0<mI\leqslant A,B\leqslant MI$, where I represents an identity, $v\in[0,1]$ for $m,M\in\mathbb{R}$, then

$$||(1-v)AX+vXB||_2^2\leqslant K(h,2)^{R'}||A^{1-v}XB^v||_2^2+r^2||AX-XB||_2^2$$

and

$$||AX+XB||_2^2\leqslant K(h,2)^{R'}||A^{1-v}XB^v+A^vXB^{1-v}||_2^2+2r||AX-XB||_2^2$$

where $h=\frac{M}{m}$, $r=\min\{v,1-v\}$ and $R'=\max\{2r,1-2r\}$.

THEOREM 4.1. *Suppose $A,B,X\in M_n$ such that A,B are two positive definite matrices and satisfy $0<mI\leqslant A,B\leqslant MI$, where I represents an identity matrix and $m,M\in R$ for any $v\in[0,1]$, then*

$$\begin{aligned}
||(1-v)AX+vXB||_2^2\leqslant{}&(v^2-v+1)||AX-XB||_2^2+||A^{1-v}XB^v||_2^2\\
&-\sum_{j=1}^{N}s_j(v)||A^{\frac{k_j(v)}{2^{j-1}}}XB^{1-\frac{k_j(v)}{2^{j-1}}}-A^{\frac{k_j(v)+1}{2^{j-1}}}XB^{1-\frac{k_j(v)+1}{2^{j-1}}}||_2^2.
\end{aligned}$$

Proof. Since A and B are positive definite, it follows by spectral theorem that there exist unitary matrices $U,V \in M_n$, such that

$$A = U\Lambda_1 U^*, B = V\Lambda_2 V^*,$$

where $\Lambda_1 = diag(\lambda_1,\lambda_2,\cdots,\lambda_n)$, $\Lambda_2 = diag(\nu_1,\nu_2,\cdots,\nu_n)$, λ_i, $\nu_i > 0$, $i = 1,2,\cdots,n$.

Let $Y = U^*XV = [y_{il}]$, then

$$(1-v)AX + vXB = U[(1-v)\Lambda_1 Y + vY\Lambda_2]V^* = U[((1-v)\lambda_i + v\nu_l)y_{il}]V^*,$$

$$AX - XB = U[(\lambda_i - \nu_l)y_{il}]V^*, \quad A^{1-v}XB^v = U((\lambda_i^{1-v}\nu_l^v)y_{il})V^*$$

and

$$A^{\frac{k_j(v)}{2^{j-1}}}XB^{1-\frac{k_j(v)}{2^{j-1}}} - A^{\frac{k_j(v)+1}{2^{j-1}}}XB^{1-\frac{k_j(v)+1}{2^{j-1}}} = U(\lambda_i^{\frac{k_j(v)}{2^{j-1}}}\nu_l^{1-\frac{k_j(v)}{2^{j-1}}} - \lambda_i^{\frac{k_j(v)+1}{2^{j-1}}}\nu_l^{1-\frac{k_j(v)+1}{2^{j-1}}})V^*.$$

Now, by (2.2) and the unitarily invariant of the Hilbert-Schmidt norm, we have

$$\begin{aligned}
&||(1-v)AX + vXB||_2^2 \\
=& \sum_{i,l=1}^{n} ((1-v)\lambda_i + v\nu_l)^2 |y_{il}|^2 \\
\leqslant& \sum_{i,l=1}^{n} \{\max K\left(\sqrt[2^{N-1}]{t_{il}},2\right)^{-\beta_N(v)} (\lambda_i^{1-v}\nu_l^v)^2 + (v^2 - v + 1)(\lambda_i - \nu_l)^2 \\
&- \sum_{j=1}^{N} s_j(v)(\lambda_i^{\frac{k_j(v)}{2^{j-1}}}\nu_l^{1-\frac{k_j(v)}{2^{j-1}}} - \lambda_i^{\frac{k_j(v)+1}{2^{j-1}}}\nu_l^{1-\frac{k_j(v)+1}{2^{j-1}}})^2\}|y_{il}|^2
\end{aligned}$$

where $t_{il} = \frac{\lambda_i}{\nu_l}$.

According to the conditions $0 < mI \leqslant A,B \leqslant MI$, $\frac{m}{M} = \frac{1}{h} \leqslant t_{il} = \frac{\lambda_i}{\nu_l} \leqslant h = \frac{M}{m}$ and the properties of the Kantorovich constant ($K(\sqrt[2^{N-1}]{t_{il}},2)^{-\beta_N(v)} \leqslant 1$), we can get

$$\begin{aligned}
&||(1-v)AX + vXB||_2^2 \\
\leqslant& \sum_{i,l=1}^{n} \{(\lambda_i^{1-v}\nu_l^v)^2 + (v^2 - v + 1)(\lambda_i - \nu_l)^2 \\
&- \sum_{j=1}^{N} s_j(v)(\lambda_i^{\frac{k_j(v)}{2^{j-1}}}\nu_l^{1-\frac{k_j(v)}{2^{j-1}}} - \lambda_i^{\frac{k_j(v)+1}{2^{j-1}}}\nu_l^{1-\frac{k_j(v)+1}{2^{j-1}}})^2\}|y_{il}|^2 \\
=& (v^2 - v + 1)||AX - XB||_2^2 + ||A^{1-v}XB^v||_2^2 \\
&- \sum_{j=1}^{N} s_j(v)||A^{\frac{k_j(v)}{2^{j-1}}}XB^{1-\frac{k_j(v)}{2^{j-1}}} - A^{\frac{k_j(v)+1}{2^{j-1}}}XB^{1-\frac{k_j(v)+1}{2^{j-1}}}||_2^2,
\end{aligned}$$

here we completed the proof. □

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