Journal of Mathematical Inequalities
Volume 13, Number 1 (2019), 77–93 doi:10.7153/jmi-2019-13-06

ON CERTAIN CONJECTURES FOR THE TWO SEIFFERT MEANS

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(*Communicated by T. Burić*)

Abstract. In 2015 Vukšić, by using the asymptotic expansion method, conjectured certain inequalities related to the first and second Seiffert means. In this paper, we prove certain conjectures given by Vukšić.

1. Introduction

Throughout this paper we assume that the numbers x and y are positive and unequal. The first and second Seiffert means $P(x,y)$ and $T(x,y)$ are defined in [19] and [20], respectively by

$$P(x,y) = \frac{x-y}{2\arcsin\frac{x-y}{x+y}} \quad \text{and} \quad T(x,y) = \frac{x-y}{2\arctan\frac{x-y}{x+y}}.$$

A power mean A_r is defined by

$$A_r(x,y) = \begin{cases} \left(\frac{x^r+y^r}{2}\right)^{1/r}, & r \neq 0 \\ \sqrt{xy}, & r = 0. \end{cases}$$

As usual, the symbols H, G, L, A, Q, and N will stand, respectively, for the harmonic, geometric, logarithmic, arithmetic, root-square, and contraharmonic means of x and y,

$$H = \frac{2xy}{x+y}, \ G = \sqrt{xy}, \ L = \frac{x-y}{\ln x - \ln y}, \ A = \frac{x+y}{2}, \ Q = \sqrt{\frac{x^2+y^2}{2}}, \ N = \frac{x^2+y^2}{x+y}.$$

It is well known (see [21, 22]) that

$$H < G < L < P < A < T < Q < N.$$

Jagers [12] proved

$$\frac{A+G}{2} = A_{1/2} < P < A_{2/3}. \tag{1}$$

For the comparison of P and A_r, see [11].

Mathematics subject classification (2010): 26E60, 26D05.
Keywords and phrases: Seiffert means, inequality.

Paper JMI-13-06

Sándor [17] proved that

$$(A^2G)^{1/3} < P < \frac{G+2A}{3} \tag{2}$$

and

$$\left(\left(\frac{A+G}{2}\right)^2 A\right)^{1/3} < P < \frac{1}{3}\left(\frac{A+G}{2} + 2\sqrt{\frac{A+G}{2}A}\right). \tag{3}$$

The left-hand side of (3) is sharper than the left-hand side of (1).

By using the sequential method, Sándor[18] improved the inequality $A < T < Q$ and obtained the following results:

$$(Q^2A)^{1/3} < T < \frac{A+2Q}{3} \tag{4}$$

and

$$\left(\left(\frac{Q+A}{2}\right)^2 Q\right)^{1/3} < T < \frac{1}{3}\left(\frac{Q+A}{2} + 2\sqrt{\frac{Q+A}{2}Q}\right). \tag{5}$$

Extension of the sequential method by Sándor has been introduced for the Schwab-Borchardt means (See [14], [15]), as L, P and T are particular Schwab-Borchardt means. We note that, a new particular case of this mean, known also as the Neuman-Sándor mean, has been introduced in [14]; see also [15]. By using another method, in 2013 Witkowski [23] has proved again inequalities (2)-(5), and also other inequalities. In paricular, he proved the following results:

$$P > \frac{2}{\pi}A + \frac{\pi-2}{\pi}G \tag{6}$$

and

$$T > sA + (1-s)Q, \tag{7}$$

where

$$s = \frac{2(\pi - 2\sqrt{2})}{(2-\sqrt{2})\pi} = 0.3403413\ldots.$$

There is a large number of papers studying inequalities between Seiffert means and convex combinations of other means [3, 4, 5, 10, 13, 22]. For example, Chu et al. [3] established that the double inequality

$$\mu A + (1-\mu)H < P < \nu A + (1-\nu)H \tag{8}$$

holds if and only if $\mu \leqslant 2/\pi$ and $\nu \geqslant 5/6$. In 2011, Chu et al. [4] proved that the double inequality

$$\mu Q + (1-\mu)A < T < \nu Q + (1-\nu)A \tag{9}$$

holds if and only if $\mu \leqslant (4-\pi)/\big(\pi(\sqrt{2}-1)\big)$ and $\nu \geqslant 2/3$.

In fact, (7) can be written as

$$\left(1-\frac{4-\pi}{(\sqrt{2}-1)\pi}\right)A+\frac{4-\pi}{(\sqrt{2}-1)\pi}Q<T, \tag{10}$$

which is the left-hand side of (9).

Recently, Vukšić [22], by using the asymptotic expansion method, gave a systematic study of inequalities of the form

$$(1-\mu)M_1+\mu M_3<M_2<(1-\nu)M_1+\nu M_3,$$

where M_j are chosen from the class of elementary means given above. For example, Vukšić [22, Theorem 3.5, (3.15)] proved the following double inequality:

$$(1-\mu)H+\mu N<T<(1-\nu)H+\nu N,$$

with the best possible constants $\mu=2/\pi$ and $\nu=1/3$. See [7, 8, 9] for more details about comparison of means using asymptotic methods.

Also Vukšić [22] has conjectured certain inequalities related to the first and second Seiffert means $P(x,y)$ and $T(x,y)$. In particular, the following relations have been conjectured [22, Conjecture 3.7]:

$$\frac{3G+2T}{5}<P<\frac{G+T}{2}, \tag{11}$$

$$\frac{3L+T}{4}<P<\frac{L+T}{2}, \tag{12}$$

$$\frac{2P+T}{3}<A<\frac{(4-\pi)P+(\pi-2)T}{2}, \tag{13}$$

$$\frac{1}{4}P+\frac{3}{4}Q<T<\frac{\pi-2\sqrt{2}}{\pi-\sqrt{2}}P+\frac{\sqrt{2}}{\pi-\sqrt{2}}Q. \tag{14}$$

The first aim of this paper is to offer a proof of these inequalities (Theorems 1–4).

REMARK 1. Let $(x-y)/(x+y)=z$, and suppose $x>y$. Then $z\in(0,1)$, and the following identities hold:

$$\frac{H(x,y)}{A(x,y)}=1-z^2,\quad \frac{G(x,y)}{A(x,y)}=\sqrt{1-z^2},\quad \frac{L(x,y)}{A(x,y)}=\frac{2z}{\ln\frac{1+z}{1-z}},$$
$$\frac{P(x,y)}{A(x,y)}=\frac{z}{\arcsin z},\quad \frac{T(x,y)}{A(x,y)}=\frac{z}{\arctan z},\quad \frac{Q(x,y)}{A(x,y)}=\sqrt{1+z^2}.$$

By Remark 1, the left-hand side of (13) may be written also as

$$2\left(\frac{z}{\arcsin z}\right)+\frac{z}{\arctan z}<3,\qquad 0<z<1. \tag{15}$$

The second aim of this paper is to give an improvement of (15) (Theorem 5).

The following lemmas are needed in the sequel.

LEMMA 1. *The following inequalities hold:*

$$Q+G<2A \tag{16}$$

and

$$A\sqrt{2}<Q+(\sqrt{2}-1)G. \tag{17}$$

Proof. From the inequality $(Q+G)^2<2(Q^2+G^2)$ and the equality $Q^2+G^2=2A^2$, we obtain (16).

The proof of (17) makes use of the following inequality:

$$\sqrt{u}+(\sqrt{2}-1)\sqrt{v}>\sqrt{u+v} \quad \text{for} \quad u>v>0. \tag{18}$$

By squaring both sides of (18), it is immediately seen that (18) is equivalent to $(\sqrt{2}-1)(\sqrt{u}-\sqrt{v})>0$ for $u>v>0$. The choice $u=x^2+y^2$ and $v=2xy$ in (18) yields (17). The proof is complete. □

LEMMA 2. ([2]) *The following inequalities hold:*

$$\frac{1}{9}H+\frac{8}{9}Q<T<\frac{\pi-2\sqrt{2}}{\pi}H+\frac{2\sqrt{2}}{\pi}Q. \tag{19}$$

The double inequality (19) was conjectured by Vukšić [22, Conjecture 3.6, (3.19)]. Recently, Chen and Elezović [2] gave a proof of (19).

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2. Proofs of the inequalities (11)-(14)

THEOREM 1. *The inequalities* (11) *are true.*

Proof. By Remark 1, the left-hand side of (11) may be rewritten as

$$\frac{3}{5}\sqrt{1-z^2}+\frac{2}{5}\frac{z}{\arctan z}<\frac{z}{\arcsin z}, \qquad 0<z<1. \tag{20}$$

Using the following inequality (see [1, Lemma 3]):

$$\frac{x}{1+\frac{x^2}{3}}<\arctan x, \quad x>0, \tag{21}$$

we have

$$\frac{z}{\arcsin z}-\frac{3}{5}\sqrt{1-z^2}-\frac{2}{5}\frac{z}{\arctan z}>\frac{z}{\arcsin z}-\frac{3}{5}\sqrt{1-z^2}-\frac{2}{5}\left(1+\frac{1}{3}z^2\right).$$

In order to prove (20), it suffices to show that

$$\frac{z}{\arcsin z}-\frac{3}{5}\sqrt{1-z^2}-\frac{2}{5}\left(1+\frac{1}{3}z^2\right)>0, \qquad 0<z<1. \tag{22}$$

By an elementary change of variable $z = \sin x \, (0 < x < \pi/2)$, the inequality (22) becomes

$$g(x) > 0, \qquad 0 < x < \frac{\pi}{2},$$

where

$$g(x) = \frac{\sin x}{x} - \frac{3}{5}\cos x - \frac{2}{5}\left(1 + \frac{1}{3}\sin^2 x\right).$$

We find

$$g(x) = \frac{\sin x}{x} - \frac{3}{5}\cos x + \frac{1}{15}\cos(2x) - \frac{7}{15} = \frac{1}{36}x^4 - \frac{1}{189}x^6 + \sum_{n=4}^{\infty}(-1)^n u_n(x),$$

where

$$u_n(x) = \frac{(2n+1)4^n - 18n + 6}{15 \cdot (2n+1)!}x^{2n}.$$

Elementary calculations reveal that, for $0 < x < \pi/2$ and $n \geqslant 4$,

$$\begin{aligned}
\frac{u_{n+1}(x)}{u_n(x)} &= \frac{x^2}{n+1} \frac{(4n+6)4^n - 9n - 6}{(2n+3)\Big((2n+1)4^n - 18n + 6\Big)} \\
&< \frac{(\pi/2)^2}{n+1} \frac{(4n+6)4^n - 9n - 6}{(2n+3)\Big((2n+1)4^n - 18n + 6\Big)} \\
&< \frac{(4n+6)4^n - 9n - 6}{(2n+3)\Big((2n+1)4^n - 18n + 6\Big)}.
\end{aligned}$$

We find, for $n \geqslant 4$,

$$\begin{aligned}
&(2n+3)\Big((2n+1)4^n - 18n + 6\Big) - \Big((4n+6)4^n - 9n - 6\Big) \\
&= (4n^2 + 4n - 3)\left(4^n - \frac{36n^2 + 33n - 24}{4n^2 + 4n - 3}\right) > 0.
\end{aligned}$$

This inequality can be proved by induction on n, we omit it.

Hence, for all $0 < x < \pi/2$ and $n \geqslant 4$,

$$\frac{u_{n+1}(x)}{u_n(x)} < 1.$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \longmapsto u_n(x)$ is strictly decreasing for $n \geqslant 4$. We then obtain

$$g(x) > x^4\left(\frac{1}{36} - \frac{1}{189}x^2\right) > 0, \qquad 0 < x < \frac{\pi}{2}.$$

Hence, (20) holds.

We now prove the right-hand side of (11). In order to prove $P < (G+T)/2$, it suffices to show by (7) that

$$P < \frac{G + sA + (1-s)Q}{2},$$

i.e.,

$$\frac{2(\pi - 2\sqrt{2})}{(2-\sqrt{2})\pi} = s < \frac{G+Q-2P}{Q-A}. \tag{23}$$

By Remark 1, (23) may be rewritten as

$$\frac{2(\pi - 2\sqrt{2})}{(2-\sqrt{2})\pi} < \frac{\sqrt{1-z^2}+\sqrt{1+z^2}-\frac{2z}{\arcsin z}}{\sqrt{1+z^2}-1}, \qquad 0 < z < 1. \tag{24}$$

By an elementary change of variable $z = \sin x\,(0 < x < \pi/2)$, the inequality (24) becomes

$$\frac{2(\pi - 2\sqrt{2})}{(2-\sqrt{2})\pi} < J(x), \qquad 0 < x < \frac{\pi}{2}, \tag{25}$$

where

$$J(x) = \frac{\cos x + \sqrt{1+\sin^2 x} - \frac{2\sin x}{x}}{\sqrt{1+\sin^2 x}-1}.$$

Differentiation yields

$$J'(x) = -\frac{J_2(x) - J_1(x)}{x^2\sqrt{1+\sin^2 x}\left(\sqrt{1+\sin^2 x}-1\right)^2},$$

where

$$J_2(x) = (2x^2-4)\sin x + x^2 \sin x \cos x + 2x\cos x + 2\sin x \cos^2 x > 0$$

and

$$J_1(x) = (x^2 \sin x + 2x\cos x - 2\sin x)\sqrt{1+\sin^2 x} > 0.$$

Following the same method as was used in the proof of $g(x) > 0$, we can prove $J_1(x) > 0$ and $J_2(x) > 0$, we omit them.

Elementary calculations reveal that

$$J_2^2(x) - J_1^2(x) = 2\sin x J_3(x),$$

where

$$\begin{aligned} J_3(x) = {} & 2x^3\cos^2 x + 2x^3\cos^3 x + 2\sin x\cos^4 x + 2x^2 \sin x \cos^3 x \\ & + (x^4-6)\sin x \cos^2 x + (2x^4 - 4x^2)\sin x\cos x + (4-4x^2+x^4)\sin x. \end{aligned}$$

We find

$$\begin{aligned}
J_3(x) &= 2x^3\left(\frac{1+\cos(2x)}{2}\right)+2x^3\left(\frac{\cos(3x)+3\cos x}{4}\right)\\
&\quad +2\sin x\left(\frac{\cos(4x)+4\cos(2x)+3}{8}\right)+2x^2\sin x\left(\frac{\cos(3x)+3\cos x}{4}\right)\\
&\quad +(x^4-6)\sin x\left(\frac{1+\cos(2x)}{2}\right)+(x^4-2x^2)\sin(2x)+(4-4x^2+x^4)\sin x\\
&= x^3+x^3\cos(2x)+\frac{1}{2}x^3\cos(3x)+\frac{3}{2}x^3\cos x+\frac{1}{8}\sin(5x)+\frac{1}{4}x^2\sin(4x)\\
&\quad +\left(\frac{1}{4}x^4-\frac{9}{8}\right)\sin(3x)+\left(x^4-\frac{3}{2}x^2\right)\sin(2x)+\left(\frac{5}{4}x^4-4x^2+\frac{11}{4}\right)\sin x\\
&= \frac{13}{540}x^9-\frac{299}{18900}x^{11}+\sum_{n=6}^{\infty}(-1)^n v_n(x),
\end{aligned}$$

with

$$v_n(x)=\frac{c_n}{216\cdot(2n+1)!}x^{2n+1},$$

where

$$\begin{aligned}
c_n &= 135\cdot 25^n-27n(2n+1)16^n+(32n^4-128n^3-8n^2+32n-729)9^n\\
&\quad +108n(2n+1)(2n^2-5n+5)4^n+4320n^4-6912n^3+2376n^2+3456n+594.
\end{aligned}$$

Elementary calculations reveal that, for $0<x<\pi/2$ and $n\geqslant 6$,

$$\frac{v_{n+1}(x)}{v_n(x)}=\frac{9x^2}{2(2n+3)}\frac{a_n}{b_n}<\frac{9(\pi/2)^2}{2(2n+3)}\frac{a_n}{b_n}<\frac{a_n}{b_n},$$

where

$$\begin{aligned}
a_n &= 375\cdot 25^n-(96n^2+240n++144)16^n+(32n^4-200n^2-240n-801)9^n\\
&\quad +(192n^4+384n^3+240n^2+336n+288)4^n\\
&\quad +480n^4+1152n^3+840n^2+528n+426
\end{aligned}$$

and

$$\begin{aligned}
b_n &= (n+1)\Big(135\cdot 25^n-(54n^2+27n)16^n+(32n^4-128n^3-8n^2+32n-729)9^n\\
&\quad +(432n^4-864n^3+540n^2+540n)4^n\\
&\quad +4320n^4-6912n^3+2376n^2+3456n+594\Big).
\end{aligned}$$

Elementary calculations reveal that

$$\begin{aligned} b_n - a_n = &(135n-240)25^n - 3(n+1)(18n^2-23n-48)16^n \\ &+ (32n^5-128n^4-136n^3+224n^2-457n+72)9^n \\ &+ 12(n+1)(36n^4-88n^3+29n^2+41n-24)4^n \\ &+ 4320n^5-3072n^4-5688n^3+4992n^2+3522n+168. \end{aligned}$$

We claim that

$$b_n - a_n > 0 \quad \text{for} \quad n \geqslant 6. \tag{26}$$

Direct computations show that $b_n - a_n > 0$ holds for $n = 6$, and $n = 7$. Noting that

$$\begin{aligned} &(32n^5-128n^4-136n^3+224n^2-457n+72)9^n > 0, \\ &12(n+1)(36n^4-88n^3+29n^2+41n-24)4^n > 0, \\ &4320n^5-3072n^4-5688n^3+4992n^2+3522n+168 > 0 \end{aligned}$$

hold for $n \geqslant 8$, we have

$$\frac{b_n - a_n}{(135n-240)16^n} > \left(\frac{25}{16}\right)^n - \frac{3(n+1)(18n^2-23n-48)}{135n-240} > 0 \quad \text{for} \quad n \geqslant 8.$$

The last inequality can be proved by induction on n, we omit it. Hence, the claim (26) holds.

We then obtan, for all $0 < x < \pi/2$ and $n \geqslant 6$,

$$\frac{v_{n+1}(x)}{v_n(x)} < 1.$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \longmapsto v_n(x)$ is strictly decreasing for $n \geqslant 6$. We then obtain, for $0 < x < \pi/2$,

$$J_3(x) > x^9\left(\frac{13}{540} - \frac{299}{18900}x^2\right) > 0 \quad \text{and} \quad J'(x) < 0.$$

So, $J(x)$ is strictly decreasing for $0 < x < \pi/2$, and we have

$$\frac{2(\pi - 2\sqrt{2})}{(2-\sqrt{2})\pi} = J\left(\frac{\pi}{2}\right) < J(x), \qquad 0 < x < \frac{\pi}{2}.$$

Hence, the right side of (11) holds. The proof is complete. □

THEOREM 2. *The inequalities (12) are true.*

Proof. Noting that $G < L$ holds, we see that the upper bound in (11) is sharper than the upper bound in (12). Hence, the right-hand side of (12) holds.

By Remark 1, the left-hand side of (12) may be rewritten for $0 < x < 1$ as

$$\frac{4}{\arcsin x} > \frac{6}{\ln \frac{1+x}{1-x}} + \frac{1}{\arctan x}. \tag{27}$$

We first prove (27) for $0 < x < 0.7$. From the well known continued fraction for $\ln \frac{1+x}{1-x}$ (see [6, p. 196 Eq. (11.2.4)]), we find that for $0 < x < 1$,

$$\frac{2x(15-4x^2)}{3(5-3x^2)} = \frac{2x}{1+\frac{-\frac{1}{3}x^2}{1+\frac{-\frac{4}{15}x^2}{1}}} < \ln \frac{1+x}{1-x}. \tag{28}$$

It follows from (28) and (21) that

$$\frac{4}{\arcsin x} - \left(\frac{6}{\ln \frac{1+x}{1-x}} + \frac{1}{\arctan x}\right) > \frac{4}{\arcsin x} - \left(\frac{6}{\frac{2x(15-4x^2)}{3(5-3x^2)}} + \frac{1}{\frac{3x}{3+x^2}}\right)$$
$$= 4\left[\frac{1}{\arcsin x} - \frac{90-39x^2-2x^4}{6x(15-4x^2)}\right].$$

In order to prove (27) for $0 < x < 0.7$, it suffices to show that

$$U(x) = \frac{6x(15-4x^2)}{90-39x^2-2x^4} - \arcsin x > 0 \quad \text{for} \quad 0 < x < 0.7.$$

Differentiation yields

$$U'(x) = \frac{6(1350-495x^2+246x^4-8x^6)}{(90-39x^2-2x^4)^2} - \frac{1}{\sqrt{1-x^2}}.$$

Direct computation yields

$$\left(\frac{6(1350-495x^2+246x^4-8x^6)}{(90-39x^2-2x^4)^2}\right)^2 - \frac{1}{1-x^2} = \frac{U_1(x)+U_2(x)}{(90-39x^2-2x^4)^4(1-x^2)},$$

where

$$U_1(x) = 12757500 - 28503900x^2 + 12786255x^4 - 2911464x^6$$

and

$$U_2(x) = 110376x^8 - 3552x^{10} - 16x^{12}.$$

We now prove $U'(x) > 0$ for $0 < x < 0.7$. It suffices to show that

$$U_1(x) > 0 \quad \text{and} \quad U_2(x) > 0 \quad \text{for} \quad 0 < x < 0.7.$$

Differentiation yields

$$U_1'(x) = -x(57007800 - 51145020x^2 + 17468784x^4) < 0 \quad \text{for} \quad 0 < x < 0.7.$$

Hence, $U_1(x)$ is strictly decreasing for $0 < x < 0.7$, and we have

$$U_1(x) > U_1\left(\frac{7}{10}\right) = \frac{379509499341}{250000} > 0 \quad \text{for} \quad 0 < x < 0.7.$$

Clearly,

$$U_2(x) = x^8(110376 - 3552x^2 - 16x^4) > 0 \quad \text{for} \quad 0 < x < 0.7.$$

We then obtain $U'(x) > 0$ for $0 < x < 0.7$, and we have

$$U(x) > U(0) = 0 \quad \text{for} \quad 0 < x < 0.7.$$

Hence, (27) holds for $0 < x < 0.7$.

Second, we prove (27) for $0.7 \leqslant x < 1$. Let

$$y(x) = y_1(x) + y_2(x),$$

where

$$y_1(x) = -\left(\frac{6}{\ln\frac{1+x}{1-x}} + \frac{1}{\arctan x}\right) \quad \text{and} \quad y_2(x) = \frac{4}{\arcsin x}.$$

Let $0.7 \leqslant r \leqslant x \leqslant s < 1$. Since $y_1(x)$ is increasing and $y_2(x)$ is decreasing, we obtain

$$y(x) \geqslant y_1(r) + y_2(s) =: \sigma(r,s).$$

We divide the interval $[0.7, 1]$ into 300 subintervals:

$$[0.7, 1] = \bigcup_{k=0}^{299}\left[0.7 + \frac{k}{1000}, 0.7 + \frac{k+1}{1000}\right] \quad \text{for} \quad k = 0, 1, 2, \ldots, 299.$$

By direct computation we get

$$\sigma\left(0.7 + \frac{k}{1000}, 0.7 + \frac{k+1}{1000}\right) > 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, 299.$$

Hence,

$$y(x) > 0 \quad \text{for} \quad x \in \left[0.7 + \frac{k}{1000}, 0.7 + \frac{k+1}{1000}\right] \quad \text{and} \quad k = 0, 1, 2, \ldots, 299.$$

This implies that $y(x)$ is positive on $[0.7, 1)$. This proves (27) for $0.7 \leqslant x < 1$. Hence, (27) holds for all $0 < x < 1$. The proof is complete. □

THEOREM 3. *The inequalities* (13) *are true.*

Proof. Using the second inequalities in (2) and (4), combined with (16), we find

$$2P+T<\frac{2G+4A+A+2Q}{3}=\frac{5A+2(Q+G)}{3}<\frac{5A+4A}{3}=3A.$$

This proves the left-hand side of (13).

By (6) and (7), after some elementary computations we obtain

$$(4-\pi)P+(\pi-2)T>2Am+n\big[(\sqrt{2}Q+(2-\sqrt{2})G\big], \tag{29}$$

where

$$m=\frac{\pi^2-4\pi-\pi\sqrt{2}+8}{\pi(2-\sqrt{2})}\quad\text{and}\quad n=\frac{(\pi-2)(4-\pi)}{\pi(2-\sqrt{2})}.$$

By multiplying both sides of inequality (17) by $\sqrt{2}$, we obtain

$$\sqrt{2}Q+(2-\sqrt{2})G>2A. \tag{30}$$

Noting that $m+n=1$ holds, it follows from (2) and (30) that

$$(4-\pi)P+(\pi-2)T>2A(m+n)=2A.$$

This proves the right-hand side of (13). The proof is complete. □

THEOREM 4. *The inequalities* (14) *are true.*

Proof. By Remark 1, the left-hand side of (14) may be rewritten for $0<z<1$ as

$$\frac{z}{\arcsin z}+3\sqrt{1+z^2}<\frac{4z}{\arctan z}. \tag{31}$$

The proof of (31) makes use of the following inequality:

$$\frac{z}{\arcsin z}<\frac{3(20-9z^2)}{60-17z^2},\qquad 0<z<1 \tag{32}$$

and

$$\frac{z}{\arctan z}>\frac{3(3z^2+5)}{4z^2+15},\qquad 0<z<1. \tag{33}$$

We now prove (32) and (33). For $0<z<1$, let

$$f_1(z)=\arcsin z-\frac{z(60-17z^2)}{3(20-9z^2)}\quad\text{and}\quad f_2(z)=\frac{z(4z^2+15)}{3(3z^2+5)}-\arctan z.$$

Differentiation yields

$$f_1'(z) = \frac{1}{\sqrt{1-z^2}} - \frac{400-160z^2+51z^4}{(20-9z^2)^2} > 0 \tag{34}$$

and

$$f_2'(z) = \frac{4z^6}{(3z^2+5)^2(1+z^2)} > 0.$$

The inequality (34) holds, because

$$\frac{1}{1-z^2} - \left(\frac{400-160z^2+51z^4}{(20-9z^2)^2}\right)^2 = \frac{z^6(24400-12360z^2+2601z^4)}{(1-z^2)(20-9z^2)^4} > 0.$$

Therefore, $f_1(z)$ and $f_2(z)$ are both strictly increasing for $0<z<1$, and we have

$$f_1(z) > f_1(0) = 0 \quad \text{and} \quad f_2(z) > f_2(0) = 0 \quad \text{for} \quad 0<z<1.$$

This proves (32) and (33).

We now prove (31). For $0<z<1$, we have, by (32) and (33),

$$\begin{aligned} \frac{z}{\arcsin z} + 3\sqrt{1+z^2} - \frac{4z}{\arctan z} &< \frac{3(20-9z^2)}{60-17z^2} + 3\sqrt{1+z^2} - \frac{12(3z^2+5)}{4z^2+15} \\ &= -3\left\{\frac{3(145z^2+300-56z^4)}{(60-17z^2)(4z^2+15)} - \sqrt{1+z^2}\right\}. \end{aligned} \tag{35}$$

Elementary calculations reveal that

$$\begin{aligned} &\left(\frac{3(145z^2+300-56z^4)}{(60-17z^2)(4z^2+15)}\right)^2 - (1+z^2) \\ &\quad = \frac{x^4(36000+26025z^2+21560z^4-4624z^6)}{(60-17z^2)^2(4z^2+15)^2} > 0 \end{aligned}$$

for $0<z<1$. From (35), we obtain (31). Hence, the left-hand side of (14) holds.

We now prove the right-hand side of (14). By (6) and the right-hand side of (19), we have

$$\begin{aligned} &\frac{\pi-2\sqrt{2}}{\pi-\sqrt{2}}P + \frac{\sqrt{2}}{\pi-\sqrt{2}}Q - T \\ &\quad > \frac{\pi-2\sqrt{2}}{\pi-\sqrt{2}}\left(\frac{\pi-2}{\pi}G + \frac{2}{\pi}A\right) + \frac{\sqrt{2}}{\pi-\sqrt{2}}Q - \left(\frac{\pi-2\sqrt{2}}{\pi}H + \frac{2\sqrt{2}}{\pi}Q\right) \\ &\quad = \frac{\pi-2\sqrt{2}}{\pi(\pi-\sqrt{2})}\left\{(\pi-2)G + 2A - (\pi-\sqrt{2})H - \sqrt{2}Q\right\}. \end{aligned}$$

In order to prove the right-hand side of (14), it suffices to show that

$$(\pi-2)G+2A-(\pi-\sqrt{2})H>\sqrt{2}Q,$$

which may be rewritten, by Remark 1, as

$$(\pi-2)\sqrt{1-z^2}+2-(\pi-\sqrt{2})(1-z^2)>\sqrt{2}\sqrt{1+z^2}, \qquad 0<z<1.$$

By an elementary change of variable $x=\sqrt{1-z^2}\,(0<z<1)$, it suffices to show that

$$(\pi-2)x+2-(\pi-\sqrt{2})x^2>\sqrt{2}\sqrt{2-x^2}, \qquad 0<x<1. \tag{36}$$

Elementary calculations reveal that

$$\left((\pi-2)x+2-(\pi-\sqrt{2})x^2\right)^2-\left(\sqrt{2}\sqrt{2-x^2}\right)^2=xD(x),$$

where

$$\begin{aligned}D(x)=&-8+4\pi+(6+\pi^2-8\pi+4\sqrt{2})x\\&+(-2\pi^2+2\pi\sqrt{2}+4\pi-4\sqrt{2})x^2+(-2\pi\sqrt{2}+\pi^2+2)x^3.\end{aligned}$$

Differentiation yields

$$\begin{aligned}D'(x)=&\,6+\pi^2-8\pi+4\sqrt{2}+2(-2\pi^2+2\pi\sqrt{2}+4\pi-4\sqrt{2})x\\&+3(\pi^2-2\pi\sqrt{2}+2)x^2<0, \qquad 0<x<1.\end{aligned}$$

So, $D(x)$ is strictly decreasing for $0<x<1$, and we have

$$D(x)>D(1)=0, \qquad 0<x<1.$$

Therefore, (36) holds. Hence, the right-hand side of (14) holds. The proof is complete. □

REMARK 2. Vukšić conjectured (see the left-hand side of (3.22) of Conjecture 3.6 in [22]) that

$$\frac{L+T}{2}<A. \tag{37}$$

In fact, the left-hand side of (13) is sharper than (37), as the inequality $(L+T)/2<(2P+T)/3$ is equivalent to $(3L+T)/4<P$, which is the left-hand side of (12). Therefore, one has the following refinement of (37):

$$\frac{L+T}{2}<\frac{2P+T}{3}<A. \tag{38}$$

REMARK 3. Relation (4) can be used to prove the following Conjecture (see the right-hand side of (3.20) of Conjecture 3.6 in [22]):

$$T < \frac{H+2N}{3}. \tag{39}$$

Remark that $H = G^2/A$ and $N = Q^2/A$, so inequality (39) may be rewritten as

$$T < \frac{G^2+2Q^2}{3A}. \tag{40}$$

The inequality (40) follows by the right-hand side of (4), as the inequality $(A+2Q)/3 < (G^2+2Q^2)/(3A)$ via the identity $G^2+Q^2 = 2A^2$ may be rewritten as $2AQ < A^2+Q^2$, or $(Q-A)^2 > 0$, which is true.

REMARK 4. Vukšić conjectured (see the left-hand side of (3.23) of Conjecture 3.6 in [22]) that

$$\frac{L+4Q}{5} < T. \tag{41}$$

By the left-hand sides of (14) and (12), we have

$$T > \frac{P+3Q}{4} > \frac{(3L+T)/4+3Q}{4} = \frac{3L+T+12Q}{16},$$

which implies (41).

REMARK 5. Vukšić conjectured (see the right-hand side of (3.24) of Conjecture 3.6 of [22]) that

$$T < \frac{2}{3}A + \frac{1}{3}N \quad \text{(typo corrected)}. \tag{42}$$

Noting that the following identity holds true:

$$H+N = 2A, \tag{43}$$

we can state that (42) is the same as (39).

The left-hand side of (3.24) of Conjecture 3.6 of [22] is

$$\frac{(2\pi-4)A+(4-\pi)N}{\pi} < T, \tag{44}$$

and the left-hand side of (3.20) of Conjecture 3.6 of [22] is

$$\frac{(\pi-2)H+2N}{\pi} < T. \tag{45}$$

In fact, (44) and (45) are the same, by identity (43). The inequality (44) appears (with notation C in place of N) in [23] (Corollary 8.2).

Similarly, the right-hand side of (3.18) of Conjecture 3.6 of [22]

$$A < \frac{\pi T + (4-\pi)H}{4} \tag{46}$$

may be written as

$$T > \frac{4A - (4-\pi)H}{\pi} = \frac{(\pi-2)H + 2N}{\pi} \tag{47}$$

by identity (43). Thus inequality (47) is the same as (45), and this proves also (46).

The left-hand side of (3.18) of Conjecture 3.6 of [22]

$$A > \frac{H + 3T}{4} \tag{48}$$

can be written for $0 < x < 1$ as

$$1 - x^2 + \frac{3x}{\arctan x} < 4,$$

which can be rewritten as (21). Therefore, (48) is proved.

3. An improvement of (15)

THEOREM 5. *For* $0 < x < 1$*, we have*

$$2\left(\frac{x}{\arcsin x}\right) + \frac{x}{\arctan x} < 3 - \frac{11}{60}x^4\left(\frac{x}{\arcsin x}\right). \tag{49}$$

The constant $\frac{11}{60}$ *is the best possible.*

Proof. For $0 < x < 1$, we have

$$\begin{aligned} &\frac{2x + \frac{11}{60}x^5}{\arcsin x} + \frac{x}{\arctan x} - 3 \\ &< \frac{2x + \frac{11}{60}x^5}{x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \frac{63}{2816}x^{11}} \\ &\quad + \frac{x}{x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11}} - 3 \\ &= -\frac{15x^6 P(x)}{(887040 + 147840x^2 + 66528x^4 + 39600x^6 + 26950x^8 + 19845x^{10})Q(x)}, \end{aligned} \tag{50}$$

where

$$\begin{aligned} P(x) = {}& 6667584 + 13142052x^2 - 32340x^4 - 13134605x^6 + 2355507x^8 \\ & - 2384305x^{10} - 169785x^{12} - 1250235x^{14} \end{aligned}$$

and

$$Q(x) = 3465 - 1155x^2 + 693x^4 - 495x^6 + 385x^8 - 315x^{10}.$$

Now we prove $P(x) > 0$ and $Q(x) > 0$ for $0 < x < 1$. Define functions $F(t)$ and $G(t)$ by

$$F(t) = P(\sqrt{t}) \quad \text{and} \quad G(t) = Q(\sqrt{t}).$$

We find that for $0 < t < 1$,

$$F''(t) = -64680 - t(78807630 - 28266084t + 47686100t^2) \\ - 5093550t^4 - 52509870t^5 < 0.$$

Hence, $F(t)$ is strictly concave for $0 < t < 1$, and we have

$$F(t) > \min\{F(0), F(1)\} = 5193873 > 0, \quad 0 < t < 1 \Longrightarrow P(x) > 0, \quad 0 < x < 1.$$

We find that for $0 < t < 1$,

$$G'(t) = -1155 + 1386t - 1485t^2 + 1540t^3 - 1575t^4$$

and

$$G'''(t) = -2970 + 9240t - 18900t^2 < 0.$$

Hence, $G'(t)$ is strictly concave for $0 < t < 1$, and we have

$$G'(t) \leqslant \max_{0<t<1} \{G'(t)\} = -728.419216\ldots < 0, \quad 0 < t < 1.$$

Thus, $G(t)$ is strictly decreasing for $0 < t < 1$, and we have

$$G(t) > G(1) = 2578 > 0, \quad 0 < t < 1 \Longrightarrow Q(x) > 0, \quad 0 < x < 1.$$

From (50), we obtain (49).

Write (49) as

$$-\frac{2\left(\frac{x}{\arcsin x}\right) + \frac{x}{\arctan x} - 3}{x^5/\arcsin x} > \frac{11}{60}.$$

We find

$$\lim_{x\to 0}\left\{-\frac{2\left(\frac{x}{\arcsin x}\right) + \frac{x}{\arctan x} - 3}{x^5/\arcsin x}\right\} = \frac{11}{60}.$$

This means that inequality (49) holds with the best possible constant $\frac{11}{60}$. The proof is complete. □

Acknowledgement. We thank the referee for helpful comments.

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(Received June 21, 2016)

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