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# AN INTERPOLATION OF JENSEN'S INEQUALITY AND ITS CONVERSES WITH APPLICATIONS TO QUASI–ARITHMETIC MEAN INEQUALITIES

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*Abstract.* In this paper, we show an interpolation of Davis-Choi-Jensen operator inequality and the converse inequality for Hilbert space operators. As applications, we obtain an interpolation of quasi-arithmetic mean inequalities and the converse inequalities.

## 1. Introduction

In [8], we showed operator versions of the inequality due to Cho, Matić and Pečarić [1] in connection to Jensen's inequality for convex functions. As applications, we obtain an interpolation of the weighted arithmetic-geometric mean inequality for the Karcher mean of positive invertible operators on a Hilbert space. Moreover, we obtain an interpolation between the quasi-arithmetic means.

As a continuation of our research in [8], in this paper we show another interpolation of Davis-Choi-Jensen operator inequalities for positive linear mappings. As applications, we obtain an interpolation of both the quasi-arithmetic means of operators and the mean inequalities for the operator power means due to Lawson and Lim. Moreover, we give converses of the above results.

## 2. Results related to Jensen's inequality

Let $\mathscr{B}(\mathscr{H})$ (resp. $\mathscr{B}_h(\mathscr{H})$) be the algebra of all bounded linear operators (resp. selfadjoint operators) on a Hilbert space $\mathscr{H}$. A real valued continuous function $f$ defined on an interval $[m,M]$ is said to be operator convex if $f((1-t)A+tB) \leqslant (1-t)f(A)+tf(B)$ for all selfadjoint opetaors $A,B$ in $\mathscr{B}_h(\mathscr{H})$ with $m \leqslant A,B \leqslant M$. By the Davis-Choi-Jensen operator inequality [4, Theorem 8.9], we have $f(\sum_{j=1}^n \Phi_j(A_j)) \leqslant \sum_{j=1}^n \Phi_j(f(A_j))$ for positive linear mappings $\Phi_j$ with $\sum_{j=1}^n \Phi_j(1_{\mathscr{H}}) = 1_{\mathscr{H}}$ and selfadjoint operators $A_j$ with $m \leqslant A_j \leqslant M$, and an operator convex function $f$ on $[m,M]$ for $j=1,\ldots,n$.

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Mićić and Pečarić gave in [6] some mappings related to the Davis-Choi-Jensen operator inequality as a generalization of the result with respect to the Hermite-Hadamard inequality due to Dragomir [2, 3].

First, we give the multiple operator versions of [6, Theorem 2.2], which connects both sides of the Davis-Choi-Jensen operator inequality:

THEOREM 2.1. *For* $j=1,\ldots,n$, *let* $A_j \in \mathscr{B}_h(\mathscr{H})$ *be self-adjoint operators such that* $m \leqslant A_j \leqslant M$ *for some scalars* $m \leqslant M$ *and* $\Phi_j$ *positive linear mappings on* $\mathscr{B}(\mathscr{H})$ *such that* $\sum_{j=1}^n \Phi_j(1_{\mathscr{H}}) = 1_{\mathscr{H}}$. *If* $f(x)$ *is operator convex on* $[m,M]$ *and* $\sum_{j=1}^n \Phi_j$ *preserves the operator* $A_0 = \sum_{k=1}^n \Phi_k(A_k)$ ( *i.e.* $\sum_{j=1}^n \Phi_j(A_0) = A_0$ ), *then*

$$f\left(\sum_{j=1}^n \Phi_j(A_j)\right) \leqslant \sum_{j=1}^n \Phi_j\left(f\left(tA_j + (1-t)\sum_{k=1}^n \Phi_k(A_k)\right)\right) \leqslant \sum_{j=1}^n \Phi_j\big(f(A_j)\big) \tag{2.1}$$

*for all* $t \in [0,1]$. *Moreover, the function*

$$F(t) = \sum_{j=1}^n \Phi_j\left(f\left(tA_j + (1-t)\sum_{k=1}^n \Phi_k(A_k)\right)\right)$$

*is monotonically nondecreasing and convex on* $[0,1]$.

*Proof.* We give the direct proof for convenience. Since $\Phi_j$ is a positive linear mapping for all $j=1,\ldots,n$ and $f$ is operator convex, then we have for $t_1,t_2 \in [0,1]$ and $0<t<1$

$$\begin{aligned}
& tF(t_1) + (1-t)F(t_2) \\
& = \sum_{j=1}^n \Phi_j\Big(tf\Big(t_1A_j + (1-t_1)A_0\Big) + (1-t)f\Big(t_2A_j + (1-t_2)A_0\Big)\Big) \\
& \geqslant \sum_{j=1}^n \Phi_j f\Big(t(t_1A_j + (1-t_1)A_0) + (1-t)(t_2A_j + (1-t_2)A_0)\Big) \\
& = \sum_{j=1}^n \Phi_j f\Big((tt_1 + (1-t)t_2)A_j + (1-tt_1-(1-t)t_2)A_0\Big) \\
& = F(tt_1 + (1-t)t_2)
\end{aligned}$$

and so $F$ is convex on $[0,1]$.

Next, since $\sum_{j=1}^n \Phi_j$ is a unital positive linear mapping, it follows from Davis-Choi-Jensen operator inequality [4, Theorem 8.9] that

$$f\left(\sum_{j=1}^n \Phi_j(B_j)\right) \leqslant \sum_{j=1}^n \Phi_j\big(f(B_j)\big)$$

and thus

$$\begin{aligned}
&\sum_{j=1}^{n} \Phi_j\left(f\left(tA_j + (1-t)\sum_{k=1}^{n}\Phi_k(A_k)\right)\right) \\
&\geqslant f\left(\sum_{j=1}^{n} \Phi_j\left(tA_j + (1-t)\sum_{k=1}^{n}\Phi_k(A_k)\right)\right) \\
&= f\left(t\sum_{j=1}^{n} \Phi_j(A_j) + (1-t)\sum_{j=1}^{n}\Phi_j(A_0)\right) \\
&= f\left(\sum_{j=1}^{n} \Phi_j(A_j)\right) \qquad \text{by } \textstyle\sum_{j=1}^{n}\Phi_j(A_0) = A_0.
\end{aligned}$$

So $F(0) \leqslant F(t)$. If $0 \leqslant s < t \leqslant 1$, then $s = \frac{t-s}{t}\cdot 0 + \frac{s}{t}\cdot t$, and the convexity of $F$ implies

$$F(s) \leqslant \frac{t-s}{t}F(0) + \frac{s}{t}F(t) \leqslant F(t)$$

and so $F$ is monotonically nondecreasing on $[0,1]$. Hence it follows that $F(0) \leqslant F(t) \leqslant F(1)$ for all $t \in [0,1]$ and therefore (2.1) is valid. $\square$

A vector $\omega = (\omega_1, \ldots, \omega_n)$ is called *a weight vector* if $\omega_j \geqslant 0$ for all $j = 1, \ldots, n$ and $\sum_{j=1}^{n} \omega_j = 1$. A $n$-tuple $\Phi = (\Phi_1, \ldots, \Phi_n)$ of positive linear mappings on $\mathscr{B}(\mathscr{H})$ is called *totally unital* if $\sum_{j=1}^{n} \Phi_j(1_{\mathscr{H}}) = 1_{\mathscr{H}}$.

REMARK 2.2. We present a simple example which satisfies the conditions of Theorem 2.1. Let $\mathbb{M}_n(\mathscr{B}(\mathscr{H}))$ be the algebra of all $n\times n$ matrices with entries from $\mathscr{B}(\mathscr{H})$, and $P_j = 0 \oplus \cdots \oplus 1_{\mathscr{H}} \oplus \cdots \oplus 0$ an orthogonal projection for $j = 1, \ldots, n$ which is $1_{\mathscr{H}}$ at the $j$th position and zeros everywhere else. Put $\Phi_j(X) = P_jXP_j$ for $j = 1, \ldots, n$. Then $\Phi = (\Phi_1, \ldots, \Phi_n)$ is totally unital and $\sum_{j=1}^{n} \Phi_j$ preserves the operator $A_0 = \sum_{k=1}^{n} \Phi_k(A_k)$ for $A_k = (A_{ij}^k) \in \mathbb{M}_n(\mathscr{B}(\mathscr{H}))$ and $k = 1, \ldots, n$. In fact,

$$\sum_{j=1}^{n} \Phi_j(A_0) = \sum_{j=1}^{n} \Phi_j(A_{11}^1 \oplus \cdots \oplus A_{nn}^n) = A_{11}^1 \oplus \cdots \oplus A_{nn}^n = A_0.$$

In this case, if $f(x)$ is operator convex on $[m,M]$ and $f(A_k) = (f(A_k)_{ij}) \in \mathbb{M}_n(\mathscr{B}(\mathscr{H}))$ for $k = 1, \ldots, n$, then Theorem 2.1 implies

$$f(A_{jj}^j) \leqslant f(tA_j + (1-t)A_0)_{jj} \leqslant f(A_j)_{jj}$$

for all $t \in [0,1]$ and $j = 1, \ldots, n$.

Since $\sum_{j=1}^{n} \omega_j \left(\sum_{k=1}^{n} \omega_k A_k\right) = \sum_{k=1}^{n} \omega_k A_k$, putting $\Phi_j(X) = \omega_j X$ for $X \in \mathscr{B}(\mathscr{H})$ in Theorem 2.1, we obtain a version of (2.1) with unital positive linear mappings as follows (see [8, Theorem 2.2]):

COROLLARY 2.3. *For $j=1,\dots,n$, let $A_j$ be self-adjoint operators with $m \leqslant A_j \leqslant M$ for some scalars $m \leqslant M$, $\Phi_j$ unital positive linear mappings on $\mathscr{B}(\mathscr{H})$, and $\omega=(\omega_1,\dots,\omega_n)$ a weight vector. If $f(x)$ is operator convex on $[m,M]$, then*

$$\begin{aligned} f\left(\sum_{j=1}^n \omega_j \Phi_j(A_j)\right) &\leqslant \sum_{j=1}^n \omega_j f\left(t\Phi_j(A_j)+(1-t)\sum_{k=1}^n \omega_k \Phi_k(A_k)\right) \\ &\leqslant \sum_{j=1}^n \omega_j \Phi_j\big(f(A_j)\big) \end{aligned} \tag{2.2}$$

*for all $t \in [0,1]$. Moreover, the function*

$$F(t)=\sum_{j=1}^n \omega_j f\left(t\Phi_j(A_j)+(1-t)\sum_{k=1}^n \omega_k \Phi_k(A_k)\right)$$

*is monotonically nondecreasing and convex on $[0,1]$.*

*Proof.* If we put $\Phi_j(X)=\omega_j X$ in Theorem 2.1, then

$$f\left(\sum_{j=1}^n \omega_j A_j\right) \leqslant \sum_{j=1}^n \omega_j \left(f\left(tA_j+(1-t)\sum_{k=1}^n \omega_k A_k\right)\right) \leqslant \sum_{j=1}^n \omega_j \big(f(A_j)\big)$$

holds for all $t \in [0,1]$ and the function

$$t \mapsto \sum_{j=1}^n \omega_j \left(f\left(tA_j+(1-t)\sum_{k=1}^n \omega_k (A_k)\right)\right)$$

is monotonically nondecreasing and convex on $[0,1]$. Now, replacing $A_j$ by $\Phi_j(A_j)$ in the above results, where $\Phi_j$ is a unital positive linear mapping, we obtain

$$\begin{aligned} f\left(\sum_{j=1}^n \omega_j \Phi_j(A_j)\right) &\leqslant \sum_{j=1}^n \omega_j f\left(t\Phi_j(A_j)+(1-t)\sum_{k=1}^n \omega_k \Phi_k(A_k)\right) \\ &\leqslant \sum_{j=1}^n \omega_j f\big(\Phi_j(A_j)\big) \\ &\leqslant \sum_{j=1}^n \omega_j \Phi_j\big(f(A_j)\big) \quad \text{(by the Davis-Choi-Jensen inequality)} \end{aligned}$$

for all $t \in [0,1]$ and the function $F$ is monotonically nondecreasing and convex on $[0,1]$. $\square$

## 3. Application of Jensen's inequality

As an application, we obtain an interpolation between quasi-arithmetic means. We define the quasi-arithmetic mean of operators:

$$\mathfrak{M}_\varphi(\mathbf{A};\mathbf{\Phi}) := \varphi^{-1}\left(\sum_{j=1}^{n}\Phi_j\left(\varphi(A_j)\right)\right), \tag{3.1}$$

where $\mathbf{A} = (A_1,\ldots,A_k)$ is an $n$-tuple of self-adjoint operators in $\mathscr{B}_h(\mathscr{H})$ with spectra in an interval $I$, $\mathbf{\Phi} = (\Phi_1,\ldots,\Phi_k)$ is an $n$-tuple of positive linear mappings $\Phi_j : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ such that $\sum_{j=1}^{n}\Phi_j(1_{\mathscr{H}}) = 1_{\mathscr{K}}$, and $\varphi : I \to \mathbb{R}$ is a strictly monotone function. If $\varphi^{-1}$ is operator concave on $\varphi(I)$, then

$$\mathfrak{M}_\varphi(\mathbf{A};\Phi) \geqslant \mathfrak{M}_1(\mathbf{A};\Phi) = \sum_{j=1}^{n}\Phi_j(A_j). \tag{3.2}$$

The power mean is a special case of the quasi-arithmetic mean

$$\mathfrak{M}_r(\mathbf{A};\mathbf{\Phi}) := \begin{cases} \left(\sum_{j=1}^{n}\Phi_j\left(A_j^r\right)\right)^{1/r}, & r \in \mathbb{R}\backslash\{0\}, \\ \exp\left(\sum_{j=1}^{n}\Phi_j\left(\log(A_j)\right)\right), & r = 0\,, \end{cases} \tag{3.3}$$

where $\mathbf{A} = (A_1,\ldots,A_n)$ is an $n$-tuple of positive invertible operators.

Replacing $\Phi_j$ by $\omega_j\Phi_j$ in (3.1), where $\Phi_j : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ are unital positive linear mappings for all $j = 1,\ldots,n$, and $\omega = (\omega_1,\ldots,\omega_n)$ is a weight vector, we have special cases of (3.1)

$$\mathfrak{M}_\varphi(\omega;\mathbf{A};\mathbf{\Phi}) := \varphi^{-1}\left(\sum_{j=1}^{n}\omega_j\Phi_j\left(\varphi(A_j)\right)\right). \tag{3.4}$$

We can define analogue a mean $\mathfrak{M}_r(\omega;\mathbf{A};\mathbf{\Phi})$ by using (3.3).

By virtue of Theorem 2.1, we have an interpolation of the quasi-arithmetic mean and the arithmetic mean $\mathfrak{M}_1(\mathbf{A};\Phi)$ in (3.2).

THEOREM 3.1. *Let $\mathbf{A} = (A_1,\ldots,A_n)$ be an $n$-tuple of self-adjoint operators in $\mathscr{B}_h(\mathscr{H})$ with spectra in an interval $I$, $\mathbf{\Phi} = (\Phi_1,\ldots,\Phi_n)$ a totally unital $n$-tuple of positive linear mappings $\Phi_j$ on $\mathscr{B}(\mathscr{H})$, and $\mathbf{e_j}$, $j = 1,\ldots,n$ the standard basis vector in $\mathbb{R}^n$ ( i.e. $\mathbf{e_j}$ be an $n$-tuple that has $1$ at the $j$th position and zeros everywhere else ).*

*If $\varphi : I \to \mathbb{R}$ is a strictly monotone function such that $\varphi^{-1}$ is operator concave on $\varphi(I)$ and $\sum_{j=1}^{n}\Phi_j$ preserves the operator $A_0 = \sum_{k=1}^{n}\Phi_k(A_k)$, then*

$$\mathfrak{M}_\varphi(\mathbf{A};\mathbf{\Phi}) \geqslant \sum_{j=1}^{n}\Phi_j\left(\mathfrak{M}_\varphi(\mathbf{A};t\mathbf{e_j} + (1-t)\mathbf{\Phi})\right) \geqslant \mathfrak{M}_1(\mathbf{A};\mathbf{\Phi}) \tag{3.5}$$

*for all* $t \in [0,1]$. *Moreover,*

$$M(t) := \sum_{j=1}^{n} \Phi_j\big(\mathfrak{M}_\varphi(\mathbf{A}; t\mathbf{e_j} + (1-t)\mathbf{\Phi})\big)$$

*is monotonically nonincreasing and concave on* $[0,1]$.

*But, if* $\varphi^{-1}$ *is operator convex on* $\varphi(I)$, *then the reverse inequalities are valid in* (3.5) *and* $M(t)$ *is monotonically nondecreasing and convex on [0,1].*

*Proof.* If we replace $A_j$ by $\varphi(A_j)$ in Theorem 2.1 we obtain

$$\begin{aligned} f\left(\sum_{j=1}^{n} \Phi_j(\varphi(A_j))\right) &\geqslant \sum_{j=1}^{n} \Phi_j\left(f\left(t\varphi(A_j) + (1-t)\sum_{k=1}^{n} \Phi_k(\varphi(A_k))\right)\right) \\ &\geqslant \sum_{j=1}^{n} \Phi_j\big(f(\varphi(A_j))\big) \end{aligned}$$

for all $t \in [0,1]$, where $f$ is operator concave on $\varphi(I)$. Since $\varphi^{-1}$ is operator concave on $\varphi(I)$, then

$$\begin{aligned} &\varphi^{-1}\left(\sum_{j=1}^{n} \Phi_j(\varphi(A_j))\right) \\ &\geqslant \sum_{j=1}^{n} \Phi_j\left(\varphi^{-1}\Big(t\varphi(A_j) + (1-t)\Phi_1(\varphi(A_1)) + \ldots + (1-t)\Phi_n(\varphi(A_n))\Big)\right) \\ &\geqslant \sum_{j=1}^{n} \Phi_j(A_j), \end{aligned}$$

which give the desired inequality (3.5). Next, it follows from Theorem 2.1 that

$$\begin{aligned} M(t) &= \sum_{j=1}^{n} \Phi_j\left(\varphi^{-1}\left(t\varphi(A_j) + (1-t)\sum_{k=1}^{n} \Phi_k(\varphi(A_k))\right)\right) \\ &= \sum_{j=1}^{n} \Phi_j\big(\mathfrak{M}_\varphi(\mathbf{A}; t\mathbf{e_j} + (1-t)\mathbf{\Phi}\big) \end{aligned}$$

is monotonically nonincreasing and concave on $[0,1]$. □

Combining inequalities in Theorem 3.1 we obtain the following corollary.

COROLLARY 3.2. *Let the assumptions of Theorem* 3.1 *hold and* $\sum_{j=1}^{n} \Phi_j$ *preserves the operator* $A_0 = \sum_{k=1}^{n} \Phi_k(A_k)$. *If* $\varphi, \psi : I \to \mathbb{R}$ *is a strictly monotone functions such that* $\varphi^{-1}$ *is operator concave and* $\psi^{-1}$ *is operator convex, then*

$$\begin{aligned} \mathfrak{M}_\varphi(\mathbf{A};\mathbf{\Phi}) &\geqslant \sum_{j=1}^{n} \Phi_j\big(\mathfrak{M}_\varphi(\mathbf{A}; t\mathbf{e_j} + (1-t)\mathbf{\Phi})\big) \geqslant \mathfrak{M}_1(\mathbf{A};\mathbf{\Phi}) \\ &\geqslant \sum_{j=1}^{n} \Phi_j\big(\mathfrak{M}_\psi(\mathbf{A}; t\mathbf{e_j} + (1-t)\mathbf{\Phi})\big) \geqslant \mathfrak{M}_\Psi(\mathbf{A};\mathbf{\Phi}) \end{aligned}$$

*for all* $t \in [0,1]$.

Now, we give a generalization of inequalities in Corollary3.2.

COROLLARY 3.3. *Let* $\mathbf{A} = (A_1,\dots,A_n)$ *and* $\mathbf{B} = (B_1,\dots,B_n)$ *be two* $n$*-tuples of self-adjoint operators in* $\mathscr{B}_h(H)$ *with spectra in* $J_A$ *and* $J_B$*, respectively, and* $\mathbf{\Phi} = (\Phi_1,\dots,\Phi_n)$ *and* $\mathbf{\Psi} = (\Psi_1,\dots,\Psi_n)$ *be two totally unital* $n-$*tuples of positive linear mappings* $\Phi_j, \Psi_j$ *on* $\mathscr{B}(\mathscr{H})$. *If* $\sum_{j=1}^n \Phi_j$ *preserves the operator* $A_0 = \sum_{k=1}^n \Phi_k(A_k)$, $\sum_{j=1}^n \Psi_j$ *preserves the operator* $B_0 = \sum_{k=1}^n \Psi_k(B_k)$ *and* $A_0 \geqslant B_0$*, then*

$$\begin{aligned}\mathfrak{M}_\varphi(\mathbf{A};\mathbf{\Phi}) &\geqslant \sum_{j=1}^n \Phi_j\big(\mathfrak{M}_\varphi(\mathbf{A};t\mathbf{e_j}+(1-t)\mathbf{\Phi})\big) \geqslant \mathfrak{M}_1(\mathbf{A};\mathbf{\Phi})\\ &\geqslant \mathfrak{M}_1(\mathbf{B};\mathbf{\Psi}) \geqslant \sum_{j=1}^n \Psi_j\big(\mathfrak{M}_\psi(\mathbf{B};s\mathbf{e_j}+(1-s)\mathbf{\Psi})\big) \geqslant \mathfrak{M}_\Psi(\mathbf{B};\mathbf{\Psi})\end{aligned}$$

*for all* $t,s \in [0,1]$ *and every strictly monotone functions* $\varphi : J_A \to \mathbb{R}$, $\psi : J_B \to \mathbb{R}$ *such that* $\varphi^{-1}$ *is operator concave and* $\psi^{-1}$ *is operator convex.*

By virtue of Corollary 2.3, similar to Corollary 3.3 we have interpolations of the quasi-arithmetic mean (3.4) and the weighted arithmetic mean $\mathfrak{M}_1(\omega;\mathbf{A};\Phi)$ (see [8, Theorem 3.4]).

COROLLARY 3.4. *Let* $\mathbf{A} = (A_1,\dots,A_n)$ *and* $\mathbf{B} = (B_1,\dots,B_n)$ *be two* $n$*-tuples of self-adjoint operators in* $\mathscr{B}_h(\mathscr{H})$ *with spectra in* $[m_A, M_A]$ *and* $[m_B, M_B]$*, respectively, such that* $m_A \geqslant M_B$. *Let* $\Phi = (\Phi_1,\dots,\Phi_n)$ *be an* $n$*-tuple of unital positive linear mappings* $\Phi_j : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$. *If* $\omega = (\omega_1,\dots,\omega_n)$ *and* $\nu = (\nu_1,\dots,\nu_n)$ *are weight vectors, then*

$$\begin{aligned}\mathfrak{M}_\varphi(\omega;\mathbf{A};\Phi) &\geqslant \sum_{j=1}^n \omega_j \mathfrak{M}_\varphi\big(te_j+(1-t)\omega;\mathbf{A};\Phi\big) \geqslant \mathfrak{M}_1(\omega;\mathbf{A};\Phi)\\ &\geqslant \mathfrak{M}_1(\nu;\mathbf{B};\Phi) \geqslant \sum_{j=1}^n \nu_j \mathfrak{M}_\psi\big(se_j+(1-s)\nu;\mathbf{B};\Phi\big) \geqslant \mathfrak{M}_\psi(\omega;\mathbf{B};\Phi)\end{aligned}$$

*holds for all* $t,s \in [0,1]$ *and every strictly monotone functions* $\varphi : [m_A, M_A] \to \mathbb{R}$, $\psi : [m_B, M_B] \to \mathbb{R}$ *such that* $\varphi^{-1}$ *is operator concave and* $\psi^{-1}$ *is operator convex.*

Applying the above results we can obtain an interpolation of the power-arithmetic mean inequalities. For example, we give the following corollary (see also [8, Remark 1]).

COROLLARY 3.5. *Let* $\mathbf{A} = (A_1,\dots,A_n)$ *be an* $n$*-tuple of positive invertible operators in* $\mathscr{B}_h^+(H)$, $\Phi = (\Phi_1,\dots,\Phi_n)$ *an* $n$*-tuple of unital positive linear mappings*

*$\Phi_j : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$, and $\omega = (\omega_1, \ldots, \omega_n)$ a weight vector. Then*

$$\begin{aligned}
\left(\sum_{j=1}^{n} \omega_j \Phi_j\left(A_j^r\right)\right)^{\frac{1}{r}} &\geqslant \sum_{j=1}^{n} \omega_j \left(t_1 \Phi_j(A_j^r) + (1-t_1)\sum_{k=1}^{n} \omega_k \Phi_k(A_k^r)\right)^{\frac{1}{r}} \\
&\geqslant \sum_{j=1}^{n} \omega_j \Phi_j\left(A_j\right) \\
&\geqslant \sum_{j=1}^{n} \omega_j \left(t_2 \Phi_j(A_j^s) + (1-t_2)\sum_{k=1}^{n} \omega_k \Phi_k(A_k^s)\right)^{\frac{1}{s}} \geqslant \left(\sum_{j=1}^{n} \omega_j \Phi_j\left(A_j^s\right)\right)^{\frac{1}{s}}
\end{aligned}$$

*for all $t_1, t_2 \in [0,1]$, $r \in [1,\infty)$ and $s \in (-\infty,-1] \cup [1/2,1]$. Moreover, if $r \in [1,\infty)$ then*

$$M(t) := \sum_{j=1}^{n} \omega_j \left(t \Phi_j(A_j^r) + (1-t)\sum_{k=1}^{n} \omega_k \Phi_k(A_k^r)\right)^{\frac{1}{r}}$$

*is monotonically nonincreasing and concave on $[0,1]$, but if $r \in (-\infty,-1] \cup [1/2,1]$ then $M(t)$ is monotonically nondecreasing and convex on $[0,1]$*

## 4. Application of Operator power means

Let $\mathbf{A} = (A_1, \ldots, A_n)$ be an $n$-tuple of positive invertible operators on a Hilbert space and $\omega = (\omega_1, \ldots, \omega_n)$ a weight vector. Putting $\Phi_j(X) = \omega_j X$ in (3.3), we have the ordinary power means

$$\mathfrak{M}_r(\omega; \mathbf{A}) = \left(\sum_{j=1}^{n} \omega_j A_j^r\right)^{1/r} \quad \text{for } r \neq 0. \tag{4.1}$$

It is known that the power mean $\mathfrak{M}_r(\omega; \mathbf{A})$ is not operator mean except $r = 1$. In this case, Corollary 3.5 says that

$$\left(\sum_{j=1}^{n} \omega_j A_j^r\right)^{1/r} \leqslant \sum_{j=1}^{n} \omega_j \left(t A_j^r + (1-t)\sum_{k=1}^{n} \omega_k A_k^r\right)^{1/r} \leqslant \sum_{j=1}^{n} \omega_j A_j$$

for $1/2 \leqslant r \leqslant 1$ and $t \in [0,1]$. If the $A_j$'s commute, then we have

$$\left(\sum_{j=1}^{n} \omega_j A_j^r\right)^{1/r} \leqslant \left[\sum_{j=1}^{n} \omega_j \left(t A_j + (1-t)\sum_{k=1}^{n} \omega_k A_k\right)^{r}\right]^{1/r} \leqslant \sum_{j=1}^{n} \omega_j A_j \tag{4.2}$$

for $0 < r \leqslant 1$ and $t \in [0,1]$.

We try to study a noncommutative operator version of (4.2). For this, we recall the operator power means for positive invertible operators. In 2014, Lawson and Lim [5] established a new definition of operator power means for positive invertible operators, which is an extension of $\mathfrak{M}_r(\omega;\mathbf{A})$ for $0 < r \leqslant 1$ and the commuting $\mathbf{A}$, that is, the $A_j$'s commute each other. They showed that there exists the unique positive invertible solition of the power mean equation

$$X = \sum_{j=1}^{n} \omega_j X \sharp_r A_j \tag{4.3}$$

for $0 < r \leqslant 1$, where the opertor geometric mean $\sharp_r$ is defined by

$$X \sharp_r Y = X^{1/2}(X^{-1/2}YX^{-1/2})^r X^{1/2}.$$

For each $0 < r \leqslant 1$, we say that the solution $X$ of (4.3) is the operator power mean for $\mathbf{A} = (A_1,\ldots,A_n)$ and denote it by $P_r(\omega;\mathbf{A}) = P_r(\omega;A_1,\ldots,A_n)$. In the case of $n = 2$, the operator power mean $P_r((1-u,u);A,B)$ coincides with

$$A \sharp_{r,u} B = A^{1/2}\left((1-u)1_{\mathscr{H}} + u(A^{-1/2}BA^{-1/2})^r\right)^{1/r} A^{1/2}$$

for $0 < r \leqslant 1$ and $u \in [0,1]$. We list some properties of the operator power means which we need later: For each $0 < r \leqslant 1$

(P1) Consistency with scalars: $P_r(\omega;\mathbf{A}) = \left(\sum_{j=1}^n \omega_j A_j^r\right)^{1/r}$ if the $A_j$'s commute;

(P2) Joint concavity:

$$(1-u)P_r(\omega;\mathbf{A}) + uP_r(\omega;\mathbf{B}) \leqslant P_r(\omega;(1-u)\mathbf{A}+u\mathbf{B})$$

for any $u \in [0,1]$;

(P3) Arithmetic-Geometric mean inequality: $P_r(\omega;\mathbf{A}) \leqslant \sum_{j=1}^n \omega_j A_j$.

Now, we show an interpolation of (P3), which is a noncommutative operator version of (4.2):

THEOREM 4.1. *Let $A_1,\ldots,A_n$ be positive invertible operators and $\omega = (\omega_1,\ldots,\omega_n)$ a weight vector. Then for each $0 < r \leqslant 1$*

$$\begin{aligned} &P_r(\omega;A_1,\ldots,A_n) \\ &\leqslant P_r(\omega;tA_1 + (1-t)\sum_{k=1}^n \omega_k A_k,\ldots,tA_n + (1-t)\sum_{k=1}^n \omega_k A_k) \\ &\leqslant \sum_{j=1}^n \omega_j A_j \end{aligned}$$

*for all $t \in [0,1]$. Moreover, for each $0 < r \leqslant 1$*

$$M_r(t) = P_r(\omega;tA_1 + (1-t)\sum_{k=1}^n \omega_k A_k,\ldots,tA_n + (1-t)\sum_{k=1}^n \omega_k A_k)$$

*is monotonically nonincreasing and concave on $[0,1]$.*

*Proof.* Put $A_0=\sum_{k=1}^n \omega_k A_k$. By the joint concavity (P2) of the operator power means, we have

$$
\begin{aligned}
&P_r(\omega;tA_1+(1-t)A_0,\dots,tA_n+(1-t)A_0)\\
&\geqslant tP_r(\omega;A_1,\dots,A_n)+(1-t)P_r(\omega;A_0,\dots,A_0)\\
&=tP_r(\omega;A_1,\dots,A_n)+(1-t)A_0 \quad \text{by (P1)}\\
&\geqslant tP_r(\omega;A_1,\dots,A_n)+(1-t)P_r(\omega;A_1,\dots,A_n) \quad \text{by (P3)}\\
&=P_r(\omega;A_1,\dots,A_n).
\end{aligned}
$$

The second inequality follows from (P3):

$$
\begin{aligned}
&P_r(\omega;tA_1+(1-t)A_0,\dots,tA_n+(1-t)A_0)\\
&\leqslant \sum_{j=1}^n \omega_j(tA_j+(1-t)A_0)=\sum_{j=1}^n \omega_j A_j.
\end{aligned}
$$

For the concavity of $M_r$, we have for $t_1,t_2\in[0,1]$ and $0<t<1$

$$
\begin{aligned}
&tM_r(t_1)+(1-t)M_r(t_2)\\
&\leqslant P_r(\omega;t[t_1A_1+(1-t_1)A_0]+(1-t)[t_2A_1+(1-t_2)A_0],\dots,\\
&\qquad t[t_1A_n+(1-t_1)A_0]+(1-t)[t_2A_n+(1-t_2)A_0])\\
&=M_r(tt_1+(1-t)t_2)
\end{aligned}
$$

and so $M_r$ is concave on $[0,1]$.

For monotonically nonincreasing of $M_r$, if $0<s<t<1$, then $s=\frac{t-s}{t}\cdot 0+\frac{s}{t}\cdot t$ and the concavity of $M_r$ implies

$$
M_r(s)\geqslant \frac{t-s}{t}M_r(0)+\frac{s}{t}M_r(t)\geqslant M_r(t)
$$

because $M_r(0)=P_r(\omega;A_0)=A_0\geqslant M_r(t)$, and the proof is complete. □

REMARK 4.2. Lawson and Lim [5] showed that the Karcher mean is the strong operator limit as $r\to 0$ of the operator power mean $P_r$. Therefore, as $r\to 0$ in Theorem 4.1, we obtain [8, Theorem 3.2].

## 5. Results related to converses of Jensen's inequality

In this section we observe converses of the inequalities obtained in Section 2. For given $f:[m,M]\to\mathbb{R}$, $m<M$, let $l_f(x)$ denote a linear function through $(m,f(m))$ and $(M,f(M))$, i.e.

$$
l_f(x)=a_f x+b_f=\frac{f(M)-f(m)}{M-m}x+\frac{Mf(m)-mf(M)}{M-m}.
$$

To obtain our results, we will need a discrete version of [7, Lemma 4].

LEMMA 5.1. *For $j=1,\ldots,n$, let $A_j \in \mathscr{B}_h(\mathscr{H})$ be self-adjoint operators such that $m \leqslant A_j \leqslant M$ for some scalars $m<M$ and $\Phi_j : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ positive linear mappings such that $\sum_{j=1}^n \Phi_j(1_{\mathscr{H}}) = 1_{\mathscr{K}}$. Then*

$$\sum_{j=1}^n \Phi_j\big(f(A_j)\big) \leqslant l_f\Big(\sum_{k=1}^n \Phi_k(A_k)\Big) - \delta\widetilde{A} \leqslant l_f\Big(\sum_{k=1}^n \Phi_k(A_k)\Big) \tag{5.1}$$

*for every continuous convex function $f:[m,M]\to\mathbb{R}$, where*

$$\delta = f(m)+f(M)-2f\left(\frac{m+M}{2}\right) \geqslant 0,$$

$$\widetilde{A} = \frac{1}{2}1_{\mathscr{K}} - \frac{1}{M-m}\sum_{j=1}^n \Phi_j\Big(\Big|A_j - \frac{m+M}{2}1_{\mathscr{H}}\Big|\Big) \geqslant 0.$$

*If $f$ is concave, then the reverse inequality is valid in* (5.1).

Now, we give results related to converses of Jensen's inequality given in Theorem 2.1. We start with the difference case of a converse of (2.1):

THEOREM 5.2. *For $j=1,\ldots,n$, let $A_j \in \mathscr{B}_h(\mathscr{H})$ be self-adjoint operators such that $m \leqslant A_j \leqslant M$ for some scalars $m<M$ and $\Phi=(\Phi_1,\ldots,\Phi_n)$ a totally unital $n$-tuple of positive linear mappings on $\mathscr{B}(\mathscr{H})$. If $f(x)$ is convex on $[m,M]$ and $\sum_{j=1}^n \Phi_j$ preserves the operator $A_0=\sum_{k=1}^n \Phi_k(A_k)$, then*

$$\begin{aligned}\sum_{j=1}^n \Phi_j\big(f(A_j)\big) &\leqslant \sum_{j=1}^n \Phi_j\Big(f(tA_j+(1-t)\sum_{k=1}^n \Phi_k(A_k)\Big) + \beta 1_{\mathscr{H}} - \delta\widetilde{A} \\ &\leqslant f\Big(\sum_{j=1}^n \Phi_j(A_j)\Big) + 2\beta 1_{\mathscr{H}} - \delta(\widetilde{A}+\widetilde{A}_t)\end{aligned} \tag{5.2}$$

*for all $t\in[0,1]$, where*

$$\beta = \max_{m\leqslant x\leqslant M}\{l_f(x)-f(x)\},\ \delta = f(m)+f(M)-2f\left(\frac{m+M}{2}\right),$$

$$\widetilde{A}_t = \frac{1}{2}1_{\mathscr{H}} - \frac{1}{M-m}\sum_{j=1}^n \Phi_j\left(\Big|tA_j+(1-t)A_0-\frac{m+M}{2}1_{\mathscr{H}}\Big|\right)$$

*and* $\widetilde{A} = \frac{1}{2}1_{\mathscr{H}} - \frac{1}{M-m}\sum_{j=1}^n \Phi_j\Big(\Big|A_j-\frac{m+M}{2}1_{\mathscr{H}}\Big|\Big).$

*If $f(x)$ is concave then the reverse inequality is valid in* (5.2).

*Proof.* By using Lemma 5.1 and since $A_0=\sum_{j=1}^n \Phi_j(A_0)$ we have:

$$\sum_{j=1}^n \Phi_j\big(f(A_j)\big) \leqslant l_f\Big(\sum_{j=1}^n \Phi_j(A_j)\Big) - \delta\widetilde{A} = l_f\Big(\sum_{j=1}^n \Phi_j\big(tA_j+(1-t)A_0\big)\Big) - \delta\widetilde{A}. \tag{5.3}$$

By using that $m\leqslant tA_j+(1-t)A_0 \leqslant M$ for $j=1,\ldots,n$, we have

$$l_f\big(tA_j+(1-t)A_0\big) - f\big(tA_j+(1-t)A_0\big) \leqslant \max_{m\leqslant x\leqslant M}\{l_f(x)-f(x)\}1_{\mathscr{H}} = \beta 1_{\mathscr{H}}$$

and so

$$
\begin{aligned}
l_f\Big(\sum_{j=1}^{n}\Phi_j\big(tA_j+(1-t)A_0\big)\Big) &= \sum_{j=1}^{n}\Phi_j\Big(l_f\big(tA_j+(1-t)A_0\big)\Big)\\
&\leqslant \sum_{j=1}^{n}\Phi_j\Big(f(tA_j+(1-t)A_0\Big)+\beta 1_{\mathscr{H}}.
\end{aligned}
\tag{5.4}
$$

Combining (5.3) and (5.4) we obtain

$$
\sum_{j=1}^{n}\Phi_j\big(f(A_j)\big) \leqslant \sum_{j=1}^{n}\Phi_j\Big(f\big(tA_j+(1-t)A_0\big)\Big)+\beta 1_{\mathscr{H}}-\delta\widetilde{A}. \tag{5.5}
$$

Next, replacing $A_j$ by $tA_j+(1-t)A_0$ in (5.1) we obtain

$$
\begin{aligned}
&\sum_{j=1}^{n}\Phi_j\Big(f\big(tA_j+(1-t)A_0\big)\Big)-f\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big)\\
&\leqslant l_f\Big(\sum_{j=1}^{n}\Phi_j\big(tA_j+(1-t)A_0\big)\Big)-\delta\widetilde{A}_t-f\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big)\\
&= l_f\Big(\sum_{j=1}^{n}\Phi_j\big(tA_j\big)\Big)-f\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big)-\delta\widetilde{A}_t\\
&\leqslant \max_{m\leqslant x\leqslant M}\big\{l_f(x)-f(x)\big\}1_{\mathscr{H}}-\delta\widetilde{A}_t=\beta 1_{\mathscr{H}}-\delta\widetilde{A}_t.
\end{aligned}
\tag{5.6}
$$

Now, combining (5.5) and (5.6) we obtain (5.2). □

Next, we give the ratio case of a converse of (2.1).

THEOREM 5.3. *Let the assumptions of Theorem 5.2 hold. If $f(x)$ is strictly positive convex on $[m,M]$, and $\sum_{j=1}^{n}\Phi_j$ preserves the operator $A_0$, then*

$$
\sum_{j=1}^{n}\Phi_j\big(f(A_j)\big)\leqslant \gamma_1\sum_{j=1}^{n}\Phi_j\Big(f(tA_j+(1-t)\sum_{k=1}^{n}\Phi_k(A_k)\Big)\leqslant \gamma_1\gamma_2\, f\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big) \tag{5.7}
$$

*for all $t\in[0,1]$, where*
$\gamma_1=\max_{m\leqslant x\leqslant M}\left\{\frac{l_f(x)-\delta m_{\widetilde{A}}}{f(x)}\right\}$, $\gamma_2=\max_{m_A\leqslant x\leqslant M_A}\left\{\frac{l_f(x)-\delta m_{\widetilde{A}_t}}{f(x)}\right\}$, *$m_{\widetilde{A}}$ and $m_{\widetilde{A}_t}$ are the lower bound of the operator $\widetilde{A}$ and $\widetilde{A}_t$, respectively, and $\widetilde{A}$ and $\widetilde{A}_t$ are as in Theorem 5.2.*

*If $f(x)$ is strictly positive concave then the reverse inequality is valid in* (5.7).

*Proof.* We use the same technique as in the proof of Theorem 5.2. Ineq. (5.3) give

$$
\begin{aligned}
\Sigma_{j=1}^{n}\Phi_j\big(f(A_j)\big) &\leqslant \sum_{j=1}^{n}\Phi_j\Big(l_f\big(tA_j+(1-t)A_0\big)\Big)-\delta m_{\widetilde{A}}\\
&= \sum_{j=1}^{n}\Phi_j\Big(l_f\big(tA_j+(1-t)A_0\big)-\delta m_{\widetilde{A}}\Big)
\end{aligned}
\tag{5.8}
$$

Since

$$l_f\big(tA_j+(1-t)A_0\big)-\delta m_{\widetilde{A}} \leqslant \max_{m\leqslant x\leqslant M}\left\{\frac{l_f(x)-\delta m_{\widetilde{A}}}{f(x)}\right\} f\big(tA_j+(1-t)A_0\big) \\ =\gamma_1 f\big(tA_j+(1-t)A_0\big),$$

we obtain

$$\sum_{j=1}^{n}\Phi_j\Big(l_f\big(tA_j+(1-t)A_0\big)-\delta m_{\widetilde{A}}\Big)\leqslant \gamma_1\sum_{j=1}^{n}\Phi_j\Big(f\big(tA_j+(1-t)A_0\big)\Big). \tag{5.9}$$

Combining (5.8) and (5.9) we obtain

$$\sum_{j=1}^{n}\Phi_j\big(f(A_j)\big)\leqslant \gamma_1\sum_{j=1}^{n}\Phi_j\Big(f\big(tA_j+(1-t)A_0\big)\Big). \tag{5.10}$$

Next, replacing $A_j$ by $tA_j+(1-t)A_0$ in (5.1) we obtain

$$\sum_{j=1}^{n}\Phi_j\Big(f\big(tA_j+(1-t)A_0\big)\Big)\leqslant l_f\Big(\sum_{j=1}^{n}\Phi_j\big(tA_j+(1-t)A_0\big)\Big)-\delta m_{\widetilde{A_t}} \\ = l_f\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big)-\delta m_{\widetilde{A_t}}$$

and so

$$\sum_{j=1}^{n}\Phi_j\Big(f\big(tA_j+(1-t)A_0\big)\Big) \\ \leqslant \max_{m_A\leqslant x\leqslant M_A}\left\{\frac{l_f(x)-\delta m_{\widetilde{A_t}}}{f(x)}\right\} f\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big)=\gamma_2 f\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big). \tag{5.11}$$

Now, combining (5.10) and (5.11) we obtain (5.7). $\square$

REMARK 5.4. Let the assumptions of Theorem 5.3 hold. Similar to (5.7) we can obtain the following inequalities:

$$\sum_{j=1}^{n}\Phi_j\Big(f\big(tA_j+(1-t)A_0\big)\Big)\leqslant \gamma\sum_{j=1}^{n}\Phi_j\Big(f\big(tA_j+(1-t)A_0\big)-\delta\widetilde{A} \\ \leqslant \gamma\gamma_0\, f\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big)-\delta\left(\widetilde{A}+\gamma\widetilde{A_t}\right)$$

and

$$\sum_{j=1}^{n}\Phi_j\Big(f\big(tA_j+(1-t)A_0\big)\Big)\leqslant \gamma\sum_{j=1}^{n}\Phi_j\Big(f\big(tA_j+(1-t)A_0\big)-\delta\widetilde{A} \\ \leqslant \gamma\gamma_2\, f\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big)-\delta\widetilde{A}$$

for all $t \in [0,1]$, where $\gamma = \max_{m \leqslant x \leqslant M} \{ l_f(x)/f(x) \}$, $\gamma_0 = \max_{m_A \leqslant x \leqslant M_A} \{ l_f(x)/f(x) \}$ and $\gamma_2 = \max_{m_A \leqslant x \leqslant M_A} \left\{ \frac{l_f(x) - \delta m_{\widetilde{A}_t}}{f(x)} \right\}$.

Now we give converses of (2.2).

COROLLARY 5.5. *For $j = 1, \dots, n$, let $A_j$ be self-adjoint operators with $m \leqslant A_j \leqslant M$ for some scalars $m < M$, $\Phi_j$ be unital positive linear mappings, and $\omega = (\omega_1, \dots, \omega_n)$ be a weight vector. If $f(x)$ is convex on $[m,M]$, then*

$$\begin{aligned} \sum_{j=1}^n \omega_j \Phi_j\big(f(A_j)\big) &\leqslant \sum_{j=1}^n \omega_j \Big( f(t\Phi_j(A_j) + (1-t) \sum_{k=1}^n \Phi_k(A_k) \Big) + \beta 1_{\mathscr{H}} - \delta \widetilde{A} \\ &\leqslant f\Big( \sum_{j=1}^n \omega_j \Phi_j(A_j) \Big) + 2\beta 1_{\mathscr{H}} - \delta(\widetilde{A} + \widetilde{A}_t) \end{aligned} \tag{5.12}$$

*for all $t \in [0,1]$, where $\beta$, $\widetilde{A}_t$ and $\widetilde{A}$ are as in Theorem 5.2.*

*Additionally, if $f(x)$ is strictly positive convex on $[m,M]$, then*

$$\sum_{j=1}^n \omega_j \Phi_j\big(f(A_j)\big) \leqslant \gamma_1 \sum_{j=1}^n \Big( f(t\Phi_j(A_j) + (1-t) \sum_{k=1}^n \omega_j \Phi_k(A_k) \Big) \leqslant \gamma_1 \gamma_2 f\Big( \sum_{j=1}^n \omega_j \Phi_j(A_j) \Big) \tag{5.13}$$

*for all $t \in [0,1]$, where $\gamma_1$, $\gamma_2$ are as in Theorem 5.3.*

## 6. Application of converses of Jensen's inequality

As an application, we give converses of (3.5).

THEOREM 6.1. *(Difference case) Let $\mathbf{A} = (A_1, \dots, A_k)$ be an $n$-tuple of self-adjoint operators in $\mathscr{B}_h(\mathscr{H})$ such that $m \leqslant A_j \leqslant M$ for some scalars $m < M$, $\mathbf{\Phi} = (\Phi_1, \dots, \Phi_k)$ be a totally unital $n$-tuple of positive linear mappings $\Phi_j$ on $\mathscr{B}(\mathscr{H})$ and $\mathbf{e_j}$, $j = 1, \dots, n$, be the standard basis vector in $\mathbb{R}^n$.*
*If $\varphi : [m,M] \to \mathbb{R}$ is a strictly monotone function such that $\varphi^{-1}$ is concave on $\varphi([m,M])$ and $\sum_{j=1}^n \Phi_j$ preserves the operator $A_0 = \sum_{k=1}^n \Phi_k(A_k)$, then*

$$\begin{aligned} \mathfrak{M}_\varphi(\mathbf{A};\mathbf{\Phi}) + (\beta_0 + \beta) 1_{\mathscr{H}} - \delta(\widetilde{A} + \widetilde{A}_t) &\leqslant \sum_{j=1}^n \Phi_j\big( \mathfrak{M}_\varphi(\mathbf{A}; t\mathbf{e_j} + (1-t)\mathbf{\Phi}) \big) + \beta 1_{\mathscr{H}} - \delta \widetilde{A} \\ &\leqslant \mathfrak{M}_1(\mathbf{A};\mathbf{\Phi}) \end{aligned} \tag{6.1}$$

*for all $t \in [0,1]$, where $\beta$, $\beta_0$, $\widetilde{A}_t$ and $\widetilde{A}$ are as in Theorem 5.2.*

*But, if $\varphi^{-1}$ is convex on $\varphi([m,M])$, then the reverse inequalities are valid in (6.1).*

THEOREM 6.2. *(Ratio case) Let the assumptions of Theorem 5.2 hold. If $\varphi : I \to \mathbb{R}$ is a strictly monotone function such that $\varphi^{-1}$ is strictly positive concave on $\varphi([m,M])$ and $\sum_{j=1}^{n}\Phi_j$ preserves the operator $A_0$, then*

$$\mathfrak{M}_\varphi(\mathbf{A};\mathbf{\Phi}) \leqslant \gamma_1 \sum_{j=1}^{n} \Phi_j\big(\mathfrak{M}_\varphi(\mathbf{A};t\mathbf{e_j}+(1-t)\mathbf{\Phi})\big) \leqslant \gamma_1\gamma_2\,\mathfrak{M}_1(\mathbf{A};\mathbf{\Phi}) \tag{6.2}$$

*for all $t \in [0,1]$, , where $\gamma_1$, $\gamma_2$ are as in Theorem 5.3.*

*But, if $\varphi^{-1}$ is convex on $\varphi([m,M])$, then the reverse inequalities are valid in (6.2).*

*Proof.* We omit the proofs of the above two theorems because it is proved in a similar method as Theorem 3.1. □

REMARK 6.3. The interested reader can obtain other results. E.g.

1) Combining inequalities in Theorem 6.1 can be obtained a converse of inequalities in Corollary 3.2. Combining inequalities in Theorem 6.2 can be obtained another converse of these inequalities.

2) Similar, can be obtained converses of inequalities in Corollary 3.3 and 3.4.

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