

## MARTINGALE INEQUALITIES ON HARDY-LORENTZ-KARAMATA SPACES

QINGFANG WU, DEJIAN ZHOU AND LIHUA PENG

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*Abstract.* We establish several martingale inequalities on Hardy-Lorentz-Karamata spaces via atomic decompositions. We also prove that all martingale Hardy-Lorentz-Karamata spaces are equivalent if the stochastic basis is regular.

### 1. Introduction

The study of martingale inequalities has a long history. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space with non-decreasing sub-algebras  $(\mathcal{F}_n)_{n \geq 0}$  satisfying  $\mathcal{F} = \sigma(\cup_n \mathcal{F}_n)$ . For a martingale  $f = (f_n)_{n \geq 0}$ , we define  $M(f) = \sup_{n \geq 0} |f_n|$  and  $S(f) = (\sum_{n=0}^{\infty} |f_{n+1} - f_n|^2)^{1/2}$ . In [2], Burkholder and Gundy proved the well-known inequality

$$\|M(f)\|_p \approx \|S(f)\|_p, \quad 1 < p < \infty.$$

The case  $p = 1$  for this inequality was due to Davis [4]. The martingale version of the John-Nirenberg inequality was investigated by Garsia [6]. For more martingale inequalities, we refer the reader to [3] and [13].

The atomic decomposition, which is a useful tool to prove martingale inequalities and dualities, of martingale Hardy spaces were studied by many authors; see for example [7], [17] and [16]. In particular, Jiao et al. [10] obtained the following result (see next section for any unexplained notations).

**THEOREM 1.1.** ([10, Theorem 3.4]) *Suppose that  $0 < p < \infty, 0 < q \leq \infty$ . Then the following inequalities hold:*

$$\begin{aligned} \|f\|_{H_{p,q}} &\leq C \|f\|_{H_{p,q}^s}, & \|f\|_{H_{p,q}^s} &\leq C \|f\|_{H_{p,q}^s}, & 0 < p < 2; \\ \|f\|_{H_{p,q}} &\leq \|f\|_{\mathcal{D}_{p,q}}, & \|f\|_{H_{p,q}^s} &\leq \|f\|_{\mathcal{D}_{p,q}}; \\ \|f\|_{H_{p,q}^s} &\leq C \|f\|_{\mathcal{D}_{p,q}}, & \|f\|_{H_{p,q}} &\leq C \|f\|_{\mathcal{D}_{p,q}}; \\ \|f\|_{H_{p,q}^s} &\leq C \|f\|_{\mathcal{D}_{p,q}}, & \|f\|_{H_{p,q}^s} &\leq C \|f\|_{\mathcal{D}_{p,q}}. \end{aligned}$$

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It is well known that the family of Lorentz-Karamata spaces, defined via the slowly varying functions, is a generalization of the Lorentz spaces and the Lorentz-Zygmund spaces; see [5]. Recently, Ho [8] introduced the martingale Hardy-Lorentz-Karamata spaces, established duality and interpolation results by atomic decompositions. Jiao et al. [12], via atomic decomposition, proved a John-Nirenberg inequality in the frame of Lorentz-Karamata spaces. Liu and Zhou [14] showed the dual space of weak martingale Hardy-Karamata space. In this paper, we continue this line of research. More precisely, we devote to extending Theorem 1.1 to the frame of Lorentz-Karamata spaces. Our main result reads as follows.

**THEOREM 1.2.** *Let  $b$  be a non-decreasing slowly varying function. Suppose that  $0 < p < \infty, 0 < q \leq \infty$ . Then the following inequalities hold:*

$$\|f\|_{H_{p,q,b}} \leq C\|f\|_{H_{p,q,b}^s}, \quad \|f\|_{H_{p,q,b}^s} \leq C\|f\|_{H_{p,q,b}^s} \quad 0 < p < 2; \tag{1.1}$$

$$\|f\|_{H_{p,q,b}} \leq \|f\|_{\mathcal{D}_{p,q,b}}, \quad \|f\|_{H_{p,q,b}^s} \leq \|f\|_{\mathcal{D}_{p,q,b}}; \tag{1.2}$$

$$\|f\|_{H_{p,q,b}^s} \leq C\|f\|_{\mathcal{D}_{p,q,b}}, \quad \|f\|_{H_{p,q,b}} \leq C\|f\|_{\mathcal{D}_{p,q,b}}; \tag{1.3}$$

$$\|f\|_{H_{p,q,b}^s} \leq C\|f\|_{\mathcal{D}_{p,q,b}}, \quad \|f\|_{H_{p,q,b}^s} \leq C\|f\|_{\mathcal{D}_{p,q,b}}; \tag{1.4}$$

$$\|f\|_{\mathcal{D}_{p,q,b}} \leq C\|f\|_{\mathcal{D}_{p,q,b}} \leq C\|f\|_{\mathcal{D}_{p,q,b}}. \tag{1.5}$$

If  $b \equiv 1$ , then the above theorem reduces to Theorem 1.1.

Recall that the stochastic basis  $\{\mathcal{F}_n\}_{n \geq 0}$  is said to be regular, if there exists an absolute constant  $R > 0$  such that

$$f_n \leq Rf_{n-1}, \quad \forall n \in \mathbb{N},$$

holds for all non-negative adapted martingales  $f = (f_n)_{n \geq 0}$ . We refer the reader to [15] for more information.

With the help of Theorem 1.2, we find that all five martingale Hardy-Lorentz-Karamata spaces are equivalent if  $\{\mathcal{F}_n\}_{n \geq 0}$  is regular.

**THEOREM 1.3.** *Let  $b$  be a non-decreasing slowly varying function. Suppose that  $0 < p < \infty, 0 < q \leq \infty$ . If  $\{\mathcal{F}_n\}_{n \geq 0}$  is regular, then*

$$H_{p,q,b}^s = \mathcal{D}_{p,q,b} = \mathcal{D}_{p,q,b} = H_{p,q,b} = H_{p,q,b}^s$$

with equivalent quasi-norms.

This article is organized as follows. In next section, we present preliminaries, definitions and lemmas used throughout the paper. The proofs of Theorem 1.2 and Corollary 1.3 are given in Section 3.

In the paper, the set of integers and the set of non-negative integers are always denoted by  $\mathbb{Z}$  and  $\mathbb{N}$ , respectively. We use  $C$  to denote the absolute constant which may vary from line to line. If we write  $f \approx g$ , then it stands for  $C_1f \leq g \leq C_2f$ . We call that  $f$  is equivalent to  $g$  if  $f \approx g$ .

### 2. Preliminaries

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space. We denote by  $L_0(\Omega, \mathcal{F}, \mathbb{P})$ , or simply  $L_0(\Omega)$ , the space of all measurable functions on  $(\Omega, \mathcal{F}, \mathbb{P})$ . We firstly recall some notations and properties of Lorentz-Karamata function spaces.

DEFINITION 2.1. ([5]) A Lebesgue measurable function  $b : [1, \infty) \rightarrow (0, \infty)$  is said to be a slowly varying function, if for any given  $\varepsilon > 0$ , the function  $t^\varepsilon b(t)$  is equivalent to a non-decreasing function and the function  $t^{-\varepsilon} b(t)$  is equivalent to a non-increasing function on  $[1, \infty)$ .

Let  $b$  be a slowly varying function on  $[1, \infty)$ . Define  $\gamma_b$  on  $(0, \infty)$  by

$$\gamma_b(t) = b(\max\{t, 1/t\}), \quad t > 0.$$

REMARK 2.2. ([5]) (1) If  $b$  is a non-decreasing function, by the definition of  $\gamma_b$ , we know that  $\gamma_b$  is non-increasing on  $(0, 1]$ . For any given  $\varepsilon > 0$ , the function  $t^\varepsilon \gamma_b(t)$  is equivalent to a non-decreasing function and the function  $t^{-\varepsilon} \gamma_b(t)$  is equivalent to a non-increasing function on  $(0, \infty)$ .

(2) Let  $r > 0$ . Then  $\gamma_b(rt) \approx \gamma_b(t)$  for all  $t > 0$ .

For any  $f \in L_0(\Omega, \mathcal{F}, \mathbb{P})$ , the distribution function of  $f$  is defined by

$$\lambda_s(f) = \mathbb{P}(\{\omega \in \Omega : |f(\omega)| > s\}), \quad s \geq 0.$$

Denote by  $f^*(t)$  the decreasing rearrangement of  $f$ , defined by

$$f^*(t) = \inf\{s \geq 0 : \lambda_s(f) \leq t\}, \quad t \geq 0,$$

with the convention that  $\inf \emptyset = \infty$ .

DEFINITION 2.3. Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $b$  be a slowly varying function. The Lorentz-Karamata space  $L_{p,q,b}$  consists of those measurable functions on  $(\Omega, \mathcal{F}, \mathbb{P})$  such that  $\|f\|_{p,q,b} < \infty$ , where

$$\|f\|_{p,q,b} = \left[ \int_0^1 \left( t^{\frac{1}{p}} \gamma_b(t) f^*(t) \right)^q \frac{dt}{t} \right]^{1/q}, \quad 0 < q < \infty,$$

and

$$\|f\|_{p,\infty,b} = \sup_{0 \leq t \leq 1} t^{\frac{1}{p}} \gamma_b(t) f^*(t), \quad q = \infty.$$

The Lorentz-Karamata space  $L_{p,q,b}$  is a rearrangement invariant quasi-Banach function space (see [9]), and the quasi-norm  $\|\cdot\|_{p,q,b}$  has the following equivalent characterization [8, Lemma 2.4]:

$$\|f\|_{p,q,b} \approx \left[ \int_0^\infty \left( \lambda_s(f)^{\frac{1}{p}} \gamma_b(\lambda_s(f)) s \right)^q \frac{ds}{s} \right]^{1/q}, \quad 0 < q < \infty,$$

and

$$\|f\|_{p,\infty,b} \approx \sup_{s>0} \lambda_s(f)^{\frac{1}{p}} \gamma_b(\lambda_s(f))s, \quad q = \infty.$$

We now introduce the martingale Hardy-Lorentz-Karamata spaces. In the sequel, let  $\{\mathcal{F}_n\}_{n \geq 0}$  be a non-decreasing sequence of sub- $\sigma$ -algebra of  $\mathcal{F}$  such that  $\mathcal{F} = \sigma(\bigcup_{n=0}^{\infty} \mathcal{F}_n)$ . Denote by  $\mathcal{M}$  the set of all martingales  $f = (f_n)_{n \geq 0}$  relative to  $\{\mathcal{F}_n\}_{n \geq 0}$  such that  $f_0 = 0$ . The expectation operator and the conditional expectation operator relative to  $\mathcal{F}_n$  are denoted by  $\mathbb{E}$  and  $\mathbb{E}_n$ , respectively. For  $f \in \mathcal{M}$ , denote its martingale difference by  $d_n f = f_n - f_{n-1}$  ( $n \geq 0$ , with convention  $f_{-1} = 0$ ). Then the maximal function, the quadratic variation and the conditional quadratic variation of a martingale  $f$  are respectively defined by

$$\begin{aligned} M_n(f) &= \sup_{0 \leq i \leq n} |f_i|, \quad M(f) = \sup_{n \geq 0} |f_n|, \\ S_n(f) &= \left( \sum_{i=1}^n |d_i f|^2 \right)^{\frac{1}{2}}, \quad S(f) = \left( \sum_{i=1}^{\infty} |d_i f|^2 \right)^{\frac{1}{2}}, \\ s_n(f) &= \left( \sum_{i=1}^n \mathbb{E}_{i-1} |d_i f|^2 \right)^{\frac{1}{2}}, \quad s(f) = \left( \sum_{i=1}^{\infty} \mathbb{E}_{i-1} |d_i f|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $b$  be a slowly varying function. Let  $\Lambda_{p,q,b}$  be the collection of all sequences  $\lambda = (\lambda_n)_{n \geq 0}$  of non-decreasing, non-negative and adapted functions with  $\lambda_{\infty} = \lim_{n \rightarrow \infty} \lambda_n \in L_{p,q,b}$ . We define martingale Hardy-Lorentz-Karamata spaces as follows.

DEFINITION 2.4. Let  $0 < p < \infty, 0 < q \leq \infty$  and  $b$  be a slowly varying function. Define

$$\begin{aligned} H_{p,q,b} &= \{f \in \mathcal{M} : \|f\|_{H_{p,q,b}} = \|M(f)\|_{p,q,b} < \infty\}, \\ H_{p,q,b}^S &= \{f \in \mathcal{M} : \|f\|_{H_{p,q,b}^S} = \|S(f)\|_{p,q,b} < \infty\}, \\ H_{p,q,b}^s &= \{f \in \mathcal{M} : \|f\|_{H_{p,q,b}^s} = \|s(f)\|_{p,q,b} < \infty\}, \\ \mathcal{Q}_{p,q,b} &= \{f \in \mathcal{M} : \exists \lambda = (\lambda_n)_{n \geq 0} \in \Lambda_{p,q,b} \text{ s.t. } S_n(f) \leq \lambda_{n-1}\} \\ &\quad \text{with } \|f\|_{\mathcal{Q}_{p,q,b}} = \inf_{\lambda} \|\lambda_{\infty}\|_{p,q,b}, \\ \mathcal{P}_{p,q,b} &= \{f \in \mathcal{M} : \exists \lambda = (\lambda_n)_{n \geq 0} \in \Lambda_{p,q,b} \text{ s.t. } |f_n| \leq \lambda_{n-1}\} \\ &\quad \text{with } \|f\|_{\mathcal{P}_{p,q,b}} = \inf_{\lambda} \|\lambda_{\infty}\|_{p,q,b}. \end{aligned}$$

If  $b \equiv 1$ , then the martingale Hardy-Lorentz-Karamata spaces return to martingale Hardy-Lorentz spaces. If  $p = q$ ,  $b \equiv 1$ , then the martingale Hardy-Lorentz spaces become the classical martingale Hardy spaces; see e.g. [15, 18].

We recall a lemma which is useful to judge whether a function belongs to the Lorentz-Karamata spaces  $L_{p,q,b}$ . By borrowing some idea from [1], it was firstly proved for  $q = \infty$  in [14]. The case  $0 < q < \infty$  can be similarly proved. We omit the details.

LEMMA 2.5. *Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $b$  be a slowly varying function. Assume that the non-negative sequence  $\{2^k \lambda_k\}_{k \in \mathbb{Z}}$  belongs to  $l_q$ . Further suppose that the non-negative function  $\varphi$  verifies the following property: there exists  $0 < \varepsilon < \min(1, q/p)$  such that, given an arbitrary integer  $k_0$ , we have  $\varphi \leq \psi_{k_0} + \eta_{k_0}$ , where  $\psi_{k_0}$  and  $\eta_{k_0}$  satisfy*

$$2^{k_0 p} \mathbb{P}^\varepsilon(\psi_{k_0} > 2^{k_0}) \gamma_b^{\varepsilon p}(\mathbb{P}(\psi_{k_0} > 2^{k_0})) \leq C \sum_{-\infty}^{k_0-1} (2^k \lambda_k^\varepsilon)^p,$$

and

$$2^{k_0 \varepsilon p} \mathbb{P}(\eta_{k_0} > 2^{k_0}) \gamma_b^p(\mathbb{P}(\eta_{k_0} > 2^{k_0})) \leq C \sum_{k=k_0}^{\infty} (2^{k\varepsilon} \lambda_k)^p.$$

Then  $\|\psi_{k_0}\|_{p,q,b} \leq C \|\{2^k \lambda_k\}\|_{l_q}$  and  $\|\eta_{k_0}\|_{p,q,b} \leq C \|\{2^k \lambda_k\}\|_{l_q}$ , and consequently,  $\varphi \in L_{p,q,b}$  and  $\|\varphi\|_{p,q,b} \leq C \|\{2^k \lambda_k\}\|_{l_q}$ .

### 3. Proof of main results

We begin this section with the concept of atoms introduced in [18].

DEFINITION 3.1. Let  $0 < p < \infty$ . A measurable function  $a$  is called a  $(1, p, \infty)$ -atom if there exists a stopping time  $\nu$  such that

- (1)  $a_n = \mathbb{E}_n(a) = 0$  if  $\nu \geq n$ ,
- (2)  $\|s(a)\|_\infty \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}$ .

Replacing (2) by  $\|S(a)\|_\infty \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}$  (or  $\|M(a)\|_\infty \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}$ ), then we get the definition of  $(2, p, \infty)$ -atom (or  $(3, p, \infty)$ -atom).

The following result is from Jiao et al. [12] which extends the atomic decomposition of [10, 11, 18].

THEOREM 3.2. *Let  $b$  be a non-decreasing slowly varying function. If martingale  $f = (f_n)_{n \geq 0} \in H_{p,q,b}^s$  for  $0 < p < \infty, 0 < q \leq \infty$ , then there exist a sequence  $(a^k)_{k \in \mathbb{Z}}$  of  $(1, p, \infty)$ -atoms and a sequence  $(\mu_k)_{k \in \mathbb{Z}}$  of real numbers satisfying  $\mu_k = A \cdot 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}$ , where  $A$  is a positive constant and  $\nu_k$  is the stopping time associated with  $a^k$ , such that*

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^k, \quad a.e., \quad n \in \mathbb{N}, \tag{3.1}$$

and

$$\|\{\gamma_b(\mathbb{P}(\nu_k < \infty))\mu_k\}\|_{l_q} \leq C \|f\|_{H_{p,q,b}^s}.$$

Conversely, if the martingale  $f$  has the above decomposition, then  $f \in H_{p,q,b}^s$  and

$$\|f\|_{H_{p,q,b}^s} \approx \inf \|\{\gamma_b(\mathbb{P}(\nu_k < \infty))\mu_k\}\|_{l_q}.$$

where the infimum is taken over all the above decompositions.

Moreover, if we replace  $H_{p,q,b}^s$ ,  $(1, p, \infty)$ -atoms by  $\mathcal{D}_{p,q,b}$ ,  $(2, p, \infty)$ -atoms (or  $\mathcal{P}_{p,q,b}$ ,  $(3, p, \infty)$ -atoms), then the above conclusions still hold.

Recall that an operator  $T : X \rightarrow Y$  is called  $\sigma$ -sublinear if for any constant  $\alpha$  it satisfies

$$\left| T \left( \sum_{k=1}^{\infty} f_k \right) \right| \leq \sum_{k=1}^{\infty} |T(f_k)| \quad \text{and} \quad |T(\alpha f)| = |\alpha| |T(f)|,$$

where  $X$  is a martingale space and  $Y$  is a measurable function space.

By Theorem 3.2, in next two lemmas, we establish a sufficient condition for a  $\sigma$ -sublinear operator to be bounded from martingale Hardy-Lorentz-Karamata spaces to Lorentz-Karamata spaces. With the help of the following two lemmas, the embeddings between different Hardy-Lorentz-Karamata spaces will be proved.

LEMMA 3.3. *Let  $b$  be a non-decreasing slowly varying function. Suppose that  $0 < p < \infty, 0 < q \leq \infty$  and  $1 < r < \infty$  satisfy  $p < r$ . If  $T : H_r^s \rightarrow L_r$  is a bounded  $\sigma$ -sublinear operator and*

$$\{|T(a)| > 0\} \subset \{v < \infty\} \tag{3.2}$$

for every  $(1, p, \infty)$ -atom  $a$  associated with the stopping time  $v$ , then

$$\|T(f)\|_{p,q,b} \leq C \|f\|_{H_{p,q,b}^s}.$$

*Proof.* Take  $f \in H_{p,q,b}^s$ . According to Theorem 3.2, we find that  $f$  has the decomposition as (3.1) such that for every  $k$ ,  $a^k$  is a  $(1, p, \infty)$ -atom and  $\mu_k = 3 \cdot 2^k \|\chi_{\{v_k < \infty\}}\|_p$ . For an arbitrary integer  $k_0$ , set

$$f = \sum_k \mu_k a^k := F_1 + F_2,$$

where

$$F_1 = \sum_{k=-\infty}^{k_0-1} \mu_k a^k \quad \text{and} \quad F_2 = \sum_{k=k_0}^{\infty} \mu_k a^k.$$

By the  $\sigma$ -sublinearity of the operator  $T$ , we have

$$|T(F_1)| \leq \sum_{k=-\infty}^{k_0-1} \mu_k |T(a^k)|, \quad |T(F_2)| \leq \sum_{k=k_0}^{\infty} \mu_k |T(a^k)|.$$

To complete the lemma, it suffices to show that  $T(F_1) \in L_{p,q,b}$  and  $T(F_2) \in L_{p,q,b}$ , separately.

**Step 1:** In this step, we prove  $T(F_2) \in L_{p,q,b}$ . According to condition (3.2),

$$\{|T(F_2)| > 2^{k_0}\} \subset \{|T(F_2)| > 0\} \subset \bigcup_{k=k_0}^{\infty} \{|T(a^k)| > 0\} \subset \bigcup_{k=k_0}^{\infty} \{v_k < \infty\}.$$

Applying Remark 2.2(1), we have

$$\begin{aligned} & 2^{k_0 \varepsilon p} \mathbb{P}(|T(F_2)| > 2^{k_0}) \gamma_b^p(\mathbb{P}(|T(F_2)| > 2^{k_0})) \\ & \leq 2^{k_0 \varepsilon p} \sum_{k=k_0}^{\infty} \mathbb{P}(v_k < \infty) \gamma_b^p(\mathbb{P}(v_k < \infty)) \\ & \leq \sum_{k=k_0}^{\infty} [2^{k \varepsilon} \mathbb{P}(v_k < \infty)^{\frac{1}{p}} \gamma_b(\mathbb{P}(v_k < \infty))]^p. \end{aligned}$$

By Lemma 2.5 and Theorem 3.2, we obtain

$$\|T(F_2)\|_{p,q,b} \leq C \|\{2^k \mathbb{P}(v_k < \infty)^{\frac{1}{p}} \gamma_b(\mathbb{P}(v_k < \infty))\}\|_{\ell_q} \leq C \|f\|_{H_{p,q,b}^s}.$$

**Step 2:** In this step, we estimate  $\|T(F_1)\|_{p,q,b}$ . We prove this into two cases:  $\frac{q}{p} \in [1, \infty]$  and  $\frac{q}{p} \in (0, 1)$ .

Case 1:  $\frac{q}{p} \in [1, \infty]$ . For  $0 < \ell < 1$ , by Hölder’s inequality, we have

$$\begin{aligned} |T(F_1)| & \leq \sum_{k=-\infty}^{k_0-1} \mu_k |T(a^k)| \chi_{\{v_k < \infty\}} \\ & \leq \left( \sum_{k=-\infty}^{k_0-1} 2^{k \ell r'} \right)^{1/r'} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{-k \ell r} \left( \mu_k |T(a^k)| \chi_{\{v_k < \infty\}} \right)^r \right\}^{1/r} \\ & \leq C 2^{k_0 \ell} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{-k \ell r} \left( \mu_k |T(a^k)| \chi_{\{v_k < \infty\}} \right)^r \right\}^{1/r}. \end{aligned}$$

Since  $a^k$  is a  $(1, p, \infty)$ -atom for each  $k \in \mathbb{Z}$ , we have  $\|s(a^k)\|_{\infty} \leq \mathbb{P}(v_k < \infty)^{-\frac{1}{p}}$ . Note that the boundedness of  $T$  implies  $\|Tf\|_r \leq C \|f\|_{H_{p,r}^s}$ . Then, by Chebyshev’s inequality,

$$\begin{aligned} \mathbb{P}(T(F_1) > 2^{k_0}) & \leq \left\| \frac{|T(F_1)|^r}{2^{k_0 r}} \right\|_1 \leq C 2^{k_0 r(\ell-1)} \sum_{k=-\infty}^{k_0-1} 2^{-k \ell r} \mu_k^r \|T(a^k) \chi_{\{v_k < \infty\}}\|_r^r \\ & \leq C 2^{k_0 r(\ell-1)} \sum_{k=-\infty}^{k_0-1} 2^{-k \ell r} \mu_k^r \|s(a^k)\|_r^r \\ & \leq C 2^{k_0 r(\ell-1)} \sum_{k=-\infty}^{k_0-1} 2^{k(1-\ell)r} \mathbb{P}(v_k < \infty). \end{aligned}$$

By Remark 2.2, for  $0 < \varepsilon < 1$ , we deduce that

$$\begin{aligned} & \mathbb{P}(T(F_1) > 2^{k_0})^\varepsilon \gamma_b^{\varepsilon P}(\mathbb{P}(T(F_1) > 2^{k_0})) \\ & \leq C \left( 2^{k_0 r(\ell-1)} \sum_{k=-\infty}^{k_0-1} 2^{k(1-\ell)r} \mathbb{P}(v_k < \infty) \right)^\varepsilon \\ & \quad \times \gamma_b^{\varepsilon P} \left( \sum_{k=-\infty}^{k_0-1} C 2^{(k-k_0)(1-\ell)r} \mathbb{P}(v_k < \infty) \right) \\ & \leq C \sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)(1-\ell)r\varepsilon} \mathbb{P}(v_k < \infty)^\varepsilon \times \gamma_b^{\varepsilon P} \left( C 2^{(k-k_0)(1-\ell)r} \mathbb{P}(v_k < \infty) \right). \end{aligned}$$

Take  $\varepsilon \in (\frac{p}{r}, 1)$  (this implies  $\varepsilon < \frac{q}{p}$ ) and  $\ell \in (0, 1 - \frac{p}{\varepsilon r})$ . Set

$$z := \varepsilon - \frac{p}{(1-\ell)r} > 0.$$

Then  $(1-\ell)r(\varepsilon-z) = p$ . Since  $t^z \gamma_b(t)$  is equivalent to a non-decreasing function, we obtain

$$\begin{aligned} & 2^{k_0 p} \mathbb{P}(T(F_1) > 2^{k_0})^\varepsilon \gamma_b^{\varepsilon P}(\mathbb{P}(T(F_1) > 2^{k_0})) \\ & \leq C 2^{k_0 p} \sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)(1-\ell)r\varepsilon} \mathbb{P}(v_k < \infty)^\varepsilon 2^{-(k-k_0)(1-\ell)rz} \times \gamma_b^{\varepsilon P}(\mathbb{P}(v_k < \infty)) \\ & = C \sum_{k=-\infty}^{k_0-1} \left( 2^k \mathbb{P}(v_k < \infty)^\varepsilon \frac{1}{p} \gamma_b^\varepsilon(\mathbb{P}(v_k < \infty)) \right)^p. \end{aligned} \tag{3.3}$$

By Lemma 2.5, we have  $T(F_1) \in L_{p,q,b}$ , which finishes the proof of Case 1.

Case 2:  $\frac{q}{p} \in (0, 1)$ . Take  $\ell \in (0, 1 - \frac{p}{r})$ ,  $\varepsilon \in (\frac{p}{(1-\ell)r}, 1)$  and set

$$z := 1 - \frac{p}{(1-\ell)r\varepsilon}.$$

Similar to (3.3), we have

$$\begin{aligned} & 2^{k_0 p} \mathbb{P}(T(F_1) > 2^{k_0})^\varepsilon \gamma_b^{\varepsilon P}(\mathbb{P}(T(F_1) > 2^{k_0})) \\ & \leq C \left( \sum_{k=-\infty}^{k_0-1} \left( 2^{\frac{k}{\varepsilon}} \mathbb{P}(v_k < \infty)^{\frac{1}{p}} \gamma_b^\varepsilon(\mathbb{P}(v_k < \infty)) \right)^p \right)^\varepsilon, \end{aligned}$$



which further implies that

$$\begin{aligned}
 & \sum_{k_0=-\infty}^{\infty} 2^{k_0 q} \mathbb{P}(|T(F_1)| > 2^{k_0})^{\frac{q}{p}} \gamma_b^q(\mathbb{P}(|T(F_1)| > 2^{k_0})) \\
 & \leq C \sum_{k_0=-\infty}^{\infty} 2^{k_0(1-\frac{1}{\varepsilon})q} \left( \sum_{k=-\infty}^{k_0-1} \left( 2^{\frac{k}{\varepsilon}} \mathbb{P}(v_k < \infty)^{\frac{1}{p}} \gamma_b(\mathbb{P}(v_k < \infty)) \right)^p \right)^{\frac{q}{p}} \\
 & \leq C \sum_{k_0=-\infty}^{\infty} 2^{k_0(1-\frac{1}{\varepsilon})q} \sum_{k=-\infty}^{k_0-1} 2^{k\frac{1}{\varepsilon}q} \mathbb{P}(v_k < \infty)^{\frac{q}{p}} \gamma_b^q(\mathbb{P}(v_k < \infty)) \\
 & = \sum_{k=-\infty}^{\infty} 2^{k\frac{1}{\varepsilon}q} \mathbb{P}(v_k < \infty)^{\frac{q}{p}} \gamma_b^q(\mathbb{P}(v_k < \infty)) \sum_{k_0=k+1}^{\infty} 2^{k_0(1-\frac{1}{\varepsilon})q} \\
 & \leq C \sum_{k=-\infty}^{\infty} 2^{kq} \mathbb{P}(v_k < \infty)^{\frac{q}{p}} \gamma_b^q(\mathbb{P}(v_k < \infty)).
 \end{aligned}$$

By Theorem 3.2, we obtain

$$\|T(F_1)\|_{p,q,b} \leq C \|f\|_{H_{p,q,b}^s}.$$

This finishes the proof of Case 2.  $\square$

An argument similar to that used above allows to prove the following result. We omit the proof.

LEMMA 3.4. *Let  $b$  be a non-decreasing slowly varying function. Suppose that  $0 < p < \infty, 0 < q \leq \infty$  and  $1 < r < \infty$  with  $p < r$ . If  $T : H_r^s \rightarrow L_r$  (or  $H_r \rightarrow L_r$ ) is a bounded  $\sigma$ -sublinear operator and*

$$\{|T(a)| > 0\} \subset \{v < \infty\} \tag{3.4}$$

for every  $(2, p, \infty)$ -atom (or  $(3, p, \infty)$ -atom)  $a$  associated with the stopping time  $v$ , then

$$\|T(f)\|_{p,q,b} \leq C \|f\|_{\mathcal{D}_{p,q,b}} \quad (\text{or } \|T(f)\|_{p,q,b} \leq C \|f\|_{\mathcal{D}_{p,q,b}}).$$

Now we prove our first main result of the paper.

*Proof of Theorem 1.2.* Note that the operators  $M, S$  and  $s$  are all  $\sigma$ -sublinear and satisfy (3.2). First we show (1.1). Indeed, it follows from the facts  $\|M(f)\|_2 \leq C \|s(f)\|_2$  ([18, Theorem 2.11(i)],  $\|S(f)\|_2 = \|s(f)\|_2$  and Lemma 3.3 that the two inequalities in (1.1) hold. The inequalities (1.2) comes easily from the definitions of these martingale spaces.

Consider  $T(f) = M(f)$  or  $S(f)$ . Then (1.3) follows from the combination of the Burkholder-Gundy inequalities ([18, Theorem 2.12]), Doob’s maximal inequalities

$$\|S(f)\|_r \approx \|M(f)\|_r \approx \|f\|_r \quad (1 < r < \infty)$$

and Lemma 3.4.

Applying the inequalities (see [18, Theorem 2.11(ii)])

$$\|s(f)\|_r \leq C\|M(f)\|_r, \quad \|s(f)\|_r \leq C\|S(f)\|_r, \quad 2 < r < \infty,$$

and Lemma 3.4, we obtain (1.4).

In order to prove (1.5), we use (1.3). Take  $f = (f_n)_{n \geq 0} \in \mathcal{D}_{p,q,b}$ . Then there exists an optimal control  $(\lambda_n^{(1)})_{n \geq 0}$  such that  $S_n(f) \leq \lambda_{n-1}^{(1)}$  with  $\lambda_\infty^{(1)} \in L_{p,q,b}$ . Since

$$|f_n| \leq M_{n-1}(f) + \lambda_{n-1}^{(1)},$$

by the second inequality of (1.3) we have

$$\|f\|_{\mathcal{D}_{p,q,b}} \leq C(\|f\|_{H_{p,q,b}} + \|\lambda_\infty^{(1)}\|_{L_{p,q,b}}) \leq C\|f\|_{\mathcal{D}_{p,q,b}}.$$

Conversely, if  $f = (f_n)_{n \geq 0} \in \mathcal{D}_{p,q,b}$ , then there exists an optimal control  $(\lambda_n^{(2)})_{n \geq 0}$  such that  $|f_n| \leq \lambda_{n-1}^{(2)}$  with  $\lambda_\infty^{(2)} \in L_{p,q,b}$ . Note that

$$S_n(f) \leq S_{n-1}(f) + 2\lambda_{n-1}^{(2)}.$$

According to the first inequality of (1.3), we have  $\|f\|_{\mathcal{D}_{p,q,b}} \leq C\|f\|_{\mathcal{D}_{p,q,b}}$ . The proof is complete.  $\square$

In order to prove our second main result, we need the atomic decompositions of  $H_{p,q,b}$  and  $H_{p,q,b}^S$ .

LEMMA 3.5. *Let  $b$  be a non-decreasing slowly varying function. Suppose that  $0 < p < \infty, 0 < q \leq \infty$ . If  $\{\mathcal{F}_n\}_{n \geq 0}$  is regular, then*

$$H_{p,q,b}^S \subset \mathcal{D}_{p,q,b}, \quad H_{p,q,b} \subset \mathcal{D}_{p,q,b}.$$

*Proof.* We only give the proof for  $H_{p,q,b} \subset \mathcal{D}_{p,q,b}$ . The other one can be similarly shown. Take  $f \in H_{p,q,b}$ . It follows from the regularity of  $\{\mathcal{F}_n\}_{n \geq 0}$  that there exists a sequence of stopping times  $v_k$  such that

$$\{M(f) > 2^k\} \subset \{v_k < \infty\}, \quad M_{v_k}(f) \leq 2^k, \quad \mathbb{P}(v_k < \infty) \leq R\mathbb{P}(M(f) > 2^k)$$

and  $v_k \leq v_{k+1}, v_k \uparrow \infty$  (see [15, Definition 7.1.1]). It is easy to check the following decomposition

$$f_n = \sum_{k \in \mathbb{Z}} (f_n^{v_{k+1}} - f_n^{v_k}).$$

Define

$$\mu_k = 3 \cdot 2^k \|\chi_{\{v_k < \infty\}}\|_p \quad \text{and} \quad a_n^k = \frac{f_n^{v_{k+1}} - f_n^{v_k}}{\mu_k}.$$

Then  $a^k$  is a  $(3, p, \infty)$ -atom for each  $k \in \mathbb{Z}$ .

Now we know that  $f$  has a decomposition of (3.1) with a sequence  $(\mu_k)_{k \in \mathbb{Z}}$  of non-negative real numbers satisfying  $\mu_k = 3 \cdot 2^k \|\chi_{\{v_k < \infty\}}\|_p$ . Applying Theorem 3.2, we have

$$f \in \mathcal{P}_{p,q,b}.$$

The proof is complete.  $\square$

Now we are in a position to prove our second main result of this paper.

*Proof of Theorem 1.3.* According to [18, p. 33], by regularity, we have

$$S_n(f) \leq R^{1/2} s_n(f) \quad \text{and} \quad \|f\|_{H_{p,q,b}^S} \lesssim \|f\|_{H_{p,q,b}^S}.$$

Since  $s_n(f) \in \mathcal{F}_{n-1}$ , by the definition of  $\mathcal{Q}_{p,q,b}$  we have

$$\|f\|_{\mathcal{Q}_{p,q,b}} \lesssim \|s(f)\|_{L_{p,q,b}} = \|f\|_{H_{p,q,b}^S}.$$

Hence, by (1.4) we obtain

$$\mathcal{Q}_{p,q,b} = H_{p,q,b}^S.$$

Combining Lemma 3.5, the inequalities (1.2), (1.3) and (1.5), we get

$$H_{p,q,b}^S = \mathcal{Q}_{p,q,b} = \mathcal{P}_{p,q,b} = H_{p,q,b}.$$

The proof is complete.  $\square$

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Qingfang Wu  
School of Traffic and Transportation Engineering  
Central South University  
Changsha 410085, China  
e-mail: [wuqingfang@csu.edu.cn](mailto:wuqingfang@csu.edu.cn)

Dejian Zhou  
School of Mathematics and Statistics  
Central South University  
Changsha 410085, China  
e-mail: [zhoudejian@csu.edu.cn](mailto:zhoudejian@csu.edu.cn)

Lihua Peng  
School of Mathematics and Statistics  
Central South University  
Changsha 410085, China  
e-mail: [lihuap@csu.edu.cn](mailto:lihuap@csu.edu.cn)