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COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS OF EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract. In this paper, some results on complete moment convergence for weighted sums of extended negatively dependent (END, for short) random variables are established. The results extend and improve the result of Baum and Katz (1965) from complete convergence for non-weighted sums of independent random variables to the case of weighted sums of END random variables under mild conditions.

1. Introduction

In many stochastic models, the assumption that random variables are independent is not plausible. So it is of interest to extend the concept of independence to dependence cases. One of these dependence structures is extended negatively dependent structure.

Firstly, let us recall the concept of extended negatively dependent random variables.

DEFINITION 1.1. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be extended negatively dependent (END, for short) if there exists a positive constant M independent of n such that both

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leqslant M \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n) \leqslant M \prod_{i=1}^{n} P(X_i \leqslant x_i)$$

hold for each $n \geqslant 1$ and all real numbers $x_1, x_2, \ldots, x_n$. An infinite sequence $\{X_n, n \geqslant 1\}$ is said to be END if every finite subcollection is END. An array $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of random variables is said to be rowwise END, if for any fixed $n \geqslant 1$, $\{X_{ni}, 1 \leqslant i \leqslant n\}$ are END random variables.

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The concept of END sequence was introduced by Liu (2009). It is easily seen that independent and negatively orthant dependent (NOD, for short) random variables are END. Furthermore, Joag-Dev and Proschan (1983) pointed out that negatively associated (NA, for short) random variables are NOD. Meanwhile, Hu (2000) introduced the concept of negatively superadditive dependent (NSD, for short) random variables and pointed out that NSD implies NOD (see Property 2 of Hu (2000)). By the above statements, we can see that the class of END random variables includes NOD random variables, NSD random variables, NA random variables and independent random variables as special cases. Thus, it is of practical significance to further study the probability limit theorems and applications for END random variables.

Some applications for END sequence have been found. For example, Liu (2009) studied the precise large deviations for dependent random variables with heavy tails; Liu (2010) obtained the sufficient and necessary conditions of moderate deviations for dependent random variables with heavy tails; Chen et al. (2011) studied the precise large deviations of random sums in presence of NOD and consistent variation; Shen (2011) established some probability inequalities for END sequences and gave some applications; Wang and Wang (2013) obtained the precise large deviations for random sums of END real-valued random variables with consistent variation; Wang et al. (2013) studied some convergence results for weighted sums of END random variables; Wu et al. (2015) obtained L_r convergence, complete convergence and complete moment convergence for arrays of row-wise END random variables under some appropriate conditions of h-integrability; Wang et al. (2015) and Yang et al. (2017) obtained the complete consistency for the estimator of nonparametric regression models based on END errors; Shen and Volodin (2017) investigated weak and strong laws of large numbers for arrays of rowwise END random variables and gave their applications to nonparametric regression models based on END errors; Shen et al. (2017) studied the complete convergence and complete moment convergence for nonweighted and weighted sums of arrays of rowwise END random variables, and so forth.

In this paper, we aim to establish complete convergence and complete moment convergence for weighted sums of END random variables. Next, we give the concept of complete convergence, which was proposed firstly by Hsu and Robbins (1947) as follows.

DEFINITION 1.2. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to converge completely to a constant a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - a| > \varepsilon) < \infty.$$

In view of the Borel-Cantelli lemma, the above result implies that $X_n \to a$ almost surely. Therefore, the complete convergence is a very important tool in establishing almost sure convergence of summation of random variables as well as weighted sums of random variables. For more details about the complete convergence, we refer the reader to Erdös (1949), Katz (1963), Baum and Katz (1965), Chow (1973) and Gut (1992).

Furthermore, Chow (1988) first introduced the complete moment convergence, whose concept is as follows.

DEFINITION 1.3. Let $\{Z_n, n \geqslant 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. For any $\varepsilon > 0$, if

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty,$$

then $\{Z_n, n \geqslant 1\}$ is said to be complete moment convergence.

It is easily seen that complete moment convergence implies complete convergence. Thus, complete moment convergence is stronger than complete convergence. There are many articles for complete moment convergence, for example, Sung (2009) for independent (or dependent) random variables; Wang and Hu (2014) for the maximal partial sums of martingale difference sequence; Guo et al. (2013) for weighted sums of ρ^*-mixing random variables; Wu et al. (2014) for arrays of rowwise END random variables; Shen et al. (2016) for arrays of rowwise negatively superadditive dependent (NSD) random variables; Wu et al. (2017) for weighted sums of weakly dependent random variables, and so forth.

In the following, Baum and Katz (1965) obtained the following result of complete convergence for independent and identically distributed random variables.

THEOREM A. *Let $p > 1/\alpha$ and $1/2 < \alpha \leqslant 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with $EX_1 = 0$. If $E|X_1|^p < \infty$, then for $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\left|\sum_{i=1}^{n} X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{1.1}$$

The main purpose of this paper is to improve and extend Theorem A from nonweighted sums to weighted sums, and from independent random variables to END random variables.

Combined with Theorem A, our results mainly make three improvements.

(i) The results was established from complete convergence for independent random variables to complete moment convergence for END random variables;

(ii) The results was established from nonweighted sums to weighted sums, and the condition on weights is very mild;

(iii) The results was established from $1/2 < \alpha \leqslant 1$ to $\alpha > 1/2$, and from $\alpha p > 1$ to $\alpha p \geqslant 1$.

Throughout this paper, all random variables are defined on the same probability space $(\Omega, \mathscr{F}, P)$. Let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants. C and M denote positive constants not depending on n, which may be different in various places. Let $\log x = \ln\max(x, e)$, and $I(A)$ be the indicator function of the set A. Denote $x_+ = xI(x \geqslant 0)$. $a \ll b$ implies that there exists some positive constant c_1 such that $a \leqslant c_1 b$. $\lfloor x \rfloor$ stands for the integer part of x. $a \vee b$ stands for $\max(a, b)$ and $a \wedge b$ means $\min(a, b)$.

2. Preliminary lemmas

In this section, we will present some important lemmas which will be used to prove the main results of the paper.

The first one is the basic property for END random variables, which can be refered to Liu (2010).

LEMMA 2.1. *Let random variables* $X_1, X_2, \ldots, X_n$ *be END with some concrete constant* $M > 0$*. If* $f_1, f_2, \ldots, f_n$ *are all nondecreasing (or nonincreasing) functions, then random variables* $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ *are END.*

The next one comes from Wu et al. (2017) which plays an essential role to prove the result of the paper.

LEMMA 2.2. *Let* $\{Y_i, 1 \leqslant i \leqslant n\}$ *and* $\{Z_i, 1 \leqslant i \leqslant n\}$ *be two sequences of random variables. Then for any* $q > r > 0$*,* $\varepsilon > 0$*, and* $a > 0$*, the following inequality holds:*

$$E\left(\left|\sum_{i=1}^{n}(Y_i+Z_i)\right|-\varepsilon a\right)_+^r \leqslant C_r\left(\varepsilon^{-q}+\frac{r}{q-r}\right)a^{r-q}E\left|\sum_{i=1}^{n}Y_i\right|^q+C_rE\left|\sum_{i=1}^{n}Z_i\right|^r,$$

where $C_r = 1$ *if* $0 < r \leqslant 1$ *or* $C_r = 2^{r-1}$ *if* $r > 1$.

The following one is the Rosenthal type inequality for END random variables, which was obtained by Shen (2011).

LEMMA 2.3. *Let* $r \geqslant 2$ *and* $\{X_n, n \geqslant 1\}$ *be a sequence of END random variables with some concrete constant* $M > 0$*. Assume that* $EX_n = 0$ *and* $E|X_n|^r < \infty$ *for each* $n \geqslant 1$*. Then there exists a positive constant* $C(M,r)$ *depending only on* M *and* r *such that*

$$E\left(\left|\sum_{i=1}^{n}X_i\right|^r\right) \leqslant C(M,r)\left[\sum_{i=1}^{n}E|X_i|^r+\left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}\right].$$

The last one is the Marcinkiewicz-Zygmund type inequality for END random variables, which can be found in Shen et al. (2017).

LEMMA 2.4. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of END random variables with some concrete constant* $M > 0$ *and* $E|X_n|^r < \infty$ *for some* $0 < r \leqslant 2$*. Assume further that* $EX_n = 0$ *for each* $n \geqslant 1$ *if* $r > 1$*. Then there exists a positive constant* $C(M,r)$ *depending only on* M *and* r *such that*

$$E\left|\sum_{i=1}^{n}X_i\right|^r \leqslant C(M,r)\sum_{i=1}^{n}E|X_i|^r.$$

3. Main results and proofs

Now we state the main results of this paper and give their detailed proofs.

THEOREM 3.1. *Let $r > 0$, $\alpha > 1/2$ and $\alpha p > 1$. Let $\{X, X_n, n \geqslant 1\}$ be a sequence of identically distributed END random variables with $EX = 0$ if $p \vee r \geqslant 1$. Assume that $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^q \ll n$ for some $q > p \vee r$. Then*

$$\begin{cases} E|X|^p < \infty, & if\ r < p, \\ E|X|^p \log|X| < \infty, & if\ r = p, \\ E|X|^r < \infty, & if\ r > p, \end{cases} \tag{3.1}$$

implies that

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha r} E\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| - \varepsilon n^{\alpha}\right)_{+}^{r} < \infty, \tag{3.2}$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{3.3}$$

Proof. Without loss of generality, we assume that $a_{ni} > 0$ for all $1 \leqslant i \leqslant n$ and $n \geqslant 1$ (Otherwise, we use a_{ni}^{+} and a_{ni}^{-} instead of a_{ni}, respectively, and note that $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$), and $\sum_{i=1}^{n} |a_{ni}|^q \leqslant n$.

We will consider the following three cases.

Case 1: $0 < p \vee r < 1$

For fixed $n \geqslant 1$, define for $1 \leqslant i \leqslant n$,

$$Y_{ni} = -n^{\alpha} I(X_i < -n^{\alpha}) + X_i I(|X_i| \leqslant n^{\alpha}) + n^{\alpha} I(X_i > n^{\alpha}),$$

$$Z_{ni} = X_i - Y_{ni} = (X_i - n^{\alpha}) I(X_i > n^{\alpha}) + (X_i + n^{\alpha}) I(X_i < -n^{\alpha}).$$

For every $n \geqslant 1$, by Lemma 2.1, we can see that $\{Y_{ni}, 1 \leqslant i \leqslant n\}$ and $\{Z_{ni}, 1 \leqslant i \leqslant n\}$ are still END random variables, which imply that $\{a_{ni} Y_{ni}, 1 \leqslant i \leqslant n\}$ and $\{a_{ni} Z_{ni}, 1 \leqslant i \leqslant n\}$ are both END random variables. It is easily seen that

$$|Y_{ni}| = |X_i| I(|X_i| \leqslant n^{\alpha}) + n^{\alpha} I(|X_i| > n^{\alpha}) \leqslant |X_i|,$$

$$|Z_{ni}| \leqslant |X_i| I(|X_i| > n^{\alpha}) \leqslant |X_i|.$$

Taking $\gamma = q \wedge 1$, by Lemmas 2.2 and 2.4 and (3.1), we can get that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| - \varepsilon n^{\alpha}\right)_+^r \\
= &\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n} (a_{ni} Y_{ni} + a_{ni} Z_{ni})\right| - \varepsilon n^{\alpha}\right)_+^r \\
\ll &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\gamma-2} E\left|\sum_{i=1}^{n} a_{ni} Y_{ni}\right|^{\gamma} + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left|\sum_{i=1}^{n} a_{ni} Z_{ni}\right|^{r} \\
\ll &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\gamma-2} \sum_{i=1}^{n} E|a_{ni} Y_{ni}|^{\gamma} + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} E|a_{ni} Z_{ni}|^{r} \\
\ll &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\gamma-2} \sum_{i=1}^{n} |a_{ni}|^{\gamma} \left(E|X_i|^{\gamma} I(|X_i| \leqslant n^{\alpha}) + n^{\alpha\gamma} EI(|X_i| > n^{\alpha})\right) \\
&+ \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} E|a_{ni} X_i|^{r} I(|X_i| > n^{\alpha}) \\
\ll &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\gamma-2} \sum_{i=1}^{n} |a_{ni}|^{\gamma} E|X|^{\gamma} I(|X| \leqslant n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} |a_{ni}|^{r} E|X|^{r} I(|X| > n^{\alpha}) \\
\ll &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\gamma-1} E|X|^{\gamma} I(|X| \leqslant n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha r} E|X|^{r} I(|X| > n^{\alpha}) \\
=: &\, I_1 + I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha\gamma-1} E|X|^{\gamma} I(|X| \leqslant n^{\alpha}) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-\alpha\gamma-1} \sum_{m=1}^{n} E|X|^{\gamma} I((m-1)^{\alpha} < |X| \leqslant m^{\alpha}) \\
&= \sum_{m=1}^{\infty} E|X|^{\gamma} I((m-1)^{\alpha} < |X| \leqslant m^{\alpha}) \sum_{n=m}^{\infty} n^{\alpha p-\alpha\gamma-1} \\
&\ll \sum_{m=1}^{\infty} m^{\alpha p-\alpha\gamma} E|X|^{\gamma} I((m-1)^{\alpha} < |X| \leqslant m^{\alpha}) \ll E|X|^{p} < \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} E|X|^{r} I(|X| > n^{\alpha}) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} \sum_{m=n}^{\infty} E|X|^{r} I(m^{\alpha} < |X| \leqslant (m+1)^{\alpha}) \\
&= \sum_{m=1}^{\infty} E|X|^{r} I(m^{\alpha} < |X| \leqslant (m+1)^{\alpha}) \sum_{n=1}^{m} n^{\alpha p-\alpha r-1}
\end{aligned}
$$

$$
\ll \begin{cases} \sum_{m=1}^{\infty} m^{\alpha p-\alpha r} E|X|^r I(m^\alpha < |X| \leqslant (m+1)^\alpha) \ll E|X|^p, & if\ r<p, \\ \sum_{m=1}^{\infty} (\log m) E|X|^r I(m^\alpha < |X| \leqslant (m+1)^\alpha) \ll E|X|^p \log|X|, & if\ r=p, \\ \sum_{m=1}^{\infty} E|X|^r I(m^\alpha < |X| \leqslant (m+1)^\alpha) \ll E|X|^r, & if\ r>p, \end{cases}
$$
$$
< \infty.
$$

Thus, (3.2) holds.

Case 2: $1 \leqslant p \vee r < 2$

It is easily seen that $E|X|^{p\vee r} < \infty$ by (3.1). Hence, we have by $EX_i = 0$ that

$$
\begin{aligned}
n^{-\alpha} \left| \sum_{i=1}^{n} E a_{ni} Y_{ni} \right| &= n^{-\alpha} \left| \sum_{i=1}^{n} E a_{ni} Z_{ni} \right| \\
&\leqslant n^{-\alpha} \sum_{i=1}^{n} |a_{ni}| E|Z_{ni}| \\
&\leqslant n^{-\alpha} \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|X_i| > n^\alpha) \\
&\leqslant n^{1-\alpha} E|X| I(|X| > n^\alpha) \\
&\leqslant n^{1-\alpha(p\vee r)} E|X|^{p\vee r} I(|X| > n^\alpha) \\
&\to 0, \text{ as } n \to \infty. \qquad (3.4)
\end{aligned}
$$

Hence, $|\sum_{i=1}^{n} E a_{ni} Y_{ni}| \leqslant \varepsilon n^\alpha/2$ for all n large enough. Take $\beta = q \wedge 2$.

(i) $0 < r < 1$

Analogous to the proof of Case 1, by Lemmas 2.2 and 2.4, C_r-inequality, Jensen's inequality and (3.1), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left| \sum_{i=1}^{n} a_{ni} X_i \right| - \varepsilon n^\alpha \right)_+^r \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left| \sum_{i=1}^{n} (a_{ni} Y_{ni} + a_{ni} Z_{ni}) \right| - \varepsilon n^\alpha \right)_+^r \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left| \sum_{i=1}^{n} (a_{ni} Y_{ni} - E a_{ni} Y_{ni} + a_{ni} Z_{ni}) \right| - \varepsilon n^\alpha/2 \right)_+^r \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-2} E\left| \sum_{i=1}^{n} (a_{ni} Y_{ni} - E a_{ni} Y_{ni}) \right|^\beta + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left| \sum_{i=1}^{n} a_{ni} Z_{ni} \right|^r \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-2} \sum_{i=1}^{n} E|a_{ni} Y_{ni}|^\beta + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} E|a_{ni} Z_{ni}|^r
\end{aligned}
$$

$$
\begin{aligned}
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-2} \sum_{i=1}^{n} |a_{ni}|^{\beta} \left(E|X_i|^{\beta} I(|X_i| \leqslant n^{\alpha}) + n^{\alpha\beta} EI(|X_i| > n^{\alpha})\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} |a_{ni}|^{r} E|X_i|^{r} I(|X_i| > n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-2} \sum_{i=1}^{n} |a_{ni}|^{\beta} E|X|^{\beta} I(|X| \leqslant n^{\alpha}) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} |a_{ni}|^{r} E|X|^{r} I(|X| > n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-1} E|X|^{\beta} I(|X| \leqslant n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha r} E|X|^{r} I(|X| > n^{\alpha}) \\
&< \infty.
\end{aligned}
$$

(ii) $1 \leqslant r < 2$

Analogous to the proof of Case 1, by $EX_i = 0$, Lemmas 2.2 and 2.4, C_r-inequality, Jensen's inequality and (3.1), we have that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| - \varepsilon n^{\alpha}\right)_{+}^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-2} E\left(\left|\sum_{i=1}^{n} (a_{ni} Y_{ni} - E a_{ni} Y_{ni})\right|^{\beta}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n} (a_{ni} Z_{ni} - E a_{ni} Z_{ni})\right|^{r}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-2} \sum_{i=1}^{n} E|a_{ni} Y_{ni}|^{\beta} + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} E|a_{ni} Z_{ni}|^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-2} \sum_{i=1}^{n} |a_{ni}|^{\beta} \left(E|X_i|^{\beta} I(|X_i| \leqslant n^{\alpha}) + n^{\alpha\beta} EI(|X_i| > n^{\alpha})\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} |a_{ni}|^{r} E|X_i|^{r} I(|X_i| > n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-2} \sum_{i=1}^{n} |a_{ni}|^{\beta} E|X|^{\beta} I(|X| \leqslant n^{\alpha}) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} |a_{ni}|^{r} E|X|^{r} I(|X| > n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-1} E|X|^{\beta} I(|X| \leqslant n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha r} E|X|^{r} I(|X| > n^{\alpha}) \\
&< \infty.
\end{aligned}
$$

Case 3: $p \vee r \geqslant 2$

For $1 \leqslant i \leqslant n$ and $n \geqslant 1$, define $a_{ni}^{(1)} = a_{ni} I(|a_{ni}| \leqslant 1)$ and $a_{ni}^{(2)} = a_{ni} I(|a_{ni}| > 1)$. Hence,

$$\begin{aligned}
&\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| - 2\varepsilon n^{\alpha}\right)_+ \\
&\leqslant \left(\left|\sum_{i=1}^{n} a_{ni}^{(1)} X_i\right| + \left|\sum_{i=1}^{n} a_{ni}^{(2)} X_i\right| - 2\varepsilon n^{\alpha}\right)_+ \\
&\leqslant \left(\left|\sum_{i=1}^{n} a_{ni}^{(1)} X_i\right| - \varepsilon n^{\alpha}\right)_+ + \left(\left|\sum_{i=1}^{n} a_{ni}^{(2)} X_i\right| - \varepsilon n^{\alpha}\right)_+ .
\end{aligned}$$

In order to prove (3.2), it suffices to show that

$$H := \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha r} E\left(\left|\sum_{i=1}^{n} a_{ni}^{(1)} X_i\right| - \varepsilon n^{\alpha}\right)_+^r < \infty \tag{3.5}$$

and

$$G := \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha r} E\left(\left|\sum_{i=1}^{n} a_{ni}^{(2)} X_i\right| - \varepsilon n^{\alpha}\right)_+^r < \infty. \tag{3.6}$$

Now, we turn to prove $H < \infty$ and $G < \infty$.

For fixed $n \geqslant 1$, denote for $1 \leqslant i \leqslant n$ that

$$\begin{aligned}
Y_{ni}^{(1)} &= -a_{ni}^{(1)} n^{\alpha} I(X_i < -n^{\alpha}) + a_{ni}^{(1)} X_i I(|X_i| \leqslant n^{\alpha}) + a_{ni}^{(1)} n^{\alpha} I(X_i > n^{\alpha}), \\
Z_{ni}^{(1)} &= a_{ni}^{(1)} X_i - Y_{ni}^{(1)} = a_{ni}^{(1)} (X_i - n^{\alpha}) I(X_i > n^{\alpha}) + a_{ni}^{(1)} (X_i + n^{\alpha}) I(X_i < -n^{\alpha}); \\
Y_{ni}^{(2)} &= -n^{\alpha} I(a_{ni}^{(2)} X_i < -n^{\alpha}) + a_{ni}^{(2)} X_i I(|a_{ni}^{(2)} X_i| \leqslant n^{\alpha}) + n^{\alpha} I(a_{ni}^{(2)} X_i > n^{\alpha}), \\
Z_{ni}^{(2)} &= a_{ni}^{(2)} X_i - Y_{ni}^{(2)} = (a_{ni}^{(2)} X_i - n^{\alpha}) I(a_{ni}^{(2)} X_i > n^{\alpha}) + (a_{ni}^{(2)} X_i + n^{\alpha}) I(a_{ni}^{(2)} X_i < -n^{\alpha}).
\end{aligned}$$

For fixed $n \geqslant 1$, by Lemma 2.1, we can see that $\{Y_{ni}^{(j)}, 1 \leqslant i \leqslant n\}$ and $\{Z_{ni}^{(j)}, 1 \leqslant i \leqslant n\}$ $(j = 1, 2)$ are both END random variables. It is easily seen that

$$\begin{aligned}
&|Y_{ni}^{(1)}| = |a_{ni}^{(1)} X_i| I(|X_i| \leqslant n^{\alpha}) + a_{ni}^{(1)} n^{\alpha} I(|X_i| > n^{\alpha}) \leqslant |a_{ni}^{(1)} X_i|, \\
&|Z_{ni}^{(1)}| \leqslant |a_{ni}^{(1)} X_i| I(|X_i| > n^{\alpha}) \leqslant |a_{ni}^{(1)} X_i|; \\
&|Y_{ni}^{(2)}| = |a_{ni}^{(2)} X_i| I(|a_{ni}^{(2)} X_i| \leqslant n^{\alpha}) + n^{\alpha} I(|a_{ni}^{(2)} X_i| > n^{\alpha}) \leqslant |a_{ni}^{(2)} X_i|, \\
&|Z_{ni}^{(2)}| \leqslant |a_{ni}^{(2)} X_i| I(|a_{ni}^{(2)} X_i| > n^{\alpha}) \leqslant |a_{ni}^{(2)} X_i|.
\end{aligned}$$

Similar to the proof of (3.4), we can get that

$$n^{-\alpha} \left|\sum_{i=1}^{n} E Y_{ni}^{(j)}\right| = n^{-\alpha} \left|\sum_{i=1}^{n} E Z_{ni}^{(j)}\right| \to 0 \tag{3.7}$$

as $n \to \infty$, where $j = 1,2$. Note that $\left|a_{ni}^{(1)}\right| \leqslant 1$ and (3.1) implies $EX^2 < \infty$ in this case. Take $\mu > q \vee (\alpha p - 1)/(\alpha - 1/2)$ such that $\alpha p - \alpha\mu - 2 + \mu/2 < -1$.

(i) $0 < r < 1$

By Lemmas 2.2 and 2.3, C_r-inequality, Jensen's inequality, (3.1), $I_1 < \infty$ and $I_2 < \infty$, we have that

$$\begin{aligned}
H &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n}(Y_{ni}^{(1)}+Z_{ni}^{(1)})\right| - \varepsilon n^{\alpha}\right)_{+}^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} E\left(\left|\sum_{i=1}^{n}(Y_{ni}^{(1)}-EY_{ni}^{(1)})\right|^{\mu}\right) + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left|\sum_{i=1}^{n} Z_{ni}^{(1)}\right|^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\left\{\sum_{i=1}^{n} E|Y_{ni}^{(1)}-EY_{ni}^{(1)}|^{\mu} + \left(\sum_{i=1}^{n} E|Y_{ni}^{(1)}-EY_{ni}^{(1)}|^{2}\right)^{\mu/2}\right\} \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\sum_{i=1}^{n} E\left|Z_{ni}^{(1)}\right|^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\left\{\sum_{i=1}^{n} E|Y_{ni}^{(1)}|^{\mu} + \left(\sum_{i=1}^{n} E|Y_{ni}^{(1)}|^{2}\right)^{\mu/2}\right\} \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\sum_{i=1}^{n} |a_{ni}^{(1)}|^{r} E|X_i|^{r} I(|X_i| > n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\sum_{i=1}^{n} |a_{ni}^{(1)}|^{\mu}\left(E|X_i|^{\mu} I(|X_i| \leqslant n^{\alpha}) + n^{\alpha\mu} EI(|X_i| > n^{\alpha})\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\left(\sum_{i=1}^{n} |a_{ni}^{(1)}|^{2} EX_i^{2}\right)^{\mu/2} + \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha r} E|X|^{r} I(|X| > n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-1} E|X|^{\mu} I(|X| \leqslant n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\left(nEX^{2}\right)^{\mu/2} \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha r} E|X|^{r} I(|X| > n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2+\mu/2}\left(EX^{2}\right)^{\mu/2} < \infty
\end{aligned}$$

and

$$\begin{aligned}
G &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n}(Y_{ni}^{(2)}+Z_{ni}^{(2)})\right| - \varepsilon n^{\alpha}\right)_{+}^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} E\left(\left|\sum_{i=1}^{n}(Y_{ni}^{(2)}-EY_{ni}^{(2)})\right|^{\mu}\right) + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n} Z_{ni}^{(2)}\right|^{r}\right)
\end{aligned}$$

$$\begin{aligned}
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\left\{\sum_{i=1}^{n} E|Y_{ni}^{(2)}-EY_{ni}^{(2)}|^{\mu}+\left(\sum_{i=1}^{n} E|Y_{ni}^{(2)}-EY_{ni}^{(2)}|^{2}\right)^{\mu/2}\right\}\\
&\quad+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\sum_{i=1}^{n} E\left|Z_{ni}^{(2)}\right|^{r}\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\left\{\sum_{i=1}^{n} E|Y_{ni}^{(2)}|^{\mu}+\left(\sum_{i=1}^{n} E|Y_{ni}^{(2)}|^{2}\right)^{\mu/2}\right\}\\
&\quad+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\sum_{i=1}^{n} E|a_{ni}^{(2)}X_i|^{r} I(|a_{ni}^{(2)}X_i|>n^{\alpha})\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\sum_{i=1}^{n}\left(|a_{ni}^{(2)}|^{\mu}E|X_i|^{\mu}I(|a_{ni}^{(2)}X_i|\leqslant n^{\alpha})+n^{\alpha\mu}EI(|a_{ni}^{(2)}X_i|>n^{\alpha})\right)\\
&\quad+\sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\left(\sum_{i=1}^{n} E|a_{ni}^{(2)}X_i|^{2}\right)^{\mu/2}+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\sum_{i=1}^{n} E|a_{ni}^{(2)}X_i|^{r}I(|a_{ni}^{(2)}X_i|>n^{\alpha})\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\sum_{i=1}^{n} E|a_{ni}^{(2)}X|^{\mu}I(|a_{ni}^{(2)}X|\leqslant n^{\alpha})+\sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\left(\sum_{i=1}^{n} E|a_{ni}^{(2)}X|^{2}\right)^{\mu/2}\\
&\quad+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\sum_{i=1}^{n} E|a_{ni}^{(2)}X|^{r}I(|a_{ni}^{(2)}X|>n^{\alpha})\\
&=: G_1+G_2+G_3.
\end{aligned}$$

Next, we need to prove $G_i<\infty$ for $i=1,2,3$. In view of (2.21)–(2.23) in Sung (2010), we have $G_1<\infty$. Noting that $\alpha p-\alpha\mu-2+\mu/2<-1$, we have

$$\begin{aligned}
G_2 &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\left(nEX^2\right)^{\mu/2}\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2+\mu/2}\left(EX^2\right)^{\mu/2}<\infty.
\end{aligned}$$

For $G_3<\infty$, denote for $n\geqslant 2$ and $1\leqslant j\leqslant n-1$ that

$$I_{nj}=\left\{1\leqslant i\leqslant n: n^{1/q}(j+1)^{-1/q}<|a_{ni}^{(2)}|\leqslant n^{1/q}j^{-1/q}\right\}.$$

Then $\{I_{nj}, 1\leqslant j\leqslant n-1\}$ are disjoint, and $\bigcup_{j=1}^{n-1} I_{nj}=\{1\leqslant i\leqslant n: a_{ni}^{(2)}\neq 0\}$. Noting that $q>r$, we have $1\leqslant k\leqslant n-1$ that

$$n\geqslant \sum_{i=1}^{n}|a_{ni}^{(2)}|^{q}\geqslant \sum_{j=1}^{n-1}\sum_{i\in I_{nj}}|a_{ni}^{(2)}|^{q}\geqslant n\sum_{j=1}^{k}(j+1)^{-r/q}(k+1)^{-1+r/q}\sharp I_{nj},$$

and thus $\sum_{j=1}^{k} j^{-r/q}\sharp I_{nj} \leqslant C(k+1)^{1-r/q}$. Take $t = 1/(\alpha - 1/q)$. Then we have that

$$\begin{aligned}
G_3 &= \sum_{n=2}^{\infty} n^{\alpha p-\alpha r-2} \sum_{i=1}^{n} E|a_{ni}^{(2)}X|^r I(|a_{ni}^{(2)}X| > n^{\alpha}) \\
&= \sum_{n=2}^{\infty} n^{\alpha p-\alpha r-2} \sum_{j=1}^{n-1} \sum_{i\in I_{nj}} E|a_{ni}^{(2)}X|^r I(|a_{ni}^{(2)}X| > n^{\alpha}) \\
&\leqslant \sum_{n=2}^{\infty} n^{\alpha p-\alpha r-2+r/q} \sum_{j=1}^{n-1} j^{-r/q}\sharp I_{nj} \sum_{k=\lfloor nj^{t/q}\rfloor}^{\infty} E|X|^r I(k < |X|^t \leqslant k+1) \\
&\leqslant \sum_{n=2}^{\infty} n^{\alpha p-\alpha r-2+r/q} \sum_{k=n}^{\infty} E|X|^r I(k < |X|^t \leqslant k+1) \sum_{j=1}^{(n-1)\wedge\lfloor((k+1)/n)^{q/t}\rfloor} j^{-r/q}\sharp I_{nj} \\
&\leqslant C\sum_{n=2}^{\infty} n^{\alpha p-\alpha r-2+r/q} \sum_{k=n}^{\infty} \left\{(n-1)\wedge\lfloor((k+1)/n)^{q/t}\rfloor+1\right\}^{1-r/q} E|X|^r I(k < |X|^t \leqslant k+1) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+r/q} \sum_{k=n}^{\lfloor n^{1+t/q}\rfloor} \left(\lfloor((k+1)/n)^{q/t}\rfloor+1\right)^{1-r/q} E|X|^r I(k < |X|^t \leqslant k+1) \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} \sum_{k=\lfloor n^{1+t/q}\rfloor}^{\infty} E|X|^r I(k < |X|^t \leqslant k+1) \\
&=: G_{31} + G_{32}.
\end{aligned}$$

Noting that $\alpha q(p-q)/(t+q) = (p-q)/t$, we have that

$$\begin{aligned}
G_{31} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1} \sum_{k=n}^{\lfloor n^{1+t/q}\rfloor} k^{(q-r)/t} E|X|^r I(k < |X|^t \leqslant k+1) \\
&\leqslant C\sum_{k=1}^{\infty} k^{(q-r)/t+\alpha q(p-q)/(t+q)} E|X|^r I(k < |X|^t \leqslant k+1) \leqslant CE|X|^p.
\end{aligned}$$

Similarly, noting that $\alpha q(p-r)/(t+q) = (p-r)/t$, we have that

$$\begin{aligned}
G_{32} &\leqslant C\sum_{k=1}^{\infty} E|X|^r I(k < |X|^t \leqslant k+1) \sum_{n=1}^{\lfloor k^{q/(q+t)}\rfloor} n^{\alpha p-\alpha r-1} \\
&\leqslant \begin{cases} C\sum_{k=1}^{\infty} k^{\alpha q(p-r)/(q+t)} E|X|^r I(k < |X|^t \leqslant k+1), & if\ r < p, \\ C\sum_{k=1}^{\infty} (\log k) E|X|^r I(k < |X|^t \leqslant k+1), & if\ r = p, \\ C\sum_{k=1}^{\infty} E|X|^r I(k < |X|^t \leqslant k+1), & if\ r > p, \end{cases} \\
&\leqslant \begin{cases} CE|X|^p, & if\ r < p, \\ CE|X|^p \log|X|, & if\ r = p, \\ CE|X|^r, & if\ r > p, \end{cases}
\end{aligned}$$

Hence $G_3 < \infty$.

(ii) $1 \leqslant r \leqslant 2$

By Lemmas 2.2, 2.3 and 2.4, Jensen's inequality, C_r-inequality, the proof of $H < \infty$ and $G < \infty$, we have for $j = 1,2$ that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n} a_{ni}^{(j)} X_i\right| - \varepsilon n^{\alpha}\right)_+^r \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n} (Y_{ni}^{(j)} + Z_{ni}^{(j)})\right| - \varepsilon n^{\alpha}\right)_+^r \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} E\left(\left|\sum_{i=1}^{n} (Y_{ni}^{(j)} - EY_{ni}^{(j)})\right|^{\mu}\right) + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n} (Z_{ni}^{(j)} - EZ_{ni}^{(j)})\right|^{r}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} \left\{\sum_{i=1}^{n} E|Y_{ni}^{(j)} - EY_{ni}^{(j)}|^{\mu} + \left(\sum_{i=1}^{n} E|Y_{ni}^{(j)} - EY_{ni}^{(j)}|^{2}\right)^{\mu/2}\right\} \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} E|Z_{ni}^{(j)} - EZ_{ni}^{(j)}|^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} \left\{\sum_{i=1}^{n} E|Y_{ni}^{(j)}|^{\mu} + \left(\sum_{i=1}^{n} E|Y_{ni}^{(j)}|^{2}\right)^{\mu/2}\right\} + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} E|Z_{ni}^{(j)}|^{r} \\
&< \infty.
\end{aligned}$$

(iii) $r > 2$

Noting that $\alpha p - 2 - (\alpha r - 1)r/2 < \alpha p - \alpha r - 1$, thus for $j = 1,2$, by Lemma 2.3 and (3.1), Jensen's inequality, C_r-inequality, the proof of $H < \infty$ and $G < \infty$, we can get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n} a_{ni}^{(j)} X_i\right| - \varepsilon n^{\alpha}\right)_+^r \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} E\left(\left|\sum_{i=1}^{n} (Y_{ni}^{(j)} - EY_{ni}^{(j)})\right|^{\mu}\right) + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\left|\sum_{i=1}^{n} (Z_{ni}^{(j)} - EZ_{ni}^{(j)})\right|^{r}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} \left\{\sum_{i=1}^{n} E|Y_{ni}^{(j)}|^{\mu} + \left(\sum_{i=1}^{n} E|Y_{ni}^{(j)}|^{2}\right)^{\mu/2}\right\} \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \left\{\sum_{i=1}^{n} E|Z_{ni}^{(j)}|^{r} + \left(\sum_{i=1}^{n} E|Z_{ni}^{(j)}|^{2}\right)^{r/2}\right\} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \left(\sum_{i=1}^{n} E|Z_{ni}^{(j)}|^{2}\right)^{r/2} \\
&\ll \begin{cases} \sum_{n=1}^{\infty} n^{(\alpha p-1)(1-r/2)-1} (E|X|^{p})^{r/2} < \infty, & if\ p \geqslant 2, \\ \sum_{n=1}^{\infty} n^{\alpha p-2-(\alpha r-1)r/2} (E|X|^{r})^{r/2} < \infty, & if\ 0 < p < 2. \end{cases}
\end{aligned}$$

From the statements above, we have proved the result (3.2).

At last, we prove that (3.3). For $\forall\varepsilon > 0$, we can get that

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} E\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\alpha})\right)_+^r\\ &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_0^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\alpha} > t^{1/r}\right) dt\\ &\geqslant \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_0^{\varepsilon^r n^{\alpha r}} P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\alpha} > t^{1/r}\right) dt\\ &\geqslant \varepsilon^r \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| > 2\varepsilon n^{\alpha}\right).\end{aligned}$$

Hence, (3.2) implies (3.3) immediately. This completes the proof of the theorem. □

REMARK 3.1. Our result is much more exact and comprehensive as compared to Theorem A, because Theorem A is obtained by taking $r = 1$ and $a_{ni} = 1$ ($\forall\ 1 \leqslant i \leqslant n,\ n \geqslant 1$) in Theorem 3.1. Here we considered the case of weighted average. The condition $1/2 < \alpha \leqslant 1$ is also extended to $\alpha > 1/2$ in this paper. Moreover, the method used in this paper is different from that in Baum and Katz (1965).

Taking $\alpha p = 2$ in Theorem 3.1, we can get the following corollary.

COROLLARY 3.1. *Let $0 < p < 4$. Let $\{X, X_n, n \geqslant 1\}$ be a sequence of identically distributed END random variables with $E|X|^p < \infty$ and $EX = 0$ if $p \geqslant 1$. Assume further that $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^q \ll n$ for some $q > p$. Then*

$$\frac{1}{n^{2/p}} \sum_{i=1}^{n} a_{ni}X_i \to 0\ \ a.s.,\ \ n \to \infty. \tag{3.8}$$

The result of the case $\alpha p = 1$ is also obtained as follows.

THEOREM 3.2. *Let $r > 0$ and $0 < p < 2$. Let $\{X, X_n, n \geqslant 1\}$ be a sequence of identically distributed END random variables with $EX = 0$ if $p \vee r \geqslant 1$. Assume that $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^q \ll n$ for some $q > p \vee r$. Then (3.1) implies that*

$$\sum_{n=1}^{\infty} n^{-1-r/p} E\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{1/p}\right)_+^r < \infty, \tag{3.9}$$

and thus

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| > \varepsilon n^{1/p}\right) < \infty. \tag{3.10}$$

Proof. According to the proof of Theorem 3.1, we only need to show that (3.4) and (3.7) also hold when $p\vee r\geqslant 1$ for the case $\alpha p=1$. Noting that $E|X|^{p\vee r}<\infty$ by (3.1), we have by Dominated Convergence Theorem that

$$\begin{aligned}
n^{-1/p}\left|\sum_{i=1}^{n}Ea_{ni}Y_{ni}\right| &= n^{-1/p}\left|\sum_{i=1}^{n}Ea_{ni}Z_{ni}\right|\\
&\leqslant n^{-1/p}\sum_{i=1}^{n}|a_{ni}|E|X_i|I(|X_i|>n^{1/p})\\
&\leqslant n^{1-1/p}E|X|I(|X|>n^{1/p})\\
&\leqslant n^{0\wedge(1-r/p)}E|X|^{p\vee r}I(|X|>n^{1/p})\\
&\leqslant n^{0\wedge(1-r/p)}E|X|^{p\vee r}I(|X|>n^{1/p})\\
&\to 0,\ \text{as}\ n\to\infty.
\end{aligned}$$

For $\alpha p=1$, we will prove (3.7) .

(i) For $j=1$, we have

$$\begin{aligned}
n^{-1/p}\left|\sum_{i=1}^{n}EY_{ni}^{(1)}\right| &= n^{-1/p}\left|\sum_{i=1}^{n}EZ_{ni}^{(1)}\right|\\
&\leqslant n^{-1/p}\sum_{i=1}^{n}|a_{ni}^{(1)}|E|X_i|I(|X_i|>n^{1/p})\\
&\leqslant n^{1-1/p}E|X|I(|X|>n^{1/p})\\
&\leqslant n^{0\wedge(1-r/p)}E|X|^{p\vee r}I(|X|>n^{1/p})\\
&\leqslant n^{0\wedge(1-r/p)}E|X|^{p\vee r}I(|X|>n^{1/p})\\
&\to 0,\ \text{as}\ n\to\infty.
\end{aligned}$$

(ii) For $j=2$, we have

$$\begin{aligned}
n^{-1/p}\left|\sum_{i=1}^{n}EY_{ni}^{(2)}\right| &= n^{-1/p}\left|\sum_{i=1}^{n}EZ_{ni}^{(2)}\right|\\
&\leqslant n^{-1/p}\sum_{i=1}^{n}E|a_{ni}^{(2)}X_i|I(|a_{ni}^{(2)}X_i|>n^{1/p})\\
&\leqslant n^{-1\wedge(-r/p)}\sum_{i=1}^{n}|a_{ni}^{(2)}|^{p\vee r}E|X_i|^{p\vee r}I(|a_{ni}^{(2)}X_i|>n^{1/p})\\
&\leqslant n^{0\wedge(1-r/p)}E|X|^{p\vee r}I(|X|>n^{1/p-1/q})\\
&\to 0,\ \text{as}\ n\to\infty.
\end{aligned}$$

The proof is completed. □

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