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ON THE COMPLETE MOMENT CONVERGENCE OF WEIGHTED SUMS OF ρ^*–MIXING RANDOM VARIABLES

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Abstract. In this paper, the authors study the complete moment convergence of the weighted sum of ρ^*-mixing sequences which are stochastically dominated by a random variable X. The obtained results improve the corresponding ones of Sung (2013) and Wu (2014).

1. Introduction

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{F}, P)$. Denote $\mathscr{F}_S = \sigma(X_i, i \in S \subset \mathbb{N})$. For given sub-$\sigma$-algebras $\mathscr{F}_S$, $\mathscr{F}_T$ of $\mathscr{F}$, let

$$\rho(\mathscr{F}_S, \mathscr{F}_T) = \sup_{X \in L_2(\mathscr{F}_S), Y \in L_2(\mathscr{F}_T)} \frac{|EXY - EXEY|}{(Var(X) \cdot Var(Y))^{1/2}}$$

Define

$$\rho^*(k) = \sup \rho(\mathscr{F}_S, \mathscr{F}_T)$$

where the supremum is taken over all finite subsets $S, T \in \mathbb{N}$ such that

$$\text{dist}(S,T) = \min_{j \in S, h \in T} |j - h| \geqslant k, k \geqslant 0$$

DEFINITION 1.1. A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be a ρ^*-mixing sequence if there exists $k \in \mathbb{N}$ such that $\rho^*(k) < 1$.

An array of random variables $\{X_{nk}, k \in \mathbb{N}, n \in \mathbb{N}\}$ is said to be rowwise ρ^*-mixing, if for every $n \in \mathbb{N}$, $\{X_{nk}, k \in \mathbb{N}\}$ is a ρ^*-mixing sequence of random variables.

The concept of ρ^*-mixing sequence can be dated back to Stein [1]. Bradley [2] studied the properties of ρ^*-mixing sequence and obtained the central limit theorem. Since the article of Bradley [2], many authors studied the convergence properties for sequences or arrays of ρ^*-mixing random variables. We refer the reader to [2,3,4,5,6] for more details.

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A sequence of random variables $\{U_n, n \geqslant 1\}$ is said to convergence completely to a constant a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty$$

This notion was first given by Hsu and Robbins [7], further studied by Baum and Katz [8], and it has been an important basic tool to study the convergence in probability and statistics.

Let $\{Z_n, n \geqslant 1\}$ be a sequence of random variables and $a_n > 0, b_n > 0, q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \tag{1.1}$$

for all $\varepsilon > 0$, then $\{Z_n, n \geqslant 1\}$ was called a complete moment convergence sequence by Chow [9]. Zhou et al. [10] and Sung [11] obtained the following complete convergence results for weighted sums of ρ^*-mixing sequence, respectively.

THEOREM A. (Zhou et al. [10]) *Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed ρ^*-mixing sequence, and let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\max\{\alpha,\gamma\}} = O(n) \tag{1.1}$$

for some $0 < \alpha \leqslant 2$ and $\gamma > 0$ with $\alpha \neq \gamma$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. If $EX_1 = 0$ for $1 < \alpha \leqslant 2$ and

$$\begin{cases} E|X|^{\alpha} < \infty, & \alpha > \gamma, \\ E|X|^{\alpha}\log|X| < \infty, & \alpha = \gamma, \\ E|X|^{\gamma} < \infty, & \alpha < \gamma. \end{cases} \tag{1.2}$$

holds for $\alpha \neq \gamma$, then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leqslant m \leqslant n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| > b_n\varepsilon\right) < \infty \text{ for all } \varepsilon > 0 \tag{1.3}$$

THEOREM B. (Sung [11]) *Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed $\rho^\star$-mixing sequence, and let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \leqslant 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$. If $EX_1 = 0$ for $1 < \alpha \leqslant 2$ and $E|X_1|^{\alpha}\log(1+|X_1|) < \infty$, then (1.3) holds.*

As Sung [11] pointed out, theorem A. actually only studied the complete convergence of ρ^*-mixing sequence for the case $\alpha > \gamma$. Sung [11, Remark 2.2] presented an open problem whether the case $\alpha < \gamma$ of Theorem A. remains true for ρ^*-mixing sequence. Later, Wu et al. [12] solved this open problem, and get some useful inequalities.

THEOREM C. (Wu et al. [12]) *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of identically distributed* ρ^**-mixing sequence, and let* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of constants satisfying (1.1) for some* $0 < \alpha \leqslant 2$*. Let* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$*. If* $EX_1 = 0$ *for* $1 < \alpha \leqslant 2$ *and conditions (1.2) holds for* $\alpha < \gamma$*, then conclusions (1.3) holds.*

In this work, the complete moment convergence for ρ^*-mixing sequence shall be studied. The main result is given in Section 3. The formal definition of ρ^*-mixing sequence and other preliminaries are recalled in Section 2.

Throughout this paper, the symbol C always stands for a generic positive constant which may differ from one place to another.

2. Preliminaries

We first state some lemmas, which will be used in the proofs of our main result.

LEMMA 2.1. (Utev and Peligrad [13]) *Let* $0 \leqslant r < 1, p \geqslant 2$*, and* k *be a positive integer. Assume that* $\{X_n, n \geqslant 1\}$ *is a mean zero sequence of* ρ^**-mixing random variables satisfying* $\rho^*(k) \leqslant r$*. Let* $E|X_n|^p < \infty$ *for every* $n \geqslant 1$*. Then there exists a positive constant* C *not depending on* n *such that*

$$E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leqslant C\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} Var(X_i)\right)^{p/2}\right\}$$

LEMMA 2.2. (Sung [14]) *Let* $\{X_i, 1 \leqslant i \leqslant n\}$ *and* $\{Y_i, 1 \leqslant i \leqslant n\}$ *be the sequence of random variables. Then for any* $q > 1$*,* $\varepsilon > 0$*, and* $a > 0$*,*

$$\begin{aligned} E\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i+Y_i)\right| - \varepsilon a\right)_+ &\leqslant \left(\frac{1}{\varepsilon^q} + \frac{1}{q-1}\right)\frac{1}{a^{q-1}}E\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|^q\right) \\ &\quad + E\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} Y_i\right|\right) \end{aligned}$$

LEMMA 2.3. (Adler and Rosalsky [15] and Adler et al. [16]) *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of random variables, which is stochastically dominated by a random variable* X*, i.e.*

$$\sup_{n\geqslant 1} P(|X_n| > x) \leqslant CP(|X| > x), \forall x \geqslant 0$$

Then, for any $\alpha > 0$ *and* $b > 0$*, the following two statements hold:*

$$E[|X_n|^\alpha I(|X_n| \leqslant b)] \leqslant C_1\{E[|X|^\alpha I(|X| \leqslant b)] + b^\alpha P(|X| > b)\}$$

$$E[|X_n|^\alpha I(|X_n| > b)] \leqslant C_2 E[|X|^\alpha I(|X| > b)]$$

Consequently, $E|X_n|^\alpha \leqslant C_3 E|X|^\alpha$ *for all* $n \geqslant 1$*.*

By Lemma 2.3 and Lemma 2.2 of Wu et al. [12], we can get the following lemma.

LEMMA 2.4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X. Suppose $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ ba an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n) \tag{2.1}$$

for some $\alpha > 0$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$, then,

$$\sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X_i|^{\alpha} I(|a_{ni}X_i| > b_n) \leqslant H_X(\alpha, \gamma)$$

where

$$H_X(\alpha,\gamma) = \begin{cases} CE|X|^{\alpha}, & \alpha > \gamma, \\ CE|X|^{\alpha}\log(1+|X|), & \alpha = \gamma \\ CE|X|^{\gamma}, & \alpha < \gamma. \end{cases} \tag{2.2}$$

Proof. Note the fact that $\{X_n, n \geqslant 1\}$ is stochastically dominated by a random variable X, we can get,

$$P(|a_{ni}X_i| > x) \leqslant CP(|a_{ni}X| > x)$$

By Lemma 2.3, we have

$$E\left[|a_{ni}X_i|^{\alpha} I(|a_{ni}X_i| > b_n)\right] \leqslant CE\left[|a_{ni}X|^{\alpha} I(|a_{ni}X| > b_n)\right]$$

Then by Lemma 2.2 of Wu [8], we have,

$$\begin{aligned} &\sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X_i|^{\alpha} I(|a_{ni}X_i| > b_n) \\ &\leqslant C \sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I(|a_{ni}X| > b_n) \\ &\leqslant H_X(\alpha, \gamma) \end{aligned}$$

where $H_X(\alpha, \gamma)$ is defined as equation (2.2). □

The following lemma plays an important role in the proof of our main results, which improves Lemma 2.3 of Wu et al. [12].

LEMMA 2.5. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X. Suppose $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants satisfying equation (2.1) for some $\alpha > 0$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. If $q > \max\{\alpha, \gamma\}$, then*

$$\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} E|a_{ni}X_i|^{q} I(|a_{ni}X_i| \leqslant b_n) \leqslant H_X(\alpha, \gamma)$$

where $H_X(\alpha,\gamma)$ is defined as equation (2.2).

Proof. By Lemma 2.3, we can get that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} E|a_{ni}X_i|^q I(|a_{ni}X_i| \leqslant b_n) \\
&\leqslant C\left(\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} E|a_{ni}X|^q I(|a_{ni}X| \leqslant b_n) + \sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} P(|a_{ni}X| > b_n)\right) \\
&\triangleq C(J_1 + J_2)
\end{aligned}$$

By Lemma 2.3 of Wu et al. [12], we can get

$$J_1 \leqslant H_X(\alpha,\gamma)$$

where $H_X(\alpha,\gamma)$ is defined as equation (2.2). For J_2, by Lemma 2 of Wu et al. [12], we have

$$\begin{aligned}
J_2 &= \sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} P(|a_{ni}X| > b_n) \\
&= \sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} E(I(|a_{ni}X| > b_n)) \\
&\leqslant \sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} E(|a_{ni}X|^{\alpha} I(|a_{ni}X| > b_n)) \\
&\leqslant H_X(\alpha,\gamma)
\end{aligned}$$

Then, we complete the proof. □

3. Main result

In this section, we state our main theorem and its proofs, which improve Theorem A., Theorem B. and Theorem C. to the complete moment convergence under more general conditions.

THEOREM 3.1. *Let $\{X_n, n \geqslant 1\}$ be a mean zero sequence of ρ^*-mixing random variables, which is stochastically dominated by a random variable X. Suppose $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \leqslant 2$ and $\gamma > 0$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. If equation (1.2) holds, then, for all $\varepsilon > 0$,*

$$\sum_{n=2}^{\infty} (nb_n)^{-1} E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon b_n\right)_+ < \infty \tag{3.1}$$

Proof. Without loss of generality, we may assume that $\sum_{i=1}^{n} |a_{ni}|^{\max\{\alpha,\gamma\}} \leqslant n$. For all $n \geqslant 1$, let $X_{ni} = X_i I(|a_{ni}X_i| \leqslant b_n), \widetilde{X}_{ni} = X_i I(|a_{ni}X_i| > b_n)$, $1 \leqslant i \leqslant n$. Obviously,

$$a_{ni}X_i = a_{ni}X_{ni} + a_{ni}\widetilde{X}_{ni} = [a_{ni}X_{ni} - E(a_{ni}X_{ni})] + E(a_{ni}X_{ni}) + a_{ni}\widetilde{X}_{ni} \tag{3.2}$$

Then, by equation (3.2) and Lemma 2.2 with $a = b_n$ and $q > \max\{2, 2\gamma/\alpha\}$, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}(nb_n)^{-1}E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|-\varepsilon b_n\right)_+ \\
\leqslant\ & C\sum_{n=2}^{\infty}n^{-1}(b_n)^{-q}E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}[a_{ni}X_{ni}-E(a_{ni}X_{ni})]\right|^q\right) \\
&+\sum_{n=2}^{\infty}(nb_n)^{-1}E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}\widetilde{X}_{ni}\right|\right)+\sum_{n=2}^{\infty}(nb_n)^{-1}\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}E(a_{ni}X_{ni})\right|\right) \\
=:\ & I_1+I_2+I_3
\end{aligned}$$

First, for I_3, for $0<\alpha<1, 0<\gamma<1$, then, $1>\max\{\alpha,\gamma\}$. Then, by equation (1.2) and Lemma 2.5, we have,

$$\begin{aligned}
I_3 &\leqslant \sum_{n=2}^{\infty}(nb_n)^{-1}\sum_{i=1}^{n}E|a_{ni}X_{ni}| \\
&= \sum_{n=2}^{\infty}n^{-1}b_n^{-1}\sum_{i=1}^{n}E|a_{ni}X_i|I(|a_{ni}X_i|\leqslant b_n) \\
&\leqslant H_X(\alpha,\gamma) \\
&< \infty
\end{aligned}$$

where $H_X(\alpha,\gamma)$ is defined as equation (2.2). For $0<\alpha<1$, $\gamma\geqslant 1$, then $\alpha<\gamma$ and $\sum_{i=1}^{n}|a_{ni}|^\gamma \leqslant n$. By Lemma 2.3 and $E(X_n)=0, n\geqslant 1$,

$$\begin{aligned}
I_3 &\leqslant \sum_{n=2}^{\infty}(nb_n)^{-1}\sum_{i=1}^{n}E|a_{ni}X_i|I(|a_{ni}X_i|\leqslant b_n) \\
&\leqslant \sum_{n=2}^{\infty}(nb_n)^{-1}\sum_{i=1}^{n}E|a_{ni}X_i|I(|a_{ni}X_i|> b_n) \\
&\leqslant \sum_{n=2}^{\infty}n^{-1}b_n^{-\gamma}\sum_{i=1}^{n}E|a_{ni}X_i|^\gamma I(|a_{ni}X_i|> b_n) \\
&\leqslant \sum_{n=2}^{\infty}n^{-1}b_n^{-\gamma}\sum_{i=1}^{n}E|a_{ni}X|^\gamma I(|a_{ni}X|> b_n) \\
&= \sum_{n=2}^{\infty}n^{-1-\gamma/\alpha}(\log n)^{-1}\sum_{i=1}^{n}E|a_{ni}X|^\gamma I(|a_{ni}X|> b_n) \\
&\leqslant \sum_{n=2}^{\infty}n^{-\gamma/\alpha}(\log n)^{-1}E|X|^\gamma \\
&\leqslant \sum_{n=2}^{\infty}n^{-1}(\log n)^{-\gamma/\alpha}E|X|^\gamma \\
&\leqslant CE|X|^\gamma
\end{aligned}$$

$$< \infty$$

For $1 \leqslant \alpha \leqslant 2$, noting $E(X_n) = 0$ for all $n \geqslant 1$, by equation (1.2) and Lemma 2.4, we get that

$$\begin{aligned}
I_3 &= \sum_{n=2}^{\infty} (nb_n)^{-1} \left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} E(a_{ni}\widetilde{X}_{ni}) \right| \right) \\
&\leqslant \sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X_i|^{\alpha} I(|a_{ni}X_i| > b_n) \\
&\leqslant H_X(\alpha, \gamma) \\
&< \infty.
\end{aligned}$$

For I_2, it can be easily get that

$$I_2 \leqslant \sum_{n=2}^{\infty} (nb_n)^{-1} \sum_{i=1}^{n} E|a_{ni}X_i| I(|a_{ni}X_i| > b_n)$$

Similarly to the proof of $I_3 < \infty$, we can get that $I_2 < \infty$ for $0 < \alpha \leqslant 2$.

By Lemma 2.1, we can get that

$$\begin{aligned}
I_1 &= C \sum_{n=2}^{\infty} n^{-1} (b_n)^{-q} E \left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} [a_{ni}X_{ni} - E(a_{ni}X_{ni})] \right|^q \right) \\
&\leqslant C \sum_{n=2}^{\infty} n^{-1} (b_n)^{-q} \sum_{i=1}^{n} E|a_{ni}X_{ni}|^q + C \sum_{n=2}^{\infty} n^{-1} (b_n)^{-q} \left(\sum_{i=1}^{n} E(a_{ni}X_{ni})^2 \right)^{q/2} \\
&=: I_{11} + I_{12}
\end{aligned}$$

By Lemma 2.5, we can get $I_{11} \leqslant H_X(\alpha, \gamma) < \infty$. For I_{12}, if $\alpha \neq \gamma$, by $\alpha \leqslant 2$, $\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leqslant n$ and $q > 2\gamma/\alpha$,

$$\begin{aligned}
I_{12} &\leqslant C \sum_{n=2}^{\infty} n^{-1} \left(b_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X_i|^{\alpha} I(|a_{ni}X_i| \leqslant b_n) \right)^{q/2} \\
&\leqslant C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-\alpha q/2\gamma} (E|X|^{\alpha})^{q/2} < \infty
\end{aligned}$$

If $\alpha = \gamma$, let $q = 2$, by Lemma 2.5, we get that,

$$\begin{aligned}
I_{12} &= C \sum_{n=2}^{\infty} n^{-1} (b_n)^{-2} \sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \leqslant b_n) \\
&\leqslant CE|X|^{\alpha} \log(1 + |X|) < \infty
\end{aligned}$$

Therefore, the proof is completed. $\square$

4. Conclusions

The main result study the complete moment convergence for the weighted sums of ρ^*-mixing sequences which are stochastically dominated by a random variable X. Sung (2013) and Wu et al. (2014) studied the complete convergence for the identically distributed random variables. So, our results improve the corresponding ones of Sung (2013) and Wu et al. (2014) under more general conditions.

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