PROBABILITY INEQUALITIES FOR SUMS OF WUOD RANDOM VARIABLES AND THEIR APPLICATIONS

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Abstract. Let n be a positive integer, $X_1,...,X_n$ be real-valued random variables and $S_n = \sum_{i=1}^n X_i$. When $X_1,...,X_n$ are widely upper orthant dependent, some inequalities for the tail probability of S_n have been given. The obtained results extend some existing results. As applications, the complete convergence of WOD random variables has been investigated.

1. Introduction

Let n be a positive integer, $X_1,...,X_n$ be real-valued random variables (r.v.s) with distributions $F_1,...,F_n$, respectively. Denote the partial sum by $S_n = \sum_{i=1}^n X_i$. The partial sums are important objects in the probability theory, which can be applied to random walks, risk theory, queueing theory and so on. For example, in a renewal risk model, let the claim sizes, $\zeta_n, n \geqslant 1$, form a sequence of nonnegative r.v.s and the inter-arrival times, $\eta_n, n \geqslant 1$, constitute another sequence of r.v.s. Let $c > 0$ be the premium income rate and $x \geqslant 0$ be the initial capital. If set $X_n = \zeta_n - c\eta_n, n \geqslant 1$, then the ruin probability in infinite time

$$\psi(x) = P\left(\sup_{n\geqslant 1}\sum_{i=1}^n(\zeta_i - c\eta_i) > x\right)$$
$$= P\left(\sup_{n\geqslant 1}S_n > x\right),$$

(see, e.g. Section 1.1 of Embrechts et al. (1997)). When $X_1,...,X_n$ are independent identically distributed (i.i.d) r.v.s with $EX_1 = 0$ and $\mathrm{Var}X_1 = 1$, Nagaev (1965) gave an estimate for large deviation probability of S_n. Fuk and Nagaev (1971) extended and improved Nagaev's results to the case of independent non-identically distributed r.v.s. The above results all considers the independent r.v.s. This paper will investigate the dependent r.v.s and mainly consider the widely upper orthant dependent non-identically distributed r.v.s. The widely upper orthant dependent structure is introduced by Wang et al. (2013) when they investigated a dependent risk model.

DEFINITION 1.1. *For the r.v.s* $\{\xi_i, \ i \geqslant 1\}$, *if there exists a finite real sequence* $\{g_U(m), \ m \geqslant 1\}$ *satisfying for each integer* $m \geqslant 1$ *and for all* $x_i \in (-\infty, \infty)$, $1 \leqslant i \leqslant m$,

$$P\left(\bigcap_{i=1}^{m}\{\xi_i > x_i\}\right) \leqslant g_U(m)\prod_{i=1}^{m}P(\xi_i > x_i) \tag{1.1}$$

then we say that the r.v.s $\{\xi_i, \ i \geqslant 1\}$ *are widely upper orthant dependent (WUOD) with dominating coefficients* $g_U(m)$, $m \geqslant 1$; *if there exists a finite real sequence* $\{g_L(m), \ m \geqslant 1\}$ *satisfying for each integer* $m \geqslant 1$ *and for all* $x_i \in (-\infty, \infty)$, $1 \leqslant i \leqslant m$,

$$P\left(\bigcap_{i=1}^{m}\{\xi_i \leqslant x_i\}\right) \leqslant g_L(m)\prod_{i=1}^{m}P(\xi_i \leqslant x_i) \tag{1.2}$$

then we say that the r.v.s $\{\xi_i, \ i \geqslant 1\}$ *are widely lower orthant dependent (WLOD) with dominating coefficients* $g_L(m)$, $m \geqslant 1$; *if they are both WUOD and WLOD, then we say that the r.v.s* $\{\xi_i, \ i \geqslant 1\}$ *are widely orthant dependent (WOD).*

For the r.v.s $\xi_1, ..., \xi_n$, if for each $1 \leqslant m \leqslant n$, $\xi_1, ..., \xi_m$ satisfy (1.1), then we say that $\xi_1, ..., \xi_n$ are WUOD with dominating coefficients $g_U(m)$, $1 \leqslant m \leqslant n$; if for each $1 \leqslant m \leqslant n$, $\xi_1, ..., \xi_m$ satisfy (1.2), then we say that $\xi_1, ..., \xi_n$ are WLOD with dominating coefficients $g_L(m)$, $1 \leqslant m \leqslant n$.

Recall that when $g_U(m) = g_L(m) = 1$ for any $n \geqslant 1$ in Eqs. (1.1) and (1.2), the r.v.s $\{\xi_i, \ i \geqslant 1\}$ are called negatively upper orthant dependent (NUOD) and negatively lower orthant dependent (NLOD), respectively, and say that r.v.s $\{\xi_i, \ i \geqslant 1\}$ are negatively orthant dependent (NOD) if $\{\xi_i, \ i \geqslant 1\}$ are both NUOD and NLOD (see, Ebrahimi and Ghosh (1981) and Block et al. (1982)). Say that the r.v.s $\{\xi_i, \ i \geqslant 1\}$ are pairwise negatively quadrant dependent (NQD) or pairwise NOD, if for all positive integers $i \neq j$, the r.v.s ξ_i and ξ_j are NOD (see, Lehmann (1966)). If both Eqs. (1.1) and (1.2) hold when $g_U(m) = g_L(m) = M$ for some positive constant M and for all $m \geqslant 1$, the r.v.s $\{\xi_i, \ i \geqslant 1\}$ are called extended negatively ortnant dependent (ENOD)(see, e.g. Liu (2009) and Chen et al. (2010)). The widely orthant dependent structure has been used in risk models, regression models, limiting theory and so on. One can refer to Yang et al. (2013), Wang, Cui et al. (2012), Wang, Yang et al. (2012), Qiu and Chen (2014), Wang et al. (2014), Wang and Hu (2015), Huang et al. (2016), Li et al. (2016) , Gao et al. (2018), Xia et al. (2018), Yang et al. (2018) and so on.

Wang et al. (2013) gave the following property of the widely orthant dependent random variables.

PROPOSITION 1.1. (1) *Let* $\{\xi_i, \ i \geqslant 1\}$ *be WUOD r.v.s with dominating coefficients* $g_U(m), m \geqslant 1$. *If* $\{f_i(\cdot), \ i \geqslant 1\}$ *are nondecreasing, then* $\{f_i(\xi_i), \ i \geqslant 1\}$ *are still WUOD r.v.s with dominating coefficients* $g_U(m), m \geqslant 1$.

(2) *If the r.v.s* $\{\xi_i, \ i \geqslant 1\}$ *are nonnegative and WUOD with dominating coefficients* $g_U(m)$, $m \geqslant 1$, *then for each integer* $m \geqslant 1$,

$$E\prod_{i=1}^{m}\xi_i \leqslant g_U(m)\prod_{i=1}^{m}E\xi_i. \tag{1.3}$$

In particular, if the r.v.s $\{\xi_i, i \geqslant 1\}$ are WUOD with dominating coefficients $g_U(m)$, $m \geqslant 1$, then for each integer $m \geqslant 1$ and any $s > 0$,

$$\mathrm{E}\exp\left\{s\sum_{i=1}^{m}\xi_i\right\} \leqslant g_U(m)\prod_{i=1}^{m}\mathrm{E}\exp\{s\xi_i\}. \tag{1.4}$$

2. Main results

For the probability inequalities for S_n of dependent r.v.s, Asadian et al. (2006) considered the NOD r.v.s and obtained some Rosenthal's type inequalities. Shen (2011) investigated the ENOD r.v.s and gave some probability inequalities and their application. This paper will consider the r.v.s with a widely orthant dependent structure.

In the following, we assume that $X_1,...,X_n$ are WUOD r.v.s with dominating coefficients $g_U(m), 1 \leqslant m \leqslant n$. $y_1,...,y_n$ are n positive numbers and $y = \max\{y_1,\cdots,y_n\}$. Let $Y = (y_1,\cdots, y_n)$ and $c_n = \sum_{i=1}^{n}\overline{F}_i(y_i)$. Now we give some inequalities for the tail probability of S_n.

THEOREM 2.1. *Let $0 < t \leqslant 1$.*
(1)For any $x \geqslant 0$ and $h > 0$,

$$P(S_n > x) \leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)\mathrm{P}_0, \tag{2.1}$$

where

$$\mathrm{P}_0 = \exp\left\{\frac{e^{hy} - 1}{y^t}M(t;0,Y) - hx + (e^{hy} - 1)c_n\right\} \tag{2.2}$$

and

$$M(t;0,Y) = \sum_{i=1}^{n} \int_{0}^{y_i} |u|^t \mathrm{d}F_i(u).$$

(2)For any $x \geqslant 0$,

$$P(S_n > x) \leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)\mathrm{P}_1, \tag{2.3}$$

where

$$\mathrm{P}_1 = \exp\left\{\frac{x}{y} - \frac{x}{y}\log\left(\frac{xy^{t-1}}{M(t;0,Y)} + 1\right) + \frac{c_n xy^{t-1}}{M(t;0,Y)}\right\}. \tag{2.4}$$

If $xy^{t-1} > M(t;0,Y) + c_n y^t$, then

$$P(S_n > x) \leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)\mathrm{P}_2, \tag{2.5}$$

where

$$P_2 = \exp\left\{ \left(\frac{xy^{t-1}}{M(t;0,Y)+c_n y^t} - 1 \right) \left(\frac{M(t;0,Y)}{y^t} + c_n \right) \right.$$
$$\left. - \frac{x}{y} \log\left(\frac{xy^{t-1}}{M(t;0,Y)+c_n y^t} \right) \right\} \tag{2.6}$$

and $P_2 \leqslant P_1$.

REMARK 2.1. *For the ENOD r.v.s, Theorem 2.1 of Shen (2011) obtained the following result:*

Assume that $X_1,...,X_n$ *are ENOD random variables with some constant* $M > 0$. *Let* $0 < t \leqslant 1$. *Then for any* $x \geqslant 0$ *and* $z > 0$,

$$P(S_n > x) \leqslant \sum_{i=1}^n P(X_i > z) + M\exp\left\{ \frac{x}{z} - \frac{x}{z}\log\left(1 + \frac{xz^{t-1}}{M_{t,n}} \right) \right\},$$

where

$$M_{t,n} = \sum_{i=1}^n E|X_i|^t.$$

We note that the above result can be obtained from Theorem 2.1(1). In fact, taking $y_i = z, i = 1,2,...,n$, *we get that*

$$P(S_n > x)$$
$$\leqslant \sum_{i=1}^n P(X_i > z) + g_U(n)e^{-hx}\exp\left\{ \sum_{i=1}^n \left(\frac{e^{hz}-1}{z^t}\int_0^z |u|^t \mathrm{d}F_i(u) + (e^{hz}-1)\overline{F}_i(z) \right) \right\}$$
$$= \sum_{i=1}^n P(X_i > z) + g_U(n)e^{-hx}\prod_{i=1}^n \exp\left\{ \frac{e^{hz}-1}{z^t}\int_0^z |u|^t \mathrm{d}F_i(u) + (e^{hz}-1)\int_z^\infty \mathrm{d}F_i(u) \right\}$$
$$\leqslant \sum_{i=1}^n P(X_i > z) + g_U(n)e^{-hx}\prod_{i=1}^n \exp\left\{ \frac{e^{hz}-1}{z^t}\int_0^z |u|^t \mathrm{d}F_i(u) + (e^{hz}-1)\int_z^\infty \frac{u^t}{z^t}\mathrm{d}F_i(u) \right\}$$
$$= \sum_{i=1}^n P(X_i > z) + g_U(n)e^{-hx}\prod_{i=1}^n \exp\left\{ \frac{e^{hz}-1}{z^t}\left(\int_0^z |u|^t \mathrm{d}F_i(u) + \int_z^\infty u^t \mathrm{d}F_i(u) \right) \right\}$$
$$\leqslant \sum_{i=1}^n P(X_i > z) + g_U(n)e^{-hx}\prod_{i=1}^n \exp\left\{ \frac{e^{hz}-1}{z^t}E|X_i|^t \right\}$$
$$= \sum_{i=1}^n P(X_i > z) + g_U(n)\exp\left\{ \frac{e^{hz}-1}{z^t}\sum_{i=1}^n E|X_i|^t - hx \right\}.$$

Now taking $h = \frac{1}{z}\log\left(1 + \frac{xz^{t-1}}{M_{t,n}} \right)$. *Then we have that*

$$P(S_n > x) \leqslant \sum_{i=1}^n P(X_i > z) + g_U(n)\exp\left\{ \frac{x}{z} - \frac{x}{z}\log\left(1 + \frac{xz^{t-1}}{M_{t,n}} \right) \right\}.$$

Since $X_1,...,X_n$ are ENOD, then $g_U(m) = g_L(m) = M, m \geqslant 1$. Thus,

$$P(S_n > x) \leqslant \sum_{i=1}^{n} P(X_i > z) + M\exp\left\{ \frac{x}{z} - \frac{x}{z}\log\left(1 + \frac{xz^{t-1}}{M_{t,n}}\right) \right\}.$$

THEOREM 2.2. *Let $1 < t \leqslant 2$. Then for any $x \geqslant 0$,*

$$P(S_n > x) \leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)P_3, \tag{2.7}$$

where

$$P_3 = \exp\left\{ \frac{x}{y} - \left(\frac{x - \mathscr{S}(-Y,Y)}{y} + \frac{M(t;-Y,Y)}{y^t} \right) \log\left(\frac{xy^{t-1}}{M(t;-Y,Y)} + 1 \right) \right.$$
$$\left. + \frac{c_n xy^{t-1}}{M(t;-Y,Y)} \right\}, \tag{2.8}$$

$$\mathscr{S}(-Y,Y) = \sum_{i=1}^{n} \int_{-y_i}^{y_i} u\,dF_i(u) \quad and \quad M(t;-Y,Y) = \sum_{i=1}^{n} \int_{-y_i}^{y_i} |u|^t\,dF_i(u).$$

THEOREM 2.3. *Let $t > 2$, $0 < \alpha < 1$ and $\beta = 1 - \alpha$. If*
$\max\left\{ t, \log\left(\frac{\beta xy^{t-1}}{M(t;0,Y)} + 1 \right) \right\} \leqslant \frac{\alpha xy}{e^t M(2;-Y,Y)}$, *then for any $x \geqslant 0$,*

$$P(S_n > x) \leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)P_4 \tag{2.9}$$

and

$$P(S_n > x) \leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)P_5, \tag{2.10}$$

where

$$P_4 = \exp\left\{ \frac{\beta x}{y} + \left(\left(\frac{\alpha}{2} - 1 \right)\frac{x}{y} + \frac{\mathscr{S}(-Y,Y)}{y} - \frac{M(t;0,Y)}{y^t} \right) \log\left(\frac{\beta xy^{t-1}}{M(t;0,Y)} + 1 \right) \right.$$
$$\left. + \frac{c_n \beta xy^{t-1}}{M(t;0,Y)} \right\} \tag{2.11}$$

and

$$P_5 = \exp\left\{ \frac{t}{y}\left(\mathscr{S}(-Y,Y) - \left(1 - \frac{\alpha}{2} \right)x - \frac{M(t;0,Y)}{y^{t-1}} \right) \right.$$
$$\left. + \left(\frac{M(t;0,Y)}{y^t} + c_n \right)(e^t - 1) \right\}. \tag{2.12}$$

If $\max\left\{t, \log\left(\frac{\beta xy^{t-1}}{M(t;0,Y)}+1\right)\right\} > \frac{\alpha xy}{e^t M(2;-Y,Y)}$, *then for any* $x \geqslant 0$,

$$P(S_n > x) \leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)P_6, \tag{2.13}$$

where

$$P_6 = \exp\left\{\frac{-\frac{1}{2}\alpha^2 x^2 + \alpha x \mathscr{S}(-Y,Y)}{e^t M(2;-Y,Y)} + \max\left\{\frac{c_n \beta xy^{t-1}}{M(t;0,Y)}, c_n(e^t-1)\right\}\right\}. \tag{2.14}$$

3. Proofs of main results

Before giving the proofs of the theorems, we let $\widehat{X}_i = \min\{X_i, y_i\}, 1 \leqslant i \leqslant n$. By Proposition 1.1(1), we know $\widehat{X}_i$ $(1 \leqslant i \leqslant n)$ are still WUOD with dominating coefficients $g_U(m), 1 \leqslant m \leqslant n$. Set $\widehat{S}_n = \sum_{i=1}^{n} \widehat{X}_i$. Therefore, for any $x \geqslant 0$,

$$P(S_n > x) = P\left(S_n > x, \bigcap_{i=1}^{n}\{X_i \leqslant y_i\}\right) + P\left(S_n > x, \bigcup_{i=1}^{n}\{X_i > y_i\}\right)$$

$$\leqslant P\left(\widehat{S}_n > x\right) + \sum_{i=1}^{n} P(X_i > y_i). \tag{3.1}$$

By Proposition 1.1(2), for any positive number h,

$$P\left(\widehat{S}_n > x\right) \leqslant e^{-hx} E \prod_{i=1}^{n} e^{h\widehat{X}_i}$$

$$\leqslant g_U(n)e^{-hx} \prod_{i=1}^{n} E e^{h\widehat{X}_i}.$$

From this and (3.1), it follows that

$$P(S_n > x) \leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)e^{-hx} \prod_{i=1}^{n} E e^{h\widehat{X}_i}. \tag{3.2}$$

In the following, we use (3.2) to prove Theorems 2.1–2.3.

Proof of Theorem 2.1.

(1) Suppose $0 < t \leqslant 1$. It is easy to verify that the functions $\frac{e^{hu}-1}{u}$ and $\frac{e^{hu}-1}{u^t}$ are nondecreasing for $u > 0$. So, we obtain

$$Ee^{h\widehat{X}_i} \leqslant \int_{-\infty}^{0} dF_i(u) + \int_{0}^{y_i} e^{hu} dF_i(u) + \int_{y_i}^{\infty} e^{hy_i} dF_i(u)$$

$$= 1 + \int_{0}^{y_i} (e^{hu}-1)dF_i(u) + \int_{y_i}^{\infty} (e^{hy_i}-1)dF_i(u)$$

$$= 1 + \int_{0}^{y_i} \frac{e^{hu}-1}{u^t} u^t dF_i(u) + (e^{hy_i}-1)\overline{F}_i(y_i)$$

$$\leqslant 1 + \frac{e^{hy}-1}{y^t} \int_{0}^{y_i} u^t dF_i(u) + (e^{hy}-1)\overline{F}_i(y_i).$$

Hence, by (3.2), we can get

$$
\begin{aligned}
P(S_n > x) &\leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)e^{-hx}\prod_{i=1}^{n}\exp\left\{\frac{e^{hy}-1}{y^t}\int_0^{y_i}u^t\,dF_i(u) + (e^{hy}-1)\overline{F}_i(y_i)\right\} \\
&= \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)e^{-hx}\exp\left\{\frac{e^{hy}-1}{y^t}\sum_{i=1}^{n}\int_0^{y_i}u^t\,dF_i(u) + (e^{hy}-1)c_n\right\} \\
&= \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)\exp\left\{\frac{e^{hy}-1}{y^t}M(t;0,Y) - hx + (e^{hy}-1)c_n\right\}, \quad (3.3)
\end{aligned}
$$

where at the first step we use an inequality $s+1 \leqslant e^s$ for all s. This shows that (2.1) holds.

(2) Now we prove (2.3) and (2.5). We set $h = h_1 = \frac{1}{y}\log\left(\frac{xy^{t-1}}{M(t;0,Y)}+1\right)$ and substitute it into (3.3), it holds that

$$
\begin{aligned}
P(S_n > x) &\leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)\exp\left\{\frac{x}{y} - \frac{x}{y}\log\left(\frac{xy^{t-1}}{M(t;0,Y)}+1\right) + \frac{c_n xy^{t-1}}{M(t;0,Y)}\right\} \\
&= \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)P_1,
\end{aligned}
$$

where P_1 is given by (2.4). Therefore, the inequality (2.3) is proved.

Now we prove (2.5). If we set

$$
Q(h) = \frac{e^{hy}-1}{y^t}M(t;0,Y) - hx + (e^{hy}-1)c_n,
$$

then it is easy to verify that the function $Q(h)$ attains a minimum value at

$$
h = h_2 = \frac{1}{y}\log\left(\frac{xy^{t-1}}{M(t;0,Y)+c_n y^t}\right),
$$

and the minimum value is

$$
Q(h_2) = \left(\frac{xy^{t-1}}{M(t;0,Y)+c_n y^t}-1\right)\left(\frac{M(t;0,Y)}{y^t}+c_n\right) - \frac{x}{y}\log\left(\frac{xy^{t-1}}{M(t;0,Y)+c_n y^t}\right),
$$

which combining with (3.3) yields that

$$
\begin{aligned}
P(S_n > x) &\leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)\exp\{Q(h_2)\} \\
&= \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)P_2,
\end{aligned}
$$

where P_2 is given by (2.6). Therefore, the inequality (2.5) is proved.

In addition, we can easily get $P_2 = \exp\{Q(h_2)\} \leqslant \exp\{Q(h_1)\} = P_1$, which holds because of $Q(h_2) \leqslant Q(h_1)$. This completes the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. Suppose $1 < t \leqslant 2$. From the monotonicity of $\frac{e^{hu}-1-hu}{u^2}$ for $u \leqslant y$ and $\frac{e^{hu}-1-hu}{u^t}$ for $u > 0$, we get

$$\mathrm{E}e^{h\widehat{X}_i} \leqslant \int_{-\infty}^{-y_i} \mathrm{d}F_i(u) + \int_{-y_i}^{y_i} e^{hu}\mathrm{d}F_i(u) + \int_{y_i}^{\infty} e^{hy_i}\mathrm{d}F_i(u)$$

$$= 1 + \int_{|u|\leqslant y_i} (e^{hu}-1)\mathrm{d}F_i(u) + \int_{y_i}^{\infty} (e^{hy_i}-1)\mathrm{d}F_i(u)$$

$$\leqslant 1 + \int_{|u|\leqslant y_i} hu\mathrm{d}F_i(u) + \int_{|u|\leqslant y_i} \frac{e^{hu}-1-hu}{u^2}u^2\mathrm{d}F_i(u) + (e^{hy}-1)\overline{F}_i(y_i)$$

$$\leqslant 1 + h\int_{|u|\leqslant y_i} u\mathrm{d}F_i(u) + \frac{e^{hy}-1-hy}{y^2}\int_{|u|\leqslant y_i} |u|^2\mathrm{d}F_i(u) + (e^{hy}-1)\overline{F}_i(y_i)$$

$$= 1 + h\int_{|u|\leqslant y_i} u\mathrm{d}F_i(u) + \left(e^{hy}-1-hy\right)\int_{|u|\leqslant y_i} \frac{|u|^2}{y^2}\mathrm{d}F_i(u) + (e^{hy}-1)\overline{F}_i(y_i)$$

$$\leqslant 1 + h\int_{|u|\leqslant y_i} u\mathrm{d}F_i(u) + \left(e^{hy}-1-hy\right)\int_{|u|\leqslant y_i} \frac{|u|^t}{y^t}\mathrm{d}F_i(u) + (e^{hy}-1)\overline{F}_i(y_i)$$

$$= 1 + h\int_{|u|\leqslant y_i} u\mathrm{d}F_i(u) + \frac{e^{hy}-1-hy}{y^t}\int_{|u|\leqslant y_i} |u|^t\mathrm{d}F_i(u) + (e^{hy}-1)\overline{F}_i(y_i),$$

where at the last step we use $\left(\frac{|u|}{y}\right)^2 \leqslant \left(\frac{|u|}{y}\right)^t$ for any $|u| \leqslant y_i$ and $1 < t \leqslant 2$. Thus, by (3.2) it holds that

$$\mathrm{P}(S_n > x) \leqslant \sum_{i=1}^{n} \mathrm{P}(X_i > y_i) + g_U(n)e^{-hx}\prod_{i=1}^{n}\exp\left\{h\int_{|u|\leqslant y_i} u\mathrm{d}F_i(u)\right.$$

$$\left. + \frac{e^{hy}-1-hy}{y^t}\int_{|u|\leqslant y_i} |u|^t\mathrm{d}F_i(u) + (e^{hy}-1)\overline{F}_i(y_i)\right\}$$

$$= \sum_{i=1}^{n} \mathrm{P}(X_i > y_i) + g_U(n)\exp\left\{-hx + h\mathscr{S}(-Y,Y) + \frac{e^{hy}-1-hy}{y^t}M(t;-Y,Y)\right.$$

$$\left. + (e^{hy}-1)c_n\right\}, \tag{3.4}$$

where the inequality $s+1 \leqslant e^s$ for all s has been used at the first step. Set

$$h = \frac{1}{y}\log\left(\frac{xy^{t-1}}{M(t;-Y,Y)}+1\right)$$

and substitute it into (3.4), it holds that

$$\mathrm{P}(S_n > x) \leqslant \sum_{i=1}^{n} \mathrm{P}(X_i > y_i) + g_U(n)\exp\left\{\frac{x}{y} - \log\left(\frac{xy^{t-1}}{M(t;-Y,Y)}+1\right)\right.$$

$$\left. \cdot\left(\frac{M(t;-Y,Y)}{y^t} + \frac{x-\mathscr{S}(-Y,Y)}{y}\right) + \frac{c_nxy^{t-1}}{M(t;-Y,Y)}\right\}$$

$$= \sum_{i=1}^{n} \mathrm{P}(X_i > y_i) + g_U(n)\mathrm{P}_3,$$

where P_3 is given by (2.8). This completes the proof of Theorem 2.2.   □

Proof of Theorem 2.3.

Suppose that $t > 2$. We first consider the case where $hy \leqslant t$, i.e. $hy_i \leqslant t$ for $i = 1, ..., n$. For any fixed $0 < \theta < 1$, it holds that

$$
\begin{aligned}
Ee^{h\widehat{X}_i} &\leqslant 1 + \int_{|u| \leqslant y_i} (e^{hu} - 1) dF_i(u) + \int_{y_i}^{\infty} (e^{hy_i} - 1) dF_i(u) \\
&\leqslant 1 + \int_{|u| \leqslant y_i} (e^{hu} - 1) dF_i(u) + (e^{hy} - 1) \overline{F}_i(y_i) \\
&= 1 + \int_{|u| \leqslant y_i} (hu + \frac{1}{2} h^2 u^2 e^{h\theta u}) dF_i(u) + (e^{hy} - 1) \overline{F}_i(y_i) \\
&= 1 + h \int_{|u| \leqslant y_i} u dF_i(u) + \frac{1}{2} h^2 \int_{|u| \leqslant y_i} u^2 e^{h\theta u} dF_i(u) + (e^{hy} - 1) \overline{F}_i(y_i) \\
&\leqslant 1 + h \int_{|u| \leqslant y_i} u dF_i(u) + \frac{1}{2} e^t h^2 \int_{|u| \leqslant y_i} u^2 dF_i(u) + (e^{hy} - 1) \overline{F}_i(y_i), \quad (3.5)
\end{aligned}
$$

where at the third step we use the Taylor's formula (with Lagrange's form of the remainder) of $e^{hu} - 1$, i.e.

$$
e^{hu} - 1 = f(u) = f(0) + f'(0)u + \frac{f''(\theta u)}{2!} u^2, u \in (-\infty, \infty).
$$

When $hy > t$, there exists an i such that $1 \leqslant i \leqslant n$ and $hy_i > t$. We now use the monotonicity of $\frac{e^{hu} - 1 - hu}{u^t}$ for $u \geqslant \frac{t}{h}$ to estimate $Ee^{h\widehat{X}_i}$. For any fixed $0 < \theta < 1$,

$$
\begin{aligned}
Ee^{h\widehat{X}_i} &\leqslant 1 + \int_{|u| \leqslant y_i} (e^{hu} - 1) dF_i(u) + (e^{hy} - 1) \overline{F}_i(y_i) \\
&= 1 + \int_{-y_i}^{\frac{t}{h}} (hu + \frac{1}{2} h^2 u^2 e^{h\theta u}) dF_i(u) + \int_{\frac{t}{h}}^{y_i} (e^{hu} - 1) dF_i(u) + (e^{hy} - 1) \overline{F}_i(y_i) \\
&= 1 + \int_{-y_i}^{\frac{t}{h}} (hu + \frac{1}{2} h^2 u^2 e^{h\theta u}) dF_i(u) + \int_{\frac{t}{h}}^{y_i} hu dF_i(u) \\
&\quad + \int_{\frac{t}{h}}^{y_i} \frac{e^{hu} - 1 - hu}{u^t} u^t dF_i(u) + (e^{hy} - 1) \overline{F}_i(y_i) \\
&\leqslant 1 + h \int_{|u| \leqslant y_i} u dF_i(u) + \frac{1}{2} e^t h^2 \int_{|u| \leqslant y_i} u^2 dF_i(u) \\
&\quad + \frac{e^{hy} - 1 - hy}{y^t} \int_0^{y_i} u^t dF_i(u) + (e^{hy} - 1) \overline{F}_i(y_i). \quad (3.6)
\end{aligned}
$$

It is clearly that the right-hand side of (3.5) is less than or equal to the right-hand side of (3.6) for any positive number h. Therefore, we can get that the inequality (3.6) holds for all $h > 0$.

By (3.2) and (3.6), it holds that for any $x \geqslant 0$ and $h > 0$,

$$
\begin{aligned}
P(S_n > x) \leqslant{} & \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) e^{-hx} \prod_{i=1}^{n} \exp\Bigg\{ h \int_{|u| \leqslant y_i} u \, dF_i(u) \\
& + \frac{1}{2} e^t h^2 \int_{|u| \leqslant y_i} u^2 \, dF_i(u) + \frac{e^{hy} - 1 - hy}{y^t} \int_0^{y_i} u^t \, dF_i(u) + (e^{hy} - 1) \overline{F}_i(y_i) \Bigg\} \\
={} & \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) \exp\Bigg\{ -hx + h \mathscr{S}(-Y,Y) + \frac{1}{2} e^t h^2 M(2; -Y, Y) \\
& + \frac{e^{hy} - 1 - hy}{y^t} M(t; 0, Y) + (e^{hy} - 1) c_n \Bigg\}.
\end{aligned}
\tag{3.7}
$$

Let $g_1(h) = \frac{1}{2} e^t h^2 M(2; -Y, Y) - \alpha h x$ and $g_2(h) = \frac{e^{hy}-1-hy}{y^t} M(t; 0, Y) - \beta h x$, where $0 < \alpha < 1$ and $\beta = 1 - \alpha$. Thus, (3.7) can be written as

$$
\begin{aligned}
P(S_n > x) \leqslant{} & \sum_{i=1}^{n} P(X_i > y_i) \\
& + g_U(n) \exp\Big\{ g_1(h) + g_2(h) + h \mathscr{S}(-Y, Y) + (e^{hy} - 1) c_n \Big\}.
\end{aligned}
\tag{3.8}
$$

Let $h_3 = \frac{\alpha x}{e^t M(2; -Y, Y)}$ and $h_4 = \max\left\{ \frac{t}{y}, \frac{1}{y} \log\left(\frac{\beta x y^{t-1}}{M(t; 0, Y)} + 1 \right) \right\}$. If $h_4 \leqslant h_3$ then letting $h = h_5 = \frac{1}{y} \log\left(\frac{\beta x y^{t-1}}{M(t; 0, Y)} + 1 \right)$ in (3.8), it holds that for any $x \geqslant 0$,

$$
\begin{aligned}
P(S_n > x) \leqslant{} & \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) \exp\Bigg\{ h_5\left(\frac{1}{2} e^t h_5 M(2; -Y, Y) - x \right) \\
& + \frac{e^{h_5 y} - 1 - h_5 y}{y^t} M(t; 0, Y) + h_5 \mathscr{S}(-Y, Y) + c_n(e^{h_5 y} - 1) \Bigg\} \\
\leqslant{} & \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) \exp\Bigg\{ h_5\left(\frac{1}{2} e^t h_3 M(2; -Y, Y) - x \right) + \frac{\beta x}{y} \\
& - \frac{M(t; 0, Y)}{y^{t-1}} h_5 + h_5 \mathscr{S}(-Y, Y) + c_n(e^{h_5 y} - 1) \Bigg\} \\
={} & \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) \exp\Bigg\{ \frac{\beta x}{y} + h_5\left(\frac{\alpha x}{2} - x - \frac{M(t; 0, Y)}{y^{t-1}} \right) \\
& + \mathscr{S}(-Y, Y) \Bigg) + c_n(e^{h_5 y} - 1) \Bigg\} \\
={} & \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) \exp\Bigg\{ \frac{\beta x}{y} + \left(\left(\frac{1}{2} \alpha - 1 \right) \frac{x}{y} + \frac{\mathscr{S}(-Y, Y)}{y} \right. \\
& - \frac{M(t; 0, Y)}{y^t} \Bigg) \log\left(\frac{\beta x y^{t-1}}{M(t; 0, Y)} + 1 \right) + \frac{c_n \beta x y^{t-1}}{M(t; 0, Y)} \Bigg\} \\
={} & \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) P_4,
\end{aligned}
$$

where P_4 is given by (2.11). Therefore, the inequality (2.9) is proved. Now we prove (2.10). Let $h_6 = \frac{t}{y}$. Then $h_6 \leqslant h_4 \leqslant h_3$. Therefore, for any $0 < \alpha < 1$ and $x \geqslant 0$,

$$\frac{1}{2}e^t h_6^2 M(2; -Y, Y) = \frac{1}{2}h_6^2 \frac{\alpha x}{h_3} \leqslant \frac{1}{2}\alpha x h_6.$$

Hence, letting $h = h_6$ in (3.7), it holds that for any $x \geqslant 0$,

$$
\begin{aligned}
P(S_n > x) &\leqslant \sum_{i=1}^n P(X_i > y_i) + g_U(n)\exp\Big\{ -h_6 x + h_6 \mathscr{S}(-Y, Y) + \frac{1}{2}\alpha x h_6 \\
&\quad + \frac{e^{h_6 y} - 1 - h_6 y}{y^t} M(t; 0, Y) + c_n(e^{h_6 y} - 1) \Big\} \\
&= \sum_{i=1}^n P(X_i > y_i) + g_U(n)\exp\Big\{ \frac{t}{y}\Big(\mathscr{S}(-Y, Y) - \Big(1 - \frac{\alpha}{2}\Big)x - \frac{M(t; 0, Y)}{y^{t-1}} \Big) \\
&\quad + \Big(\frac{M(t; 0, Y)}{y^t} + c_n \Big)(e^t - 1) \Big\} \\
&= \sum_{i=1}^n P(X_i > y_i) + g_U(n)P_5,
\end{aligned}
$$

where P_5 is given by (2.12). Therefore, the inequality (2.10) is proved.

If $h_4 > h_3$, we prove (2.13) for two cases: $h_4 > h_3 \geqslant \frac{t}{y}$ and $h_4 \geqslant \frac{t}{y} > h_3$. We consider the first case $h_4 > h_3 \geqslant \frac{t}{y}$. For the functions $g_1(h)$ and $g_2(h)$, we can easily get that $g_1(h)$ and $g_2(h)$ are convex functions, $g_1(h)$ and $g_2(h)$ attain the minimum values at $h = h_3$ and $h = h_4$, respectively, and $g_2(h_3) < 0$. Therefore, set $h = h_3$ in (3.8), it holds that

$$
\begin{aligned}
P(S_n > x) &\leqslant \sum_{i=1}^n P(X_i > y_i) + g_U(n)\exp\Big\{ -\frac{\alpha^2 x^2}{2e^t M(2; -Y, Y)} + g_2(h_3) + \frac{\alpha x \mathscr{S}(-Y, Y)}{e^t M(2; -Y, Y)} \\
&\quad + c_n(e^{h_4 y} - 1) \Big\} \\
&\leqslant \sum_{i=1}^n P(X_i > y_i) + g_U(n)\exp\Big\{ \frac{-\frac{1}{2}\alpha^2 x^2 + \alpha x \mathscr{S}(-Y, Y)}{e^t M(2; -Y, Y)} + \frac{c_n \beta x y^{t-1}}{M(t; 0, Y)} \Big\} \quad (3.9)
\end{aligned}
$$

Now we consider the second case $h_4 \geqslant \frac{t}{y} > h_3$. Set $h = h_3$, then $hy < t$. Combin-

ing (3.2) and (3.5), we obtain for any $x \geqslant 0$,

$$
P(S_n > x)
$$

$$
\leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n)e^{-h_3 x} \prod_{i=1}^{n} \exp\left\{ h_3 \int_{|u| \leqslant y_i} u \, dF_i(u) + \frac{1}{2} e^t h_3^2 \int_{|u| \leqslant y_i} u^2 \, dF_i(u) \right.
$$

$$
\left. + (e^{h_3 y} - 1) \overline{F}_i(y_i) \right\}
$$

$$
= \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) \exp\left\{ -h_3 x + h_3 \mathscr{S}(-Y,Y) + \frac{1}{2} e^t h_3^2 M(2; -Y, Y) \right.
$$

$$
\left. + c_n(e^{h_3 y} - 1) \right\}
$$

$$
= \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) \exp\left\{ \frac{1}{2} e^t h_3^2 M(2; -Y, Y) - \alpha h_3 x - \beta h_3 x + h_3 \mathscr{S}(-Y,Y) \right.
$$

$$
\left. + c_n(e^{h_3 y} - 1) \right\}
$$

$$
= \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) \exp\left\{ \frac{-\frac{1}{2}\alpha^2 x^2 + \alpha x \mathscr{S}(-Y,Y)}{e^t M(2; -Y, Y)} - \beta h_3 x + c_n(e^{h_3 y} - 1) \right\}
$$

$$
\leqslant \sum_{i=1}^{n} P(X_i > y_i) + g_U(n) \exp\left\{ \frac{-\frac{1}{2}\alpha^2 x^2 + \alpha x \mathscr{S}(-Y,Y)}{e^t M(2; -Y, Y)} + c_n(e^t - 1) \right\}. \qquad (3.10)
$$

By (3.9) and (3.10), we know that (2.13) holds. This completes the proof of Theorem 2.3. □

4. Application of main results in complete convergence

In Section 2, we have given some probability inequalities for sums of WUOD random variables. In this section, we will give an application of main results in complete convergence. For the complete convergence, Kruglov et al. (2006) obtained the following complete convergence theorem for array of rowwise independent random variables $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$, where $\{k_n, n \geqslant 1\}$ be a sequence of positive integers.

THEOREM 4.1. *Let $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ be an array of rowwise independent random variables with $EX_{ni} = 0$ for all $1 \leqslant i \leqslant k_n, n \geqslant 1$ and $\{b_n, n \geqslant 1\}$ be a sequence of positive constant. suppose the following conditions hold:*
(i) $\sum_{n=1}^{\infty} b_n \sum_{i=1}^{k_n} P\{|X_{ni}| > \varepsilon\} < \infty$ for all $\varepsilon > 0$,
(ii) there exist $j > 0$ and $p \geqslant 1$ such that:

$$
\sum_{n=1}^{\infty} b_n \left(E \left| \sum_{i=1}^{k_n} X_{ni} \right|^p \right)^j < \infty.
$$

Then

$$\sum_{n=1}^{\infty} b_n P\left\{ \max_{1 \leqslant k \leqslant k_n} \left| \sum_{i=1}^{k} X_{ni} \right| > \varepsilon \right\} < \infty$$

for any $\varepsilon > 0$.

Qiu et al. (2011) generalized the result of Kruglov et al. (2006) from independent random variables to the case of negatively dependent random variables. Recently, Shen et al. (2017) extended the result of Qiu et al. (2011) from negatively dependent random variables to the case of ENOD random variables. For more details about the complete convergence, one can refer to Gut (1992), Yan (2018), Cheng et al. (2002) and so on.

Now we consider the arrays of rowwise WOD random variables. In the following, we assume that $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ are an array of rowwise WOD random variables, i.e. for every $n \geqslant 1$, $\{X_{ni}, 1 \leqslant i \leqslant k_n\}$ is a sequence of WOD random variables.

THEOREM 4.2. *Let $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ be an array of rowwise WOD random variables and $\{b_n, n \geqslant 1\}$ be a sequence of nonnegative constants. Suppose that the following conditions hold:*
(i) $\sum_{n=1}^{\infty} b_n \sum_{i=1}^{k_n} P\{|X_{ni}| > \varepsilon\} < \infty$ for all $\varepsilon > 0$,
(ii) there exist constant $j \geqslant 1$, $0 < p \leqslant 2$ and a constant sequence $\{d_n, n \geqslant 1\}$ such that:

$$\sum_{n=1}^{\infty} b_n g(k_n) \left(\sum_{i=1}^{k_n} E|X_{ni}|^p \right)^j e^{nd_n} < \infty. \tag{4.1}$$

Then

$$\sum_{n=1}^{\infty} b_n P\left(\left| \sum_{i=1}^{k_n} X_{ni} \right| > \varepsilon \right) < \infty \tag{4.2}$$

for all $\varepsilon > 0$.

Proof. To prove Theorem 4.2, we first give the following two inequalities by Theorems 2.1 and 2.2.

(1) For (2.3) of Theorem 2.1, replacing X_i by $-X_i$, we have

$$P(-S_n > x) \leqslant \sum_{i=1}^{n} P(-X_i > y_i) + 2g(n)P_1,$$

where P_1 is given by (2.4). Therefore, when $0 < t \leqslant 1$, for any $x \geqslant 0$, we obtain

$$P(|S_n| > x) \leqslant \sum_{i=1}^{n} P(|X_i| > y_i) + 2g(n)P_1. \tag{4.3}$$

(2) For (2.7) of Theorem 2.2, replacing X_i by $-X_i$, we have

$$P(-S_n > x) \leqslant \sum_{i=1}^{n} P(-X_i > y_i) + 2g(n)P_3,$$

where P_3 is given by (2.8). Therefore, when $1 < t \leqslant 2$, for any $x \geqslant 0$, we obtain

$$P(|S_n| > x) \leqslant \sum_{i=1}^{n} P(|X_i| > y_i) + 2g(n)P_3. \tag{4.4}$$

Now we prove Theorem 4.2 for two cases $0 < p \leqslant 1$ and $1 < p \leqslant 2$. When $0 < p \leqslant 1$, for $\varepsilon > 0$, by (4.3) for $x = \varepsilon$, $t = p$, and $y_i = \frac{\varepsilon}{j}, i = 1, 2 \ldots n$, it holds that

$$
P\left(\left| \sum_{i=1}^{k_n} X_{ni} \right| > \varepsilon \right)
$$

$$
\leqslant \sum_{i=1}^{k_n} P\left(|X_{ni}| > \frac{\varepsilon}{j} \right) + 2g(k_n) \exp\left\{ j - j \log\left(1 + \frac{\varepsilon^p / j^{p-1}}{\sum_{i=1}^{n} \int_0^{\varepsilon/j} |u|^p dF_i(u)} \right) \right.
$$

$$
\left. + \frac{c_n \varepsilon^p / j^{p-1}}{\sum_{i=1}^{n} \int_0^{\varepsilon/j} |u|^p dF_i(u)} \right\}
$$

$$
\leqslant \sum_{i=1}^{k_n} P\left(|X_{ni}| > \frac{\varepsilon}{j} \right) + 2g(k_n) e^j \left(\frac{\varepsilon^p / j^{p-1}}{\sum_{i=1}^{n} \int_0^{\varepsilon/j} |u|^p dF_i(u)} \right)^{-j} \cdot e^{\frac{c_n \varepsilon^p / j^{p-1}}{\sum_{i=1}^{n} \int_0^{\varepsilon/j} |u|^p dF_i(u)}}. \tag{4.5}
$$

Thus by (4.5), it holds that

$$
\sum_{n=1}^{\infty} b_n P\left(\left| \sum_{i=1}^{k_n} X_{ni} \right| > \varepsilon \right)
$$

$$
\leqslant \sum_{n=1}^{\infty} b_n \left\{ \sum_{i=1}^{k_n} P\left(|X_{ni}| > \frac{\varepsilon}{j} \right) + 2g(k_n) e^j \varepsilon^{-jp} j^{j(p-1)} \left(\sum_{i=1}^{n} \int_0^{y_i} |u|^p dF_i(u) \right)^j \right.
$$

$$
\left. \cdot e^{\frac{c_n \varepsilon^p / j^{p-1}}{\sum_{i=1}^{n} \int_0^{\varepsilon/j} |u|^p dF_i(u)}} \right\}
$$

$$
\leqslant \sum_{n=1}^{\infty} b_n \sum_{i=1}^{k_n} P\left(|X_{ni}| > \frac{\varepsilon}{j} \right) + 2e^j \varepsilon^{-jp} j^{j(p-1)} \sum_{n=1}^{\infty} b_n g(k_n) \left(\sum_{i=1}^{n} E|X_{ni}|^p \right)^j \cdot e^{\frac{c_n \varepsilon^p / j^{p-1}}{\int_0^{\varepsilon/j} |u|^p dF_i(u)}},
$$

which implies (4.2) holds by the conditions (i) and (ii).

When $1 < p \leqslant 2$, for $\varepsilon > 0$, by (4.4) for $x = \varepsilon$, $t = p$, and $y_i = \frac{\varepsilon}{j}, i = 1, 2 \ldots n$, it

holds that

$$P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right)$$

$$\leqslant \sum_{i=1}^{k_n} P\left(|X_{ni}| > \frac{\varepsilon}{j}\right) + 2g(k_n)\exp\left\{j + \left(\frac{\sum_{i=1}^{n}\int_{-\varepsilon/j}^{\varepsilon/j} u\,dF_i(u)}{\varepsilon/j} - \frac{\sum_{i=1}^{n}\int_{-\varepsilon/j}^{\varepsilon/j} |u|^p\,dF_i(u)}{(\varepsilon/j)^p} - j\right)\right.$$

$$\left. \cdot\log\left(1 + \frac{\varepsilon^p/j^{p-1}}{\sum_{i=1}^{n}\int_{-\varepsilon/j}^{\varepsilon/j}|u|^p\,dF_i(u)}\right) + \frac{c_n\varepsilon^p/j^{p-1}}{\sum_{i=1}^{n}\int_{-\varepsilon/j}^{\varepsilon/j}|u|^p\,dF_i(u)}\right\}$$

$$\leqslant \sum_{i=1}^{k_n} P\left(|X_{ni}| > \frac{\varepsilon}{j}\right) + 2g(k_n)\exp\left\{j + \left(\sum_{i=1}^{n}\int_{-\varepsilon/j}^{\varepsilon/j} dF_i(u) - j\right)\right.$$

$$\left. \cdot\log\left(1 + \frac{\varepsilon^p/j^{p-1}}{\sum_{i=1}^{n}\int_{-\varepsilon/j}^{\varepsilon/j}|u|^p\,dF_i(u)}\right) + \frac{c_n\varepsilon^p/j^{p-1}}{\sum_{i=1}^{n}\int_{-\varepsilon/j}^{\varepsilon/j}|u|^p\,dF_i(u)}\right\}$$

$$\leqslant \sum_{i=1}^{k_n} P\left(|X_{ni}| > \frac{\varepsilon}{j}\right) + 2g(k_n)e^j(\varepsilon^p/j^{p-1})^{n-j}\left(\sum_{i=1}^{n}\int_{-\varepsilon/j}^{\varepsilon/j}|u|^p\,dF_i(u)\right)^{j-n} \cdot e^{\frac{c_n\varepsilon^p/j^{p-1}}{\int_{-\varepsilon/j}^{\varepsilon/j}|u|^p\,dF_i(u)}}$$

$$\leqslant \sum_{i=1}^{k_n} P\left(|X_{ni}| > \frac{\varepsilon}{j}\right) + 2g(k_n)\left(\sum_{i=1}^{n} E|X_{ni}|^p\right)^j e^j(\varepsilon^p/j^{p-1})^{n-j} \cdot e^{\frac{c_n\varepsilon^p/j^{p-1}}{\int_{-\varepsilon/j}^{\varepsilon/j}|u|^p\,dF_i(u)}}. \tag{4.6}$$

Thus by (4.6), it holds that

$$\sum_{n=1}^{\infty} b_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right)$$

$$\leqslant \sum_{n=1}^{\infty} b_n\left\{\sum_{i=1}^{k_n} P(|X_{ni}| > \frac{\varepsilon}{j}) + 2g(k_n)\left(\sum_{i=1}^{n} E|X_{ni}|^p\right)^j e^j(\varepsilon^p/j^{p-1})^{n-j} \cdot e^{\frac{c_n\varepsilon^p/j^{p-1}}{\int_{-\varepsilon/j}^{\varepsilon/j}|u|^p\,dF_i(u)}}\right\}$$

$$\leqslant \sum_{n=1}^{\infty} b_n\sum_{i=1}^{k_n} P(|X_{ni}| > \frac{\varepsilon}{j}) + 2e^j\varepsilon^{-jp}j^{j(p-1)}\sum_{n=1}^{\infty} b_n g(k_n)\left(\sum_{i=1}^{n} E|X_{ni}|^p\right)^j \varepsilon^{np}$$

$$\cdot e^{\frac{c_n\varepsilon^p/j^{p-1}}{\int_{-\varepsilon/j}^{\varepsilon/j}|u|^p\,dF_i(u)}},$$

which implies (4.2) holds by the conditions (i) and (ii). This complete the proof of Theorem 4.2.   □

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