Journal of Mathematical Inequalities
Volume 13, Number 1 (2019), 205–214 doi:10.7153/jmi-2019-13-15

SOME RESULTS RELATED TO BESSEL'S INEQUALITY IN INNER PRODUCT SPACES

SILVESTRU SEVER DRAGOMIR AND MOHAMED ELMURSI

(*Communicated by J. Pečarić*)

Abstract. Some inequalities related to the celebrated Bessel's inequality in inner product spaces are given. They complement the results obtained by Boas-Bellman, Bombieri, Selberg and Heilbronn in the middle of the 20^{th} century that have been applied for almost orthogonal series and in Number Theory.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\leqslant i\leqslant n}$ are orthonormal vectors in the inner product space H, i.e., $\langle e_i,e_j\rangle=\delta_{ij}$ for all $i,\ j\in\{1,\ldots,n\}$ where δ_{ij} is the Kronecker delta, then the following inequality is well known in the literature as *Bessel's inequality*:

$$\sum_{i=1}^{n}|\langle x,e_i\rangle|^2\leqslant\|x\|^2 \text{ for any } x\in H. \tag{1.1}$$

For other results related to Bessel's inequality, see [8] – [11] and Chapter XV in the book [13].

In 1941, R. P. Boas [2] and in 1944, independently, R. Bellman [1] proved the following generalization of Bessel's inequality (see also [13, p. 392]):

THEOREM 1. *If $x,\ y_1,\ldots,\ y_n$ are elements of an inner product space $(H;\langle\cdot,\cdot\rangle)$, then the following inequality holds*

$$\sum_{i=1}^{n}|\langle x,y_i\rangle|^2\leqslant\|x\|^2\left[\max_{1\leqslant i\leqslant n}\|y_i\|^2+\left(\sum_{1\leqslant i\neq j\leqslant n}\left|\langle y_i,y_j\rangle\right|^2\right)^{\frac{1}{2}}\right]. \tag{1.2}$$

Mathematics subject classification (2010): 46C05, 26D15.
Keywords and phrases: Inner product spaces, Bessel's inequality, Boas-Bellman, Bombieri, Selberg and Heilbronn inequalities.

Paper JMI-13-15

In [7] we pointed out the following Boas-Bellman type inequalities:

$$\sum_{i=1}^{n}|\langle x,y_i\rangle|^2 \leqslant \|x\| \max_{1\leqslant i\leqslant n}|\langle x,y_i\rangle| \left\{\sum_{i=1}^{n}\|y_i\|^2+\sum_{1\leqslant i\neq j\leqslant n}\left|\langle y_i,y_j\rangle\right|\right\}^{\frac{1}{2}}, \tag{1.3}$$

for any $x,\ y_1,\ldots,\ y_n$ vectors in the inner product space $(H;\langle\cdot,\cdot\rangle)$.

If we assume that $(e_i)_{1\leqslant i\leqslant n}$ is an orthonormal family in H, then by (1.3) we have

$$\sum_{i=1}^{n}|\langle x,e_i\rangle|^2 \leqslant \sqrt{n}\,\|x\| \max_{1\leqslant i\leqslant n}|\langle x,e_i\rangle|, \quad x\in H.$$

We also have, see [7]

$$\sum_{i=1}^{n}|\langle x,y_i\rangle|^2 \leqslant \|x\| \left(\sum_{i=1}^{n}|\langle x,y_i\rangle|^{2p}\right)^{\frac{1}{2p}} \tag{1.4}$$
$$\times\left\{\left(\sum_{i=1}^{n}\|y_i\|^{2q}\right)^{\frac{1}{q}}+(n-1)^{\frac{1}{p}}\left(\sum_{1\leqslant i\neq j\leqslant n}\left|\langle y_i,y_j\rangle\right|^q\right)^{\frac{1}{q}}\right\}^{\frac{1}{2}},$$

for any $x,\ y_1,\ldots,\ y_n\in H,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1.$

The above inequality (1.4) becomes, for an orthonormal family $(e_i)_{1\leqslant i\leqslant n}$,

$$\sum_{i=1}^{n}|\langle x,e_i\rangle|^2 \leqslant n^{\frac{1}{q}}\,\|x\| \left(\sum_{i=1}^{n}|\langle x,e_i\rangle|^{2p}\right)^{\frac{1}{2p}}, \quad x\in H.$$

Further, we recall [7] that

$$\sum_{i=1}^{n}|\langle x,y_i\rangle|^2 \leqslant \|x\|^2\left\{\max_{1\leqslant i\leqslant n}\|y_i\|^2+(n-1)\max_{1\leqslant i\neq j\leqslant n}\left|\langle y_i,y_j\rangle\right|\right\}, \tag{1.5}$$

for any $x,\ y_1,\ldots,\ y_n\in H$. It is obvious that (1.5) will give for orthonormal families the well known Bessel inequality.

In 1971, E. Bombieri [3] gave the following generalization of Bessel's inequality.

THEOREM 2. *If $x,\ y_1,\ldots,\ y_n$ are vectors in the inner product space $(H;(\cdot,\cdot))$, then the following inequality holds:*

$$\sum_{i=1}^{n}|\langle x,y_i\rangle|^2 \leqslant \|x\|^2 \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\left|\langle y_i,y_j\rangle\right|\right\}. \tag{1.6}$$

It is obvious that if $(y_i)_{1\leqslant i\leqslant n}$ are orthonormal, then from (1.6) one can deduce Bessel's inequality.

Another generalization of Bessel's inequality was obtained by A. Selberg (see for example [13, p. 394]):

THEOREM 3. *Let* $x,\ y_1,\dots,\ y_n$ *be vectors in* H *with* $y_i \neq 0\quad (i=1,\dots,n)$. *Then one has the inequality:*

$$\sum_{i=1}^{n} \frac{|\langle x,y_i\rangle|^2}{\sum_{j=1}^{n}\left|\left\langle y_i,y_j\right\rangle\right|} \leqslant \|x\|^2. \tag{1.7}$$

Another type of inequality related to Bessel's result, was discovered in 1958 by H. Heilbronn [12] (see also [13, p. 395]).

THEOREM 4. *With the assumptions in Theorem 2, one has*

$$\sum_{i=1}^{n} |\langle x,y_i\rangle| \leqslant \|x\| \left(\sum_{i,j=1}^{n}\left|\left\langle y_i,y_j\right\rangle\right|\right)^{\frac{1}{2}}. \tag{1.8}$$

In [8] the first author obtained the following Bombieri type inequalities

$$\sum_{i=1}^{n} |\langle x,y_i\rangle|^2 \leqslant \|x\| \max_{1\leqslant i\leqslant n} |\langle x,y_i\rangle| \left(\sum_{i,j=1}^{n}\left|\left\langle y_i,y_j\right\rangle\right|\right)^{\frac{1}{2}}, \tag{1.9}$$

$$\begin{aligned} &\sum_{i=1}^{n} |\langle x,y_i\rangle|^2 \\ &\leqslant \|x\| \max_{1\leqslant i\leqslant n} |\langle x,y_i\rangle|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\langle x,y_i\rangle|^r\right)^{\frac{1}{2r}} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|\left\langle y_i,y_j\right\rangle\right|\right)^{s}\right]^{\frac{1}{2s}}, \end{aligned} \tag{1.10}$$

where $\frac{1}{r}+\frac{1}{s}=1,\ s>1,$

$$\begin{aligned} &\sum_{i=1}^{n} |\langle x,y_i\rangle|^2 \\ &\leqslant \|x\| \max_{1\leqslant i\leqslant n} |\langle x,y_i\rangle|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\langle x,y_i\rangle|\right)^{\frac{1}{2}} \left[\max_{1\leqslant i\leqslant n}\left(\sum_{j=1}^{n}\left|\left\langle y_i,y_j\right\rangle\right|\right)\right], \end{aligned} \tag{1.11}$$

$$\begin{aligned} &\sum_{i=1}^{n} |\langle x,y_i\rangle|^2 \\ &\leqslant \|x\| \max_{1\leqslant i\leqslant n} |\langle x,y_i\rangle|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\langle x,y_i\rangle|^p\right)^{\frac{1}{2p}} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|\left\langle y_i,y_j\right\rangle\right|^q\right)^{\frac{1}{q}}\right]^{\frac{1}{2}}, \end{aligned} \tag{1.12}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$ and

$$\sum_{i=1}^{n} |\langle x,y_i\rangle|^2 \leqslant \|x\|^2 \left\{\sum_{i,j=1}^{n}\left|\left\langle y_i,y_j\right\rangle\right|^2\right\}^{\frac{1}{2}} \tag{1.13}$$

for any $x \in H$.

It has been shown that for different selection of vectors the upper bound provided by the inequality (1.13) is some time better other times worse than the one obtained by Bombieri above in (1.6).

In this paper we obtain some inequalities related to the celebrated Bessel's inequality in inner product spaces. They complement the results obtained by Boas-Bellman, Bombieri, Selberg and Heilbronn above, which have been applied for almost orthogonal series and in Number Theory.

2. The Main Results

The following generalization of Bessel's inequality may be stated.

THEOREM 5. *Let $x,\ y_j \in H$ for $j \in \{1,\ldots,n\}$, then*

$$\sum_{j=1}^{n} \left|\left\langle x,y_j\right\rangle\right|^2 \leqslant \frac{1}{2}\left(\|x\|^2+\sum_{j=1}^{n}\sum_{k=1}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\left\langle y_k,x\right\rangle\right). \tag{2.1}$$

Proof. For every scalars $\lambda_1,\ \lambda_2,\ldots,\ \lambda_n$ we have:

$$\begin{aligned}
0 &\leqslant \left\|x-\sum_{j=1}^{n}\lambda_j y_j\right\|^2 = \left\langle x-\sum_{j=1}^{n}\lambda_j y_j, x-\sum_{j=1}^{n}\lambda_j y_j\right\rangle \\
&= \langle x,x\rangle - \left\langle x,\sum_{j=1}^{n}\lambda_j y_j\right\rangle - \left\langle \sum_{j=1}^{n}\lambda_j y_j, x\right\rangle + \left\langle \sum_{j=1}^{n}\lambda_j y_j, \sum_{j=1}^{n}\lambda_j y_j\right\rangle \\
&= \|x\|^2 - \sum_{j=1}^{n}\overline{\lambda_j}\left\langle x,y_j\right\rangle - \sum_{j=1}^{n}\lambda_j\left\langle y_j,x\right\rangle + \sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_j\overline{\lambda_k}\left\langle y_j,y_k\right\rangle.
\end{aligned}$$

By choosing $\lambda_j = \left\langle x,y_j\right\rangle$ for any $1 \leqslant j \leqslant n$, we get

$$0 \leqslant \|x\|^2 - \sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2 - \sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2 + \sum_{j=1}^{n}\sum_{k=1}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\left\langle y_k,x\right\rangle,$$

which implies the desired inequality (2.1). □

REMARK 1 *If $\left\{y_j\right\}_{j=1,\ldots,n}$ is an orthonormal family in H, then we get from (2.1) that*

$$\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2 \leqslant \frac{1}{2}\left(\|x\|^2+\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2\right),$$

which is equivalent to Bessel's inequality (1.1). Also, if $n=1$ and we take $y_1 = y$, then we get from (2.1) that

$$|\langle x,y\rangle|^2 \leqslant \frac{1}{2}\left(\|x\|^2+|\langle x,y\rangle|^2\|y\|^2\right),$$

which is equivalent to

$$|\langle x,y\rangle|^2\left(2-\|y\|^2\right)\leqslant\|x\|^2 \tag{2.2}$$

and by taking $y=\frac{z}{\|z\|}$, $z\neq 0$, *we get the Schwarz inequality*

$$|\langle x,z\rangle|^2\leqslant\|x\|^2\|z\|^2,\ x,\ z\in H.$$

COROLLARY 1 *Let* $x,\ y_j\in H$ *for* $j\in\{1,\dots,n\}$, *then*

$$\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2\leqslant\frac{1}{2}\left(\|x\|^2+\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\left(\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|\right)^2\right). \tag{2.3}$$

Proof. From (2.1) we observe that $\sum_{k,j=1}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\langle y_k,x\rangle\geqslant 0$. This also can be proved directly by observing that

$$\begin{aligned}\sum_{k,j=1}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\langle y_k,x\rangle&=\sum_{k,j=1}^{n}\left\langle\left\langle x,y_j\right\rangle y_j,\overline{\langle y_k,x\rangle}y_k\right\rangle=\sum_{k,j=1}^{n}\left\langle\left\langle x,y_j\right\rangle y_j,\langle x,y_k\rangle y_k\right\rangle\\&=\left\langle\sum_{j=1}^{n}\left\langle x,y_j\right\rangle y_j,\sum_{k=1}^{n}\langle x,y_k\rangle y_k\right\rangle=\left\|\sum_{j=1}^{n}\left\langle x,y_j\right\rangle y_j\right\|^2.\end{aligned}$$

Therefore we have

$$\begin{aligned}\sum_{k,j=1}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\langle y_k,x\rangle&=\left|\sum_{k,j=1}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\langle y_k,x\rangle\right|\\&\leqslant\sum_{k,j=1}^{n}\left|\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\langle y_k,x\rangle\right|\\&=\sum_{k,j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|\left|\left\langle y_j,y_k\right\rangle\right|\left|\langle y_k,x\rangle\right|\\&\leqslant\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\sum_{i,j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|\left|\langle y_k,x\rangle\right|\\&=\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\left(\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|\right)^2.\end{aligned} \tag{2.4}$$

By utilising (2.1) we get the desired result (2.3). □

REMARK 2 *If the family* $\{y_j\}_{j=1,\dots,n}$ *is orthonormal, then* $\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|=1$ *and from (2.3) we get*

$$\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2\leqslant\frac{1}{2}\left[\|x\|^2+\left(\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|\right)^2\right]. \tag{2.5}$$

By Cauchy-Schwarz inequality we also have

$$\frac{1}{n}\left(\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|\right)^2 \leqslant \sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2. \tag{2.6}$$

Therefore (2.5) and (2.6) we have the double inequality for the orthonormal family $\{y_j\}_{j=1,\dots,n}$

$$\frac{1}{n}\left(\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|\right)^2 \leqslant \sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2 \leqslant \frac{1}{2}\left[\|x\|^2+\left(\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|\right)^2\right]. \tag{2.7}$$

We also observe that, if we use Heilbronn's inequality (1.8) then we get by (2.3) that

$$\begin{aligned}\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2 &\leqslant \frac{1}{2}\left(\|x\|^2+\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\left(\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|\right)^2\right)\\ &\leqslant \frac{1}{2}\left(\|x\|^2+\|x\|^2\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\sum_{i,j=1}^{n}\left|\left\langle y_i,y_j\right\rangle\right|\right),\end{aligned}$$

which gives

$$\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2 \leqslant \frac{1}{2}\|x\|^2\left(1+\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\sum_{i,j=1}^{n}\left|\left\langle y_i,y_j\right\rangle\right|\right) \tag{2.8}$$

for any $x\in H$.

COROLLARY 2 *Let* $x,\ y_j\in H$ *for* $j\in\{1,\dots,n\}$, *then*

$$\left(2-n\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\right)\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2 \leqslant \|x\|^2. \tag{2.9}$$

Proof. Then by (2.3) we get

$$\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2 \leqslant \frac{1}{2}\left(\|x\|^2+n\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2\right).$$

This inequality is equivalent to

$$\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2-\frac{1}{2}n\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2 \leqslant \frac{1}{2}\|x\|^2,$$

or to (2.9). □

REMARK 3 *This is an inequality of interest if the family of vectors* $\{y_j\}_{j=1,\dots,n}$ *satisfies the condition*

$$2-n\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\geqslant 1,$$

namely

$$\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\leqslant\frac{1}{n}. \tag{2.10}$$

In this situation from (2.9) we get

$$\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2\leqslant\left(2-n\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\right)\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2\leqslant\|x\|^2, \tag{2.11}$$

for any $x\in H$.

If the family of vectors $\{y_j\}_{j=1,\dots,n}$ *satisfies the condition*

$$2-n\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\geqslant 0,$$

namely

$$\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\leqslant\frac{2}{n}, \tag{2.12}$$

then we also have the meaningful inequality

$$0\leqslant\left(2-n\max_{j,k\in\{1,\dots,n\}}\left|\left\langle y_j,y_k\right\rangle\right|\right)\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2\leqslant\|x\|^2, \tag{2.13}$$

for any $x\in H$.

COROLLARY 3 *Let* $x,\ y_j\in H$ *for* $j\in\{1,\dots,n\}$, *then*

$$\begin{aligned}&\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2\left(\frac{1}{n}+\frac{1}{2n}\max_{k,\ell=1,\dots,n,k\neq\ell}\{|\langle y_\ell,y_k\rangle|\}-\frac{1}{2}\left\|y_j\right\|^2\right)\\&\leqslant\frac{1}{2}\left[\|x\|^2+\max_{k,\ell=1,\dots,n,k\neq\ell}\{|\langle y_\ell,y_k\rangle|\}\left(\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|\right)^2\right].\end{aligned} \tag{2.14}$$

Proof. We observe that

$$\begin{aligned}&\sum_{k,j=1}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\langle y_k,x\rangle\\&=\sum_{k=j=1}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\langle y_k,x\rangle+\sum_{k,j=1,k\neq j}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\langle y_k,x\rangle\\&=\sum_{j=1}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_j\right\rangle\left\langle y_j,x\right\rangle+\sum_{k,j=1,k\neq j}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\langle y_k,x\rangle\\&=\sum_{j=1}^{n}\left|\left\langle x,y_j\right\rangle\right|^2\left\|y_j\right\|^2+\sum_{k,j=1,k\neq j}^{n}\left\langle x,y_j\right\rangle\left\langle y_j,y_k\right\rangle\langle y_k,x\rangle.\end{aligned}$$

This implies that

$$
\begin{aligned}
&\sum_{k,j=1}^{n}\left\langle x,y_{j}\right\rangle\left\langle y_{j},y_{k}\right\rangle\left\langle y_{k},x\right\rangle \qquad (2.15)\\
&=\left|\sum_{k,j=1}^{n}\left\langle x,y_{j}\right\rangle\left\langle y_{j},y_{k}\right\rangle\left\langle y_{k},x\right\rangle\right|\\
&=\left|\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|^{2}\left\|y_{j}\right\|^{2}+\sum_{k,j=1,k\neq j}^{n}\left\langle x,y_{j}\right\rangle\left\langle y_{j},y_{k}\right\rangle\left\langle y_{k},x\right\rangle\right|\\
&\leqslant\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|^{2}\left\|y_{j}\right\|^{2}+\left|\sum_{k,j=1,k\neq j}^{n}\left\langle x,y_{j}\right\rangle\left\langle y_{j},y_{k}\right\rangle\left\langle y_{k},x\right\rangle\right|\\
&\leqslant\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|^{2}\left\|y_{j}\right\|^{2}+\sum_{k,j=1,k\neq j}^{n}\left|\left\langle x,y_{j}\right\rangle\left\langle y_{j},y_{k}\right\rangle\left\langle y_{k},x\right\rangle\right|.
\end{aligned}
$$

By using (2.1) we then get

$$
\begin{aligned}
\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|^{2}&\leqslant\frac{1}{2}\left(\|x\|^{2}+\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|^{2}\left\|y_{j}\right\|^{2}+\sum_{k,j=1,k\neq j}^{n}\left|\left\langle x,y_{j}\right\rangle\left\langle y_{j},y_{k}\right\rangle\left\langle y_{k},x\right\rangle\right|\right)\\
&\leqslant\frac{1}{2}\left(\|x\|^{2}+\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|^{2}\left\|y_{j}\right\|^{2}\right.\\
&\qquad\left.+\max_{k,\ell=1,\ldots,n,k\neq\ell}\left\{\left|\left\langle y_{\ell},y_{k}\right\rangle\right|\right\}\sum_{k,j=1,k\neq j}^{n}\left|\left\langle x,y_{j}\right\rangle\right|\left|\left\langle y_{k},x\right\rangle\right|\right)\\
&=\frac{1}{2}\left(\|x\|^{2}+\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|^{2}\left\|y_{j}\right\|^{2}\right.\\
&\qquad\left.+\max_{k,\ell=1,\ldots,n,k\neq\ell}\left\{\left|\left\langle y_{\ell},y_{k}\right\rangle\right|\right\}\left(\left(\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|\right)^{2}-\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|^{2}\right)\right)\\
&=\frac{1}{2}\left(\|x\|^{2}+\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|^{2}\left(\left\|y_{j}\right\|^{2}-\frac{1}{n}\max_{k,\ell=1,\ldots,n,k\neq\ell}\left\{\left|\left\langle y_{\ell},y_{k}\right\rangle\right|\right\}\right)\right.\\
&\qquad\left.+\max_{k,\ell=1,\ldots,n,k\neq\ell}\left\{\left|\left\langle y_{\ell},y_{k}\right\rangle\right|\right\}\left(\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|\right)^{2}\right),
\end{aligned}
$$

which produces the desired result (2.14). □

REMARK 4 *If the family of vectors* $\left\{y_{j}\right\}_{j=1,\ldots,n}$ *is orthogonal then* $\left\langle y_{\ell},y_{k}\right\rangle=0$ *for* $k,\ell=1,\ldots,n,k\neq\ell$ *and by (2.14) we have*

$$
\sum_{j=1}^{n}\left|\left\langle x,y_{j}\right\rangle\right|^{2}\left(\frac{2}{n}-\left\|y_{j}\right\|^{2}\right)\leqslant\|x\|^{2},\ x\in H. \qquad (2.16)
$$

This inequality is meaningful if $\|y_j\|^2 < \frac{2}{n}$ *for any* $j = 1,...,n$.

From a different perspective, we have by (2.4) that

$$\begin{aligned}
\sum_{k,j=1}^{n} \langle x,y_j\rangle \langle y_j,y_k\rangle \langle y_k,x\rangle &\leqslant \sum_{k,j=1}^{n} \left|\langle x,y_j\rangle \langle y_j,y_k\rangle \langle y_k,x\rangle\right| \\
&= \sum_{k,j=1}^{n} \left|\langle x,y_j\rangle\right| \left|\langle y_j,y_k\rangle\right| \left|\langle y_k,x\rangle\right| \\
&\leqslant \max_{k,j\in\{1,...,n\}} \left\{\left|\langle x,y_j\rangle\right| \left|\langle y_k,x\rangle\right|\right\} \sum_{k,j=1}^{n} \left|\langle y_j,y_k\rangle\right| \\
&= \max_{k\{1,...,n\}} \left\{|\langle x,y_k\rangle|^2\right\} \sum_{k,j=1}^{n} \left|\langle y_j,y_k\rangle\right|
\end{aligned}$$

for any $x \in H$.

By using the inequality (2.1) we can state the following corollary as well:

COROLLARY 4 *Let* $x,\ y_j \in H$ *for* $j \in \{1,...,n\}$, *then*

$$\begin{aligned}
\sum_{j=1}^{n} \left|\langle x,y_j\rangle\right|^2 &\leqslant \frac{1}{2}\left(\|x\|^2 + \max_{k\{1,...,n\}} \left\{|\langle x,y_k\rangle|^2\right\} \sum_{k,j=1}^{n} \left|\langle y_j,y_k\rangle\right|\right) \qquad (2.17) \\
&\leqslant \frac{1}{2}\|x\|^2 \left(1 + \max_{k\{1,...,n\}} \|y_k\|^2 \sum_{k,j=1}^{n} \left|\langle y_j,y_k\rangle\right|\right).
\end{aligned}$$

Acknowledgement. The authors would like to thank the anonymous referee for valuable suggestions that have been implemented in the final version of the paper.

REFERENCES

[1] R. BELLMAN, *Almost orthogonal series*, Bull. Amer. Math. Soc., **50** (1944), 517–519.

[2] R. P. BOAS, *A general moment problem*, Amer. J. Math., **63** (1941), 361–370.

[3] E. BOMBIERI, *A note on the large sieve*, Acta Arith., **18** (1971), 401–404.

[4] S. S. DRAGOMIR, *Discrete Inequalities of the Cauchy-Buniakowsky-Schwartz Type*, Nova Science Publishers, Inc., Hauppauge, NY, 2004. x+225 pp. ISBN: 1-59454-049-7.

[5] S. S. DRAGOMIR, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Nova Science Publishers, Inc., Hauppauge, NY, 2005. viii+249 pp. ISBN: 1-59454-202-3.

[6] S. S. DRAGOMIR, *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces*, Nova Science Publishers, Inc., New York, 2007. xii+243 pp. ISBN: 978-1-59454-903-8; 1-59454-903-6.

[7] S. S. DRAGOMIR, *On the Boas-Bellman inequality in inner product spaces*, Bull. Austral. Math. Soc., **69** (2004), no. 2, 217–225. Preprint *RGMIA Res. Rep. Coll.*, **6**(2003), Supplement, Article 14. [Online http://rgmia.org/papers/v6e/BBIIPS.pdf].

[8] S. S. DRAGOMIR, *On the Bombieri inequality in inner product spaces*, Libertas Math., **25** (2005), 13–26. Preprint RGMIA Res. Rep. Coll. 3 (2003), Art. 5 [Online http://rgmia.org/papers/v6n3/BIIPS.pdf]

[9] S. S. DRAGOMIR, *Some Bombieri type inequalities in inner product spaces*, J. Indones. Math. Soc., **10** (2004), no. 2, 91–98. Preprint *RGMIA Res. Rep. Coll.*, **6**(2003), Supplement, Article 16. [Online http://rgmia.org/papers/v6e/BTIIPS.pdf].

[10] S. S. DRAGOMIR AND B. MOND, *On the Boas-Bellman generalisation of Bessel's inequality in inner product spaces*, Italian J. of Pure & Appl. Math., **3** (1998), 29–35.
[11] S. S. DRAGOMIR, B. MOND AND J. E. PEČARIĆ, *Some remarks on Bessel's inequality in inner product spaces*, Studia Univ. Babeş-Bolyai, Mathematica, **37**(4) (1992), 77–86.
[12] H. HEILBRONN, *On the averages of some arithmetical functions of two variables*, Mathematica, **5**(1958), 1–7.
[13] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

(Received October 3, 2017)

Silvestru Sever Dragomir
Mathematics, College of Engineering & Science
Victoria University
PO Box 14428, Melbourne City, MC 8001, Australia
DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences
School of Computer Science & Applied Mathematics,
University of the Witwatersrand
Private Bag 3, Johannesburg 2050, South Africa
e-mail: sever.dragomir@vu.edu.au

Mohamed Elmursi
Mathematics & Computer Science Department
Faculty Of Science, Sohag University
Sohag City, PC 82524, Arab Republic of Egypt
e-mail: m_elmursi@science.sohag.edu.eg, *e-mail:* fedmursiex@gmail.com