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INEQUALITIES ARISING FROM GENERALIZED EULER–TYPE CONSTANTS MOTIVATED BY LIMIT SUMMABILITY OF FUNCTIONS

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Abstract. Limit summability of real functions was introduced by M.H. Hooshmand in 2001. In order to study derivation of the limit summand function, he has introduced a functional sequence corresponding to a given function f with $D_f \supseteq \mathbb{N}^*$ that is related to the Euler-type constants. In the way, we prove two main criteria for its convergence together with an extensive inequality between the limit summand function and the generalized Euler-type constants. The main inequality is also extended whenever f is a convex or concave function. Among other things, we obtain some inequalities for many special functions such as the gamma, digamma and zeta functions.

1. Introduction and preliminaries

One of the most famous and useful mathematical constants is Euler-Mascheroni constant, denoted by γ, which was introduced in 18_{th} century (see for example [2]). A type of generalized Euler constants was studied in [8]. On the other hand, in 1997, R.J. Webster [9] studied Γ-type functions which satisfy the functional equations $f(x+1) = g(x)f(x)$ ($x > 0$), and the Boher-Mollerup Theorem (see [1]) was generalized in the paper. However, in 2001 ([3]), M.H. Hooshmand introduced a new concept entitled limit summability of functions, and their summand functions for each function were defined on a subset of $\mathbb{R}$ or $\mathbb{C}$ containing all natural numbers, and he showed that Γ-type functions can be considered as a sub-topic thereof. Both in the paper and in [4], some related theorems such as the Bohr-Mollerup and a main theorem of [9] were generalized desirably, and some uniqueness conditions of the limit summand functions and their connections to the functional equations

$$\lambda(x) = f(x) + \lambda(x-1),$$

$$\phi(x) = f(x)\phi(x-1),$$

were studied. We also mention that in 2010, Muller and Schleicher ([7]) used a similar functional sequence for a type of fractional sums while they were not aware of the limit summability topic. Recently, analytic summability of functions is introduced and studied by the second author in [5].

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1.1. Limit summability of real and complex functions.

Let $\mathbb{N}^*$ denote the set of all positive integers and put $\mathbb{N} = \mathbb{N}^* \cup \{0\}$. In this paper, we assume that $f : D_f \to \mathbb{C}$, where $D_f \subseteq \mathbb{C}$ is the domain of f and in the real case, we replace $\mathbb{C}$ with $\mathbb{R}$. The summand set of D_f is defined by

$$\Sigma_f = \{x : x + \mathbb{N}^* \subseteq D_f\}.$$

Hence $x \in \Sigma_f$ if and only if $\{x+1, x+2, \cdots, x+n, \cdots\} \subseteq D_f$. If $\mathbb{N}^* \subseteq D_f$, the following functional sequences were considered in [3]

$$R_n(f,x) = R_n(x) = f(n) - f(x+n),$$
$$f_{\sigma_n}(x) \ = \ xf(n) + \sum_{k=1}^{n} R_k(x); \quad x \in \Sigma_f, \quad n \in \mathbb{N}^*.$$

Recall from [3] that a function f is limit summable at $x_0 \in \Sigma_f$ (resp. on $S \subseteq \Sigma_f$) if the functional sequence $f_{\sigma_n}(x)$ is convergent at x_0 (resp. on S). The function f is called uniformly summable on $S \subseteq \Sigma_f$ if $f_{\sigma_n}(x)$ is uniformly convergent on S. The limit function of $f_{\sigma_n}(x)$ (resp. $R_n(f,x)$) is denoted by $f_\sigma(x)$ or $\sigma(f(x))$ if $x \in D_f$ (resp. $R(f,x)$ or $R(x)$) and it is called the limit summand function of f. Note that the domain of f_σ is

$$D_{f_\sigma} = \{x \in \Sigma_f : f \text{ is limit summable at x}\}.$$

It is important to know that $\Sigma_f \cap D_f = \Sigma_f + 1 = \{x+1 : x \in \Sigma_f\}$, $f_\sigma(0) = 0$ (thus $0 \in D_{f_\sigma}$), and $f_\sigma(-1) = -f(0)$ if $0 \in D_f$. Also, $1 \in D_{f_\sigma}$ if and only if $R_n(1)$ is convergent, and if and only if $D_f \cap D_{f_\sigma} = D_{f_\sigma} + 1$. A necessary condition for limit summability of f at x is

$$\lim_{n \to \infty} \overline{R}_n(x) := \lim_{n \to \infty} (R_n(x) - xR_{n-1}(1)) = 0.$$

If $R(1) = 0$, then

$$f_\sigma(x) = f(x) + f_\sigma(x-1); \ x \in D_{f_\sigma} + 1, \tag{1}$$

$$f_\sigma(m) = \sum_{j=1}^{m} f(j); \ m \in \mathbb{N}^*. \tag{2}$$

It is proved that the following conditions are equivalent and every function satisfying one of them is called limit summable
a) $D_f \subseteq D_{f_\sigma}$, $R(1) = 0$;
b) $D_{f_\sigma} = \Sigma_f$, $D_f \subseteq D_f - 1$, $R(1) = 0$;
c) $f_\sigma(x) = f(x) + f_\sigma(x-1)$, for all $x \in D_f$.
Hence, if f is limit summable, then $\lambda = f_\sigma$ satisfies the difference functional equation

$$\lambda(x) = f(x) + \lambda(x-1); \ x \in D_f, \tag{3}$$

thus f_σ also is a solution of the difference functional equation $g(x+1) - g(x) = F(x)$, where $F(x) = f(x+1)$. If f is a real function with the domain $D_f = [1, +\infty)$, then $\Sigma_f = [0,+\infty)$, and by applying Theorem 3.3 of [4], we conclude that f is limit summable if and only if it is so on (0,1) and R(f,1)=0. One of the most important criteria for limit summability was introduced in [4] stating that convexity or concavity of f together with boundedness of $R_n(f,1)$ imply limit summability of f. The following theorem is one of the results there.

THEOREM A ([4]). *Suppose that $f:[1,+\infty)\to\mathbb{R}$ is a real function for which $R_n(1)$ is bounded. Then*
a) if f is convex or concave on $D_f=[1,+\infty)$ from a number on, then f is uniformly summable on every bounded subset of $\Sigma_f=[0,+\infty)$.
b) if f is convex on $[1,+\infty)$, then

$$f_\sigma(x) \geqslant (x+1)f(1) - f(x+1); \quad x\in[1,+\infty).$$

Another important criteria of [4] (Theorem 3.1) implies the next result for limit summability of a monotone function.

COROLLARY B ([4]). *Let $f:[1,+\infty)\to\mathbb{R}$ be a real function such that the sequence f_n is bounded. If f is monotonic on $[1,+\infty)$ from a number on, then f is absolutely and uniformly limit summable on every bounded subset of $[0,+\infty)$.*

2. Derivative of limit summand functions and its induced Euler-type constants.

In order to study the derivative of the summand function f_σ, Hooshmand faced the following functional sequence leading him to a generalization of the Euler-type constants. If f is differentiable on Σ_f, then we put

$$f_{\sigma'_n}(x) := (f_{\sigma_n}(x))' = f(n) - \sum_{k=1}^{n} f'(x+k),$$

and denote its limit function by $f_{\sigma'}(x)$ (if it exists). Since $\log_{\sigma'}(0)=-\gamma$, we also use $-\gamma_n(f,x)$, $-\gamma_n(f)$ for denoting $f_{\sigma'_n}(x)$, $f_{\sigma'_n}(0)$; respectively.

EXAMPLE 2.1. The real function $f(x)=-\log x$ has the convergent functional sequence $f_{\sigma'_n}(x)$ and we conclude that

$$\begin{aligned}\gamma(-\log,x) = \log_{\sigma'}(x) &= \lim_{n\to\infty}(\log n - \sum_{k=1}^{n}\frac{1}{k+x})\\ &= -\gamma + \sum_{k=1}^{\infty}\frac{x}{k(k+x)} = \psi(x+1) = \frac{1}{x}+\frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{x}+\psi(x),\end{aligned}$$

for all $x>0$, where ψ denotes the digamma function.

In the topic of limit summand functions, Hooshmand observed that

$$\frac{f_\sigma(m)}{m} = \frac{f(1)+f(2)+\cdots+f(m)}{m},$$

for every $m \in \mathbb{N}^*$ (if $R_n(f,1)$ is convergent), hence he introduced the "limit summand average" of f defined by

$$f_{\tilde{\sigma}}(x) = \begin{cases} \frac{1}{x} f_\sigma(x), & x \neq 0 \\ \lim\limits_{x\to 0} \dfrac{f_\sigma(x)}{x}, & x = 0. \end{cases}$$

Note that the domain of $f_{\tilde{\sigma}}$ is equal to D_{f_σ} or $D_{f_\sigma} \setminus \{0\}$. The following theorem not only gives a criterion for convergence of $\gamma_n(f,x)$, but also provides an important inequality among $\gamma(f,x)$, $f_{\tilde{\sigma}}(x)$ and $\gamma(f)$ for $x \in (0,1]$, which has many important applications.

THEOREM 2.2. *If $f : [1,+\infty) \to \mathbb{R}$ has monotonic derivative and $R(f,1) = 0$, then $f_{\sigma'_n}(x)$ is convergent on $[0,+\infty)$ and $f_{\sigma'}(x)$, $f_{\tilde{\sigma}}(x)$ satisfy the inequalities*

$$-\gamma(f,1) \leqslant f_{\sigma'}(x) \leqslant f_{\tilde{\sigma}}(x) \leqslant -\gamma(f);\ 0 < x \leqslant 1, \tag{4}$$

if f' is increasing (note that here $f'(1)$ is the same as $f'_+(1)$, and for the case f' is decreasing, the above inequalities should be reversed). Moreover, $f_{\sigma'}(x)$ is a solution of the functional equation

$$\chi(x) = f'(x) + \chi(x-1);\ \ x > 1. \tag{5}$$

Proof. Let f' be increasing (proof of decreasing case is similar). First note that the conditions imply $f'(x) \to 0$ as $x \to +\infty$ and f is decreasing on $[1,+\infty)$, hence $f(x) \leqslant f(1)$ on it. Then fix a $0 < x < 1$. For every $k \in \mathbb{N}^*$, $k < k+x < k+1$ and by applying the mean value theorem (M.V.T) on $[k,k+x]$, there is an $x_k \in (k,k+x)$ such that

$$-R_k(x) = f(k+x) - f(k) = f'(x_k)x.$$

Therefore,

$$f_{\sigma_n}(x) = xf(n) - \sum_{k=1}^{n} f'(x_k)x = x(f(n) - \sum_{k=1}^{n} f'(x_k)).$$

On the other hand, since f' is increasing on $[1,+\infty)$, for $k \in \mathbb{N}$

$$f'(k) \leqslant f'(x_k) \leqslant f'(x+k) \leqslant f'(k+1).$$

By using the equality and inequality above, we conclude that

$$-\gamma_n(f,1) \leqslant f_{\sigma'_n}(x) \leqslant f_{\tilde{\sigma}_n}(x) \leqslant -\gamma_n(f);\ \ n = 1,2,3,\ldots \tag{6}$$

Also, since the function f is convex on $[1,+\infty)$, then

$$\frac{f(n+h)-f(n)}{h} \leqslant f(n+1) - f(n) \leqslant \frac{f(n+1+h)-f(n+1)}{h},$$

where $0<h<1$, $n\in\mathbb{N}^*$ and the convexity is considered on $([n,n+h,n+1,n+1+h])$. By letting $h\to 0^+$, we will have

$$f'(n)\leqslant -R_n(1)\leqslant f'(n+1)$$

and for each $n\in\mathbb{N}^*$ we will obtain the inequalities

$$R_n(f',1)\leqslant -R_n(1)-f'(n+1)\leqslant 0. \tag{7}$$

Now, the identity

$$f_{\sigma'_n}(x)-f_{\sigma'_{n+1}}(x)=R_n(1)+f'(n+1+x)\,;\; x\geqslant 0 \tag{8}$$

together with (7) imply that

$$f_{\sigma'_{n+1}}(x)-f_{\sigma'_n}(x)\leqslant -R_n(1)-f'(n+1)\leqslant 0\,;\; x\geqslant 0,$$

which means that the sequence $f_{\sigma'_n}(x)=-\gamma_n(f,x)$ is decreasing, for all $x\geqslant 0$. Also, putting $x=0$ in (8) and by using (7), we obtain

$$R_n(f',1)\leqslant -\gamma_{n+1}(f)+\gamma_n(f)\leqslant 0 \tag{9}$$

Note that the inequalities of (6) and (9) with the identity

$$-\gamma_n(f,1)=f'(1)-f'(n+1)-\gamma_n(f);\quad n=1,2,3,...$$

imply the following inequalities.

$$\begin{aligned} f'(1)-f'(n+1)-\gamma_{n+1}(f)\leqslant f'(1)-f'(n+1)-\gamma_n(f)=-\gamma(f,1)\\ f'(1)+R_n(1)-\gamma_{n+1}(f)\leqslant f_{\sigma'_n}(x)\leqslant f_{\tilde{\sigma}_n}(x)\leqslant -\gamma_n(f)\,;\; 0<x\leqslant 1. \end{aligned} \tag{10}$$

Now, since $f_{\tilde{\sigma}_n}(x)=\frac{1}{x}f_{\sigma_n}(x)$ is convergent (Theorem 3.3 of ([4]), which means $f_{\tilde{\sigma}_n}(x)\to f_{\sigma}(x)$ as $n\to\infty$), then $f_{\sigma'_n}(x)$ becomes convergent to $f_{\sigma'}(x)$ for all $0<x\leqslant 1$. So

$$-\gamma(f,1)=f'(1)-\gamma(f)\leqslant f_{\sigma'}(x)\leqslant f_{\tilde{\sigma}}(x)\leqslant -\gamma(f)\,;\; 0<x\leqslant 1 \tag{11}$$

Now, according to the sequence $f_{\sigma'_n}(x)$ that is convergence on $(0,1]$, and by using identity

$$f_{\sigma'_n}(x)-f_{\sigma'_n}(x-1)=f'(x)-f'(x+n)$$

and also considering the property $f'(x)\to 0$ as $x\to+\infty$, we conclude that

$$f_{\sigma'}(x)=f'(x)+f_{\sigma'}(x-1);\quad 1<x\leqslant 2.$$

By continuation of the above identity, we obtain

$$f_{\sigma'}(x)=f'(x)+f'(x-1)+\cdots+f'(x-m+1)+f_{\sigma'}(x-m);\quad m<x\leqslant m+1. \tag{12}$$

Thus

$$f_{\sigma'}(x)=-\gamma(f,\{x\})+\sum_{j=0}^{[x]-1}f'(x-j), \tag{13}$$

so $f_{\sigma'_n}(x)$ is convergent on $(0,+\infty)$ and the equation 5 holds for $x>1$.
□

Note that one can extend the inequality 4 for all $x>0$, as follows.

COROLLARY 2.3. *If the conditions of Theorem 2.2 hold, then*

$$f'(1)-\gamma(f)+\sum_{j=0}^{[x]-1} f'(x-j) \leqslant f_{\sigma'}(\{x\})+\sum_{j=0}^{[x]-1} f'(x-j)$$

$$\leqslant f_{\tilde{\sigma}}(\{x\})+\sum_{j=0}^{[x]-1} f'(x-j) \leqslant -\gamma(f)+\sum_{j=0}^{[x]-1} f'(x-j); \quad x>0,$$

for the case f' is increasing.

REMARK 2.4. Note that the inequality 4 is also true for $x=0$ if and only if $f_{\tilde{\sigma}}(0)=f_{\sigma'}(0)$. Hence the next question arises.

QUESTION. Is it true that $f_{\tilde{\sigma}}(0)=f_{\sigma'}(0)$? If no, what are some sufficient (or necessary) conditions for the equality?

Now, we replace the condition $R_n(f,1)\to 0$ by the assumption that $R_n(f,1)$ is bounded and generalize Theorem 2.2 and its inequality as follows.

COROLLARY 2.5. *Assume that $f:[1,+\infty)\to\mathbb{R}$ is a function with monotonic derivative and $R_n(f,1)$ is bounded. Then*
a) $f_{\sigma_n'}(x)$ is convergent, for each $x\geqslant 0$, and $f_{\sigma'}(x)$ satisfies the following inequalities

$$f'(1)+R(1)-\gamma(f)\leqslant f_{\sigma'}(x)\leqslant f_{\tilde{\sigma}}(x)\leqslant -\gamma(f); \quad 0<x\leqslant 1, \tag{14}$$

if f' is increasing, and

$$-\gamma(f)\leqslant f_{\tilde{\sigma}}(x)\leqslant f_{\sigma'}(x)\leqslant f'(1)+R(1)-\gamma(f); \quad 0<x\leqslant 1, \tag{15}$$

if f' is decreasing.
b) $f_{\sigma'}(x)$ satisfies the following functional equation

$$\chi(x)=R(1)+f'(x)+\chi(x-1); \;\; x>1 \tag{16}$$

Proof. Since $R_n(f,1)$ is monotonic (because of convexity), $R_n(1)$ is convergent. By putting $g(x):=f(x)+R(1)x$ we get

$$g'(x)=f'(x)+R(1), \;\; g_{\sigma_n}(x)=f_{\sigma_n}(x).$$

Hence g (instead of f) satisfies the conditions of Theorem 2.2 and so

$$f'(1)+R(1)-\gamma(f)\leqslant f_{\sigma'}(x)\leqslant f_{\tilde{\sigma}}(x)\leqslant -\gamma(f); \quad 0<x\leqslant 1$$

if f' is increasing, and analogously for the decreasing case.
For the both cases, we have $f_{\sigma'}(x)=g'(x)+f_{\sigma'}(x-1)$ (since $g_{\sigma'}(x)=f_{\sigma'}(x)$). Thus

$$f_{\sigma'}(x)=f'(x)+f_{\sigma'}(x-1)+R(1); \quad x>1.$$

Note that if f' is decreasing, then $-f$ is increasing and all steps of Theorem 2.2 are confirmed. Hence the proof is complete. □

EXAMPLE 2.6. Consider the real function $f(x) = \tan^{-1} x$ (with domain $\mathbb{R}$). By applying Theorem A and 2.2, we conclude that f is limit summable on $\mathbb{R}$ (since f is concave), $f_{\sigma'_n}(x)$ is convergent, for all $x \geqslant 0$, and

$$\begin{aligned} f_{\sigma'}(x) &= \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{1}{1+(x+k)^2}, \\ -\gamma(f) &= \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{1}{k^2+1} = \frac{\pi}{2} - \frac{1}{2}(\pi\coth(\pi) - 1), \\ -\gamma(f,1) = f'(1) - \gamma(f) &= \frac{\pi+2}{2} - \frac{1}{2}\pi\coth(\pi), \\ f_{\tilde{\sigma}}(x) &= \frac{\pi}{2} + \frac{1}{x}\sum_{k=1}^{\infty}(\tan^{-1}(k) - \tan^{-1}(x+k)). \end{aligned}$$

Thus the inequalities 4 hold and so

$$\begin{aligned} \frac{1}{2}(\pi\coth(\pi) - 2) \leqslant \sum_{k=1}^{\infty} \frac{1}{1+(x+k)^2} &\leqslant \frac{1}{x}\sum_{k=1}^{\infty}(\tan^{-1}(x+k) - \tan^{-1}(k)) \\ &\leqslant \frac{1}{2}(\pi\coth(\pi) - 1)\,;\ 0 < x \leqslant 1. \end{aligned}$$

2.1. Connections and relations to some generalized Euler constants.

In [8], J. Sandor proved the existence of some type of Euler-type constants for a function $F : [1,+\infty) \to \mathbb{R}$ with certain properties. We show that Theorem 2.2 has relationships to one of his main theorem, and it is concluded from our result (for the special case $x = 0$). Let A_n and B_n be two sequences which corresponded to the integrable function $F : [1,+\infty) \to \mathbb{R}$ as follows

$$A_n = A_n(F) = \sum_{i=1}^{n} F(i) - \int_1^{n+1} F(x)dx,\ B_n = B_n(F) = \sum_{i=1}^{n+1} F(i) - \int_1^{n+1} F(x)dx \ \ (n \geqslant 1).$$

The common limit of these sequences is the well-known Euler-Mascheroni constant $\gamma = \gamma_f := \lim A_n = \lim B_n = 0.577215664\ldots$, if $F(x) = 1/x$.

THEOREM C (J. sandor [8]). *Let $F : [1,\infty) \to \mathbb{R}$ be a strictly positive and strictly decreasing and continuous function. Then (A_n) is a strictly increasing and convergent sequence. The sequence (B_n) is strictly decreasing and convergent, too. By assuming that*

$$\lim_{x\to\infty} F(x) = 0,$$

the two sequences will have the same limit.

We now want to investigate the relationship between the issues raised in the article and the above theorem. For this purpose, let $f : [1,\infty) \to \mathbb{R}$ and F be the initial function of f. Then a simple calculation shows that

$$B_{n-1}(f) = \gamma_n(F) + F(1) = \sum_{k=1}^{n} F'(k) - F(n) + F(1). \tag{17}$$

Thus, $\gamma_n(F)$ is converge if and only if $B_n(f)$ is so, and $B(f) = \gamma(F) + F(1)$. Now, we state the following claim that says the Sandor's theorem is a result of Theorem 2.2 (but the converse is not true, clearly).

CLAIM. *Theorem C is a result of Theorem* 2.2 *(for the special case* $x = 0$*).*

For proving the claim, suppose that $f : [1,\infty) \to \mathbb{R}$ satisfies the conditions of Theorem C (here f plays the role of F in the theorem). Hence f is continuous and it has primary function F (on $[1,+\infty)$). We claim that F satisfies the conditions of Theorem 2.2. For this order, it is enough to show that $R(F,1) = 0$. Since $f = F'$ is a decreasing function and according to the M.V.T, we conclude that

$$F(n) - F(n+1) = F'(c_n) = f(c_n), \quad \text{for some} \quad c_n \in (n.n+1).$$

Hence

$$\lim_{n\to\infty}(F(n) - F(n+1)) = \lim_{n\to\infty} f(c_n) = 0,$$

since $\lim_{t\to\infty} f(t) = 0$. Therefore, Theorem 2.2 implies that the functional sequence $\gamma_n(F,x)$ is convergent for all $x \geqslant 0$, in particular, $\gamma_n(F)$ is so. However, (17) together with $\lim_{t\to\infty} f(t) = 0$ guarantee that $\gamma_f = \gamma(F) + F(1)$.

Now, we extend some of the pervious results for convex and concave functions. By f'_+ (resp. f'_-) we mean the right (resp. left) derivative function of f, and when we use $f'_\pm$ in a relation it means that it is satisfied for both functions f'_+ and f'_-.
Now, if f is right/left differentiable on Σ_f (e.g., if f is convex or concave), then we put

$$f_{\sigma'_{n\pm}}(x) := (f_{\sigma_n}(x))'_\pm = f(n) - \sum_{k=1}^{n} f'_\pm(x+k),$$

and denote its limit function by $f_{\sigma'_\pm}(x)$ (if it exists). Followed by the previous notation, we use $-\gamma_{n\pm}(f,x)$, $-\gamma_{n\pm}(f)$ for denoting $f_{\sigma'_{n\pm}}(x)$, $f_{\sigma'_{n\pm}}(0)$ respectively. Note that if $f'_+ = f'_-$ (i.e., f is differentiable), then

$$f_{\sigma'_{n\pm}}(x) = f_{\sigma'_n}(x) = -\gamma_{n\pm}(f,x) = -\gamma_n(f,x),$$

and

$$f_{\sigma'_\pm}(x) = f_{\sigma'}(x) = -\gamma_\pm(f,x) = -\gamma(f,x),$$

if they are convergent.

THEOREM 2.7. *Let* $\delta < 1$ *and* $f : (\delta,+\infty) \to \mathbb{R}$ *be a convex function and* $R(f,1) = 0$. *Then*
a) $f_{\sigma'_{n\pm}}(x)$ *is convergent, for all* $x \geqslant 0$*, and the following inequalities hold*

$$-\gamma_+(f,1) \leqslant -\gamma_-(f,1) \leqslant f_{\sigma'_+}(x) \leqslant f_{\sigma'_-}(x) \leqslant f_{\tilde{\sigma}}(x) \leqslant -\gamma_+(f) \leqslant -\gamma_-(f); \quad 0 < x \leqslant 1. \tag{18}$$

b) The function $f_{\sigma'_\pm}(x)$ *is a solution of*

$$\chi(x) = f'_\pm(x) + \chi(x-1). \tag{19}$$

Proof. Fix $0<x<1$ and consider $k<k+x<k+1$, for all $k\in\mathbb{N}^*$. By applying the M.V.T for convex functions on $[k,k+x]$, there exists an $x_k\in(k,k+x)$ such that

$$f'_-(x_k)\leqslant \frac{f(x+k)-f(k)}{x}\leqslant f'_+(x_k).$$

Hence

$$-xf'_+(x_k)\leqslant f(k)-f(x+k)\leqslant -xf'_-(x_k).$$

Thus

$$f(n)-\sum_{k=1}^{n} f'_+(x_k)\leqslant \frac{f_{\sigma_n}(x)}{x}\leqslant f(n)-\sum_{k=1}^{n} f'_-(x_k). \tag{20}$$

Since f is convex, f'_+ , f'_- are increasing, $f'_-\leqslant f'_+$ and so

$$f'_-(k)\leqslant f'_+(k)\leqslant f'_-(x_k)\leqslant f'_+(x_k)\leqslant f'_-(x+k)\leqslant f'_+(x+k)\leqslant f'_-(k+1)\leqslant f'_+(k+1).$$

Therefore,

$$\begin{aligned}
f(n)-\sum_{k=1}^{n} f'_+(k+1)\leqslant f(n)-\sum_{k=1}^{n} f'_-(k+1)\leqslant f(n)-\sum_{k=1}^{n} f'_+(x+k)&\\
\leqslant f(n)-\sum_{k=1}^{n} f'_-(x+k)\leqslant f(n)-\sum_{k=1}^{n} f'_+(x_k)\leqslant f(n)-\sum_{k=1}^{n} f'_-(x_k)&\\
\leqslant f(n)-\sum_{k=1}^{n} f'_+(k)\leqslant f(n)-\sum_{k=1}^{n} f'_-(k).&
\end{aligned}$$

Now by using (20) and the above inequalities, we obtain

$$-\gamma_{n_+}(f,1)\leqslant -\gamma_{n_-}(f,1)\leqslant f_{\sigma'_{n_+}}(x)\leqslant f_{\sigma'_{n_-}}(x)\leqslant f_{\tilde{\sigma}_n}(x)\leqslant -\gamma_{n_+}(f)\leqslant -\gamma_{n_-}(f). \tag{21}$$

On the other hand, by applying the convexity of f on $[n,n+h,n+1,n+1+h]$, where $0<h<1$, we drive

$$\frac{f(n+h)-f(n)}{h}\leqslant f(n+1)-f(n)\leqslant \frac{f(n+1+h)-f(n+1)}{h}.$$

Then by letting $h\longrightarrow 0^+$, we get

$$f'_+(n)\leqslant -R_n(1)\leqslant f'_+(n+1). \tag{22}$$

In a similar way, if $-1<h<0$, then by using the convexity on $[n+h,n,n+1+h,n+1]$, we conclude that

$$f'_-(n)\leqslant -R_n(1)\leqslant f'_-(n+1). \tag{23}$$

Thereafter, by using (22) and (23), we obtain

$$f'_-(n)\leqslant f'_+(n)\leqslant -R_n(1)\leqslant f'_-(n+1)\leqslant f'_+(n+1). \tag{24}$$

Therefore, we get the inequalities

$$R_n(f'_{\pm},1) \leqslant -R_n(f,1) - f'_{\pm}(n+1) \leqslant 0. \tag{25}$$

Now, by using a method similar to the proof of (8) and (9) of Theorem 2.2, we find that the sequence $f_{\sigma'_{n\pm}}(x) = -\gamma_{n\pm}(f,x)$ is increasing, for all $x \geqslant 0$. Also, putting $x=0$ and $x=1$ in the equation, we obtain

$$R_n(f'_{\pm},1) \leqslant -\gamma_{n+1_{\pm}}(f) + \gamma_{n\pm}(f) \leqslant 0, \tag{26}$$

and the identities

$$f_{\sigma'_{n\pm}}(x) - f_{\sigma'_{n\pm}}(x-1) = f'_{\pm}(x) - f'_{\pm}(x+n)\,,\ -\gamma_{n\pm}(f,1) = f'_{\pm}(1) - f'_{\pm}(n+1) - \gamma_{n\pm}(f).$$

Hence, by letting $n \to \infty$ in (21) we arrive at

$$\begin{aligned} f'_{+}(1) - \gamma_{+}(f) = -\gamma_{+}(f,1) \leqslant f'_{-}(1) - \gamma_{-}(f) = -\gamma_{-}(f,1) \leqslant f_{\sigma'_{+}}(x) \leqslant f_{\sigma'_{-}}(x) \leqslant f_{\tilde{\sigma}}(x) \\ \leqslant -\gamma_{+}(f) \leqslant -\gamma_{-}(f), \end{aligned}$$

for all $0 < x \leqslant 1$. Now, we can complete the proof in a similar way to the proof of Theorem 2.2. $\square$

NOTE. *If* $f : [1,+\infty) \to \mathbb{R}$ *satisfies the conditions of the above theorem, then the results are valid for* f'_{+}.

COROLLARY 2.8. *Let* $\delta < 1$ *and* $f : (\delta,+\infty) \to \mathbb{R}$ *be a convex (concave) function and* $R_n(f,1)$ *be bounded. Then*
a) $f_{\sigma'_{n\pm}}(x)$ *is convergent, for each* $x \geqslant 0$, *and* $f_{\sigma'_{\pm}}(x)$ *satisfies the following inequalities*

$$f'_{\pm}(1) + R(1) - \gamma_{\pm}(f) \leqslant f_{\sigma'_{\pm}}(x) \leqslant f_{\tilde{\sigma}}(x) \leqslant -\gamma_{\pm}(f); \quad 0 < x \leqslant 1, \tag{27}$$

if f *is convex, and*

$$-\gamma_{\pm}(f) \leqslant f_{\tilde{\sigma}}(x) \leqslant f_{\sigma'_{\pm}}(x) \leqslant f'_{\pm}(1) + R(1) - \gamma_{\pm}(f); \quad 0 < x \leqslant 1, \tag{28}$$

if f *is concave.*
b) The function $f_{\sigma'_{\pm}}(x)$ *is a solution of the functional equation*

$$\chi(x) = f'_{\pm}(x) + R(f,1) + \chi(x-1). \tag{29}$$

Proof. This is similar to the proof of Corollary 2.5, by putting $g(x) := f(x) + R(1)x$. $\square$
In the following example we give a function f that does not satisfy the conditions of Theorem 2.2, but it agrees with the conditions of Theorem 2.7

EXAMPLE 2.9. Define the function $f : (\frac{1}{2}, +\infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} \log x; & x \geqslant 2 \\ \frac{2}{3}x + \log 2 - \frac{4}{3}; & \frac{1}{2} < x \leqslant 2 \end{cases}.$$

This function is concave and we obtain

$$-\gamma_{n_+}(f) = f(n) - (f'_+(1) + f'_+(2) + \cdots + f'_+(n)) = (\log n - \sum_{k=1}^{n} \frac{1}{k}) - \frac{5}{3},$$

so

$$-\gamma_+(f) = -\gamma - \frac{5}{3}.$$

Also,

$$f_{\sigma'_{n+}}(x) = (\log n - \sum_{k=1}^{n} \frac{1}{x+k}) - \frac{1}{x+1} - \frac{2}{3}; \quad x > \frac{1}{2}$$

thus

$$f_{\sigma'_+}(x) = \psi(x+1) - \frac{1}{x+1} - \frac{2}{3}.$$

On the other hand

$$f_\sigma(x) = \log\Gamma(x+1) + \log(x+1).$$

Now, Theorem 2.7 implies

$$-\gamma_+(f) \leqslant f_{\tilde{\sigma}}(x) \leqslant f_{\sigma'_+}(x) \leqslant -\gamma_+(f,1) \Rightarrow$$

$$-\gamma - \frac{5}{3} \leqslant \frac{\log\Gamma(x+1)}{x} + \frac{\log(x+1)}{x} \leqslant \psi(x+1) - \frac{1}{x+1} - \frac{2}{3} \leqslant \psi(1) - \frac{1}{6} = -\gamma - \frac{1}{6}.$$

Analogously,

$$-\gamma_-(f) = -\gamma - \frac{17}{6},$$
$$f_{\sigma'_-}(x) = \psi(x+1) - \frac{1}{x+1} - \frac{2}{3},$$
$$f_{\tilde{\sigma}}(x) = \frac{\log\Gamma(x+1)}{x} + \frac{\log(x+1)}{x},$$

hence

$$-\gamma - \frac{17}{6} \leqslant \frac{\log\Gamma(x+1)}{x} + \frac{\log(x+1)}{x} \leqslant \psi(x+1) - \frac{1}{x+1} - \frac{2}{3} \leqslant -\gamma - \frac{1}{6}; \quad 0 < x \leqslant 1$$

EXAMPLE 2.10. Define the function $f : (\frac{1}{2}, +\infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} -\sqrt{x-1}; & x \geqslant 2 \\ -x+1; & \frac{1}{2} < x \leqslant 2 \end{cases}.$$

This function is convex and we obtain

$$1-\frac{1}{2}\sum_{k=1}^{\infty}\frac{2\sqrt{k+1}-(\sqrt{k}-\sqrt{k-1})}{\sqrt{k+1}(\sqrt{k}+\sqrt{k-1})}\leqslant 1-\frac{1}{2}\sum_{k=1}^{\infty}\frac{2\sqrt{k+x}-(\sqrt{k}-\sqrt{k-1})}{\sqrt{k+x}(\sqrt{k}+\sqrt{k-1})}$$
$$\leqslant \lim_{n\to\infty}(-\sqrt{n-1}+\sum_{k=1}^{n}\frac{1}{\sqrt{x+k-1}+\sqrt{k-1}})\leqslant 1+\frac{1}{2}\zeta(\frac{1}{2});\quad 0<x\leqslant 1.$$

REMARK 2.11. Let a_n be a sequence of real numbers such that
(i) a_n is increasing,
(ii) $a_n < \frac{a_{n-1}+a_{n+1}}{2}$ (i.e., a_n is a strictly convex sequence),
(iii) $\lim_{n\to\infty}(a_{n+1}-a_n)=0$.
Then, putting $f(x)=(a_{n+1}-a_n)(x-n)+a_n$ for all $n\in\mathbb{N}$ and $n\leqslant x<n+1$, we see that $f:[0,+\infty)\to\mathbb{R}$ is a convex function such that it is not differentiable at the natural numbers. Hence, we can not apply Theorem 2.2, but by using Theorem 2.7, we conclude that f is limit summable and $f_{\sigma_n}(x)$ is convergent and the inequality 18 holds (of course, the inequality obtained from 18 is obvious for this case). Since the set of sequences a_n which satisfy the three above conditions are infinite, hence there exist infinity many sequences a_n which satisfy the conditions of Theorem 2.7, although they do not satisfy the conditions of Theorem 2.2.
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3. Some other inequalities

In Theorem 4.2 of [6] A. Laforgia and P. Natalini gave an inequality about the Gamma function. Here, we conclude it as an example of the topic.

EXAMPLE 3.1. From Theorem A and Example 2.1, we conclude that

$$\log_{\sigma}(x)=\log\Gamma(x+1)\,,\ \log_{\sigma'}(x)=-\gamma(\log,x)=\psi(x+1),\ \ \gamma=\gamma(\log,0),$$

and by using the inequality, we obtain

$$-1+\gamma\leqslant\gamma(\log,x)\leqslant\frac{-\log\Gamma(x+1)}{x}\leqslant\gamma;\quad 0<x\leqslant 1,$$
$$e^{-\gamma x}\leqslant\Gamma(x+1)\leqslant e^{x\psi(x+1)}\leqslant e^{(1-\gamma)x}. \tag{30}$$

That gives the inequality in Theorem 4.2 of [6].

The first part of the next example gives some inequalities for the zeta function and $\zeta(s,x)$ whenever $s>1$ and $0<x\leqslant 1$.

EXAMPLE 3.2. Let r be a fixed real number and put $f(x)=x^r$. We use Theorem 2.2 again, and arrive at some inequalities (as some applications of the topic) by using

the following cases.
Case 1. If $r<0$, then $f'<0,\ f''>0$ on $(0,+\infty)$ $(f(n)\to 0,\ R(1)=0)$ and we have

$$f_{\sigma_n'}(0)=n^r-\sum_{k=1}^{n} rk^{r-1},$$

thus

$$\gamma(f)=-f_{\sigma'}(0)=r\sum_{k=1}^{\infty}\frac{1}{k^{1-r}}=r\zeta(1-r)=2(2\pi)^{-r}\Gamma(r+1)cos(\frac{\pi r}{2})\zeta(r).$$

By applying the inequality, we find

$$r\zeta(1-r)\leqslant\frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{(n+x)^{-r}}-\frac{1}{n-r}\leqslant r(\zeta(1-r,x)-\frac{1}{x^{1-r}})\leqslant r\zeta(1-r)-r,$$

whenever $0<x\leqslant 1$.
Case 2. If $r=-1$, then we get

$$-\gamma(f)=\lim_{x\to\infty}(\frac{1}{n}+\sum_{k=1}^{n}\frac{1}{k^2})=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}.$$

Therefore,

$$r+\frac{\pi^2}{6}\leqslant\sum_{n=1}^{\infty}\frac{1}{(n+x)^2}\leqslant\sum_{n=1}^{\infty}\frac{1}{n^2+nx}\leqslant\frac{\pi^2}{6};\quad 0<x\leqslant 1.$$

Case 3. If $0<r<1$, then we have $R(f,1)=0$, and so

$$f_{\sigma_n'}(x)=n^r-\sum_{k=1}^{n} r(x+k)^{r-1}=1-r(x+1)^{r-1}-\sum_{k=2}^{\infty}((k-1)^r-k^r-r(x+k)^{r-1}),$$

$$-\gamma(f)=1-r-\sum_{k=2}^{\infty}((k-1)^r-k^r-rk^{r-1}).$$

Hence

$$1-\sum_{n=2}^{\infty}((n-1)^r-n^r-rn^{r-1})\leqslant 1-r(x+1)^{r-1}-\sum_{n=2}^{\infty}((n-1)^r-n^r-r(x+n)^{r-1})$$
$$\leqslant\frac{1}{x}\sum_{n=1}^{\infty}((1+x)n^r-(n+x)^r-x(n-1)^r)\leqslant 1-r-\sum_{n=2}^{\infty}((n-1)^r-n^r-rn^{r-1}),$$

whenever $0<x\leqslant 1$.

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