HYBRID ZIPF–MANDELBROT LAW

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Abstract. There is a unified approach, maximization of Shannon entropy, that naturally follows
the path of generalization from Zipf's to hybrid Zipf's law. Extending this idea we make transi-
tion from Zipf-Mandelbrot to hybrid Zipf-Mandelbrot law. It is interesting that examination of
its densities provides some new insights of Lerch's transcendent.

1. Introduction

Zipf's law [11] and more generally Zipf-Mandelbrot law [7] are probability distri-
butions with wide spread of applications from language to ecology [4].
For $N \in \mathbb{N}$, $q \geqslant 0$, $s > 0$, $k \in \{1, 2, \ldots, N\}$, Zipf-Mandelbrot law (probability mass
function) is defined with

$$f(k, N, q, s) = \frac{1/(k+q)^s}{\zeta(N, s, q)}, \tag{1}$$

where

$$\zeta(N, s, q) = \sum_{i=1}^{N} \frac{1}{(i+q)^s}, \tag{2}$$

$N \in \mathbb{N}$, $q \geqslant 0$, $s > 0$, $k \in \{1, 2, \ldots, N\}$.

If the total mass of the law is spread over all $\mathbb{N}$, then for, $q \geqslant 0$, $s > 1$, $k \in \mathbb{N}$,
density function of Zipf-Mandelbrot law becomes

$$f(k, q, s) = \frac{1/(k+q)^s}{\zeta(s, q)}, \tag{3}$$

where

$$\zeta(s, q) = \sum_{i=1}^{\infty} \frac{1}{(i+q)^s}. \tag{4}$$

Here, normalization factor $\zeta(s, q)$ we recognize as Hurwitz ζ function.
If we set $q = 0$ in (1) we get Zipf's law and in [10] Viesser made transition from Zipf's
to hybrid Zipf's law using maximum entropy approach [2, 3].

2. Shannon entropy and Zipf-Mandelbrot law

Here we extend use the maximum entropy approach in [10] to Zipf's law in order to deduce Zipf-Mandelbrot law, i.e. we maximize

$$S = -\sum_{i \in I} p_i \ln p_i \tag{5}$$

subject to some constraints. Trivial constraint is of course $\sum_{i \in I} p_i = 1$.

THEOREM 2.1. *Let* $I = \{1, \ldots, N\}$ *or* $I = \mathbb{N}$. *For a given* $q \geqslant 0$ *and* $\chi \geqslant 0$, *a probability distribution, concentrated on* I, *that maximizes Shannon entropy under additional constraint*

$$\sum_{k \in I} p_k \ln(k+q) = \chi \tag{6}$$

is Zipf-Mandelbrot law.

Proof. If $I = \{1, \ldots, N\}$, in a very standard procedure, we set two Lagrange multipliers λ and s and consider expression

$$\hat{S} = -\sum_{k=1}^{N} p_k \ln p_k - \lambda \left(\sum_{k=1}^{N} p_k - 1 \right) - s \left(\sum_{k=1}^{N} p_k \ln(k+q) - \chi \right).$$

Just for convenience we can, of course, replace $\lambda \longleftrightarrow \ln \lambda - 1$, and now conider

$$\hat{S} = -\sum_{k=1}^{N} p_k \ln p_k - (\ln \lambda - 1) \left(\sum_{k=1}^{N} p_k - 1 \right) - s \left(\sum_{k=1}^{N} p_k \ln(k+q) - \chi \right)$$

instead.
From $\hat{S}_{p_k} = 0$, $k = 1, \ldots, N$ we deduce

$$p_k = \frac{1}{\lambda (k+q)^s},$$

and combining this with $\sum_{k=1}^{N} p_k = 1$, we have

$$\lambda = \sum_{k=1}^{N} \frac{1}{(k+q)^s},$$

where $s > 0$, concluding

$$p_k = \frac{1/(k+q)^s}{\zeta(N,s,q)}, \ k = 1, \ldots, N.$$

The case $I = \mathbb{N}$ is treated in a similar manner with the restriction $s > 1$:

$$p_k = \frac{1/(k+q)^s}{\zeta(s,q)}, \ k \in \mathbb{N}.$$

□

REMARK 2.2.

(i) If X is the random variable with values at I and probability law $(p_i, \ i \in I)$, then χ from (6) is in fact expectation of the random variable $\ln(X + q)$, which depends on X.

(ii) Observe here that for Zipf-Mandelbrot law (3) Shannon entropy (5) can be bounded from above (see [8]):

$$S = -\sum_{k=1}^{\infty} f(k,q,s) \ln f(k,q,s) \leqslant -\sum_{k=1}^{\infty} f(k,q,s) \ln q_k, \qquad (7)$$

where $(q_k : \ k \in \mathbb{N})$ is any sequence of positive numbers such that $\sum\limits_{k=1}^{\infty} q_k = 1$.

3. Hybrid Zipf-Mandelbrot law

The same technique of maximum entropy we apply with one additional constraint. The derived probability law we will call *hybrid Zipf-Mandelbrot law*.

THEOREM 3.1. *Let* $I = \{1,\ldots,N\}$ *or* $I = \mathbb{N}$. *For a given* $q \geqslant 0$, $\chi \geqslant 0$ *and* $\mu \geqslant 0$, *a probability distribution, concentrated on* I, *that maximizes Shannon entropy under additional constraints*

$$\sum_{k \in I} p_k \ln(k+q) = \chi, \ \sum_{k \in I} k p_k = \mu$$

is hybrid Zipf-Mandelbrot law:

$$p_k = \frac{w^k}{(k+q)^s \Phi^*(s,q,w)}, \ k \in I,$$

where

$$\Phi_I^*(s,q,w) = \sum_{k \in I} \frac{w^k}{(k+q)^s}.$$

Proof. We consider first $I = \{1,\ldots,N\}$ and then we maximize

$$\hat{S} = -\sum_{k=1}^{N} p_k \ln p_k + \ln w \left(\sum_{k=1}^{N} k p_k - \mu \right) - (\ln \lambda - 1) \left(\sum_{k=1}^{N} p_k - 1 \right)$$
$$- s \left(\sum_{k=1}^{N} p_k \ln(k+q) - \chi \right).$$

$\hat{S}_{p_k} = 0, \ k = 1,\ldots,N$ gives us

$$-\ln p_k + k \ln w - \ln \lambda - s \ln (k+q) = 0,$$

i.e.

$$p_k = \frac{w^k}{\lambda (k+q)^s}.$$

Using $\sum\limits_{k=1}^{N} p_k = 1$, we get $\lambda = \sum\limits_{k=1}^{N} \frac{w^k}{(k+q)^s}$ and we recognize this as the partial sum of Lerch's transcendent

$$\Phi_N^*(s,q,w) = \sum_{k=1}^{N} \frac{w^k}{(k+q)^s},$$

with $w \geqslant 0, s > 0$.

In the infinite case $I = \mathbb{N}$ we have restrictions either $w < 1, s > 0$ or $w = 1, s > 1$ and

$$\lambda = \sum_{k=1}^{\infty} \frac{w^k}{(k+q)^s}$$

we recognize as Lerch's transcendent that we will denote with $\Phi^*(s,q,w)$. $\quad\square$

$$\therefore$$

Let us denote

$$f_h(w,N,k,q,s) = \frac{w^k}{(k+q)^s \Phi_N^*(s,q,w)}, \ k = 1,\dots,N \tag{8}$$

and

$$f_h(w,k,q,s) = \frac{w^k}{(k+q)^s \Phi^*(s,q,w)}, \tag{9}$$

hybrid Zipf-Mandelbrot law on finite and infinite state space, respectively.

REMARK 3.2. Some remarks are needed.

(i) Observe that constraint with the μ is in fact the expectation of the law.

(ii) There is a slight difference between Lerch's transcendent defined in [1] p. 27 and with our understanding of Lerch's transcendent: we don't have 0th summand.

(iii) We omitted the full bordered Hessian discussion in proofs of Theorems 2.1 and 3.1 as mere standard procedure.

(iv) Observe, further, that for hybrid Zipf-Mandelbrot law (9) Shannon entropy (5) can be bounded from above (see [8]):

$$S = -\sum_{k=1}^{\infty} f_h(k,q,s) \ln f_h(k,q,s) \leqslant -\sum_{k=1}^{\infty} f_h(k,q,s) \ln q_k, \tag{10}$$

where $(q_k : k \in \mathbb{N})$ is any sequence of positive numbers such that $\sum\limits_{k=1}^{\infty} q_k = 1$.

4. Properties of the hybrid Zipf-Mandelbrot law

Now we examine analytical properties of the Lerch's transcendent and the hybrid Zipf-Mandelbrot law.

THEOREM 4.1. *The functions*

$$s \mapsto \left(\frac{w - w^{N+1}}{w^k - w^{k+1}} f_h(w,N,k,q,s) \right)^{1/s} \tag{11}$$

and

$$s \mapsto \left(\frac{w}{w^k - w^{k+1}} f_h(w,k,q,s) \right)^{1/s} \tag{12}$$

are decreasing on $(0,\infty)$.

Proof. From (8) it follows

$$\frac{1}{f_h(w,N,k,q,s)} = \frac{1}{w^k} \sum_{i=1}^{N} w^i \left(\frac{k+q}{i+q} \right)^s$$

i.e.

$$\left(\frac{w^k - w^{k+1}}{(w - w^{N+1})h(w,N,k,q,s)} \right)^{1/s} = \left(\frac{1}{\frac{w - w^{N+1}}{1-w}} \sum_{i=1}^{N} w^i \left(\frac{k+q}{i+q} \right)^s \right)^{1/s}. \tag{13}$$

The right-hand side of (13) is power mean, which is increasing function on parameter s. $\square$

Now we recall well-known Lyapunov inequality, for isotonic functionals (for details see [9, pp. 117]): for $0 < r < s < t$

$$A(g^s)^{t-r} \leqslant A(g^r)^{t-s} A(g^t)^{s-r}. \tag{14}$$

THEOREM 4.2.

i) *For* $N \in \mathbb{N}$, $w > 0$, $q \geqslant 0$, $0 < r < s < t$,

$$[\Phi_N^*(s,q,w)]^{t-r} \leqslant [\Phi_N^*(s,q,w)]^{t-s} [\Phi_N^*(s,q,w)]^{s-r}.$$

ii) *For* $0 < w < 1$, $q \geqslant 0$, $1 < r < s < t$,

$$[\Phi^*(s,q,w)]^{t-r} \leqslant [\Phi^*(s,q,w)]^{t-s} [\Phi^*(s,q,w)]^{s-r}.$$

Proof. i) We apply (14) to the linear functional

$$A(g) = \sum_{k=1}^{N} w^k g(k)$$

and replace $g(k) = \frac{1}{k+q}$.

ii) Similarly, if we define

$$A(g) = \sum_{k=1}^{\infty} w^k g(k)$$

then the result follows from (14), if we choose $g(k) = \frac{1}{k+q}$.   □

We can now conclude log-convexity of Lerch's transcendent and log-concavity of hybrid Zipf-Mandelbrot law.

COROLLARY 4.3. *Let* $\lambda \in (0,1)$.

i) *For* $0 < r < t$, $N \in \mathbb{N}$, $w > 0$, $q \geqslant 0$, $0 < r < s < t$,

$$\Phi_N^*(\lambda r + (1-\lambda)t, q, w) \leqslant [\Phi_N^*(r,q,w)]^{\lambda} [\Phi_N^*(t,q,w)]^{1-\lambda}.$$

ii) *For* $1 < r < t$, $0 < w < 1$, $q \geqslant 0$

$$\Phi^*(\lambda r + (1-\lambda)t, q, w) \leqslant [\Phi^*(r,q,w)]^{\lambda} [\Phi^*(t,q,w)]^{1-\lambda}.$$

iii) *For* $N \in \mathbb{N}$, $w > 0$, $q \geqslant 0$, $0 < r < s < t$,

$$(f_h(w,N,k,q,\lambda r + (1-\lambda)t))^{-1} \leqslant (f_h(w,N,k,q,r))^{-\lambda} (f_h(w,N,k,q,t))^{-(1-\lambda)}.$$

iv) *For* $0 < w < 1$, $q \geqslant 0$, $1 < r < s < t$,

$$(f_h(w,k,q,\lambda r + (1-\lambda)t))^{-1} \leqslant (f_h(w,k,q,r))^{-\lambda} (f_h(w,k,q,t))^{-(1-\lambda)}.$$

Proof. i) and ii) follow from Theorem 4.2. iii) and iv) follow from (8) and (9) respectively.   □

The results in the previous corollary can be extended further to exponential convexity.

DEFINITION 4.4. A function $h : I \to \mathbb{R}$ is exponentially convex on an open interval $I \subseteq \mathbb{R}$ if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j h \left(\frac{x_i + x_j}{2} \right) \geqslant 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$, $x_i \in I$, $i = 1, \ldots, n$.

THEOREM 4.5. *The function* $s \mapsto \Phi^*(s,q,w)$ *is exponentially convex function on* $(1, \infty)$.

Proof. For a given $n \in \mathbb{N}$ let $\xi_m \in \mathbb{R}$, $s_m \in (1, \infty)$ $(m = 1, \ldots, n)$ we have

$$\sum_{l,m=1}^{n} \xi_l \xi_m \Phi^*(\frac{s_l + s_m}{2}, q, w) = \sum_{l,m=1}^{n} \xi_l \xi_m \sum_{i=1}^{\infty} \frac{w^i}{(i+q)^{\frac{s_l+s_m}{2}}} \tag{15}$$

$$= \sum_{i=1}^{\infty} w^i \sum_{l,m=1}^{n} \frac{\xi_l \xi_m}{(i+q)^{\frac{s_l+s_m}{2}}} \tag{16}$$

$$= \sum_{i=1}^{\infty} w^i \left(\sum_{m=1}^{n} \frac{\xi_m}{(i+q)^{\frac{s_m}{2}}} \right)^2 \geqslant 0. \tag{17}$$

Since the function $s \mapsto \Phi^*(s, q, w)$ is continuous function on $(1, \infty)$, we conclude its exponential convexity on $(1, \infty)$. $\square$

Using (9) we have also the next corollary.

COROLLARY 4.6. *The function* $s \mapsto (f_h(w, k, q, s))^{-1}$ *is exponentially convex function on* $(1, \infty)$.

COROLLARY 4.7. *The matrices* $\left[\left(\Phi^*(\frac{s_l+s_m}{2}, q, w) \right) \right]_{l,m=1}^{n}$ *and* $\left[\left(f_h(w, k, q^{\frac{s_l+s_m}{2}}) \right)^{-1} \right]_{l,m=1}^{n}$ *are positive semidefinite for all* $n \in \mathbb{N}$, $s_1, \ldots, s_n$ *in* $(1, \infty)$.

We can deduce exponential convexity for the second parameter in generalized polylogarithm function. First, we will prove theorem on integral representation of generalized polylogarithm function as a variant of Mellin transformation for Hurwitz ζ function.

LEMMA 4.8. *For* $0 < w < 1$, $q \geqslant 0$

$$\Phi^*(s, q, w) = \frac{w}{\Gamma(s)} \int_0^{\infty} \frac{u^{s-1} e^{-(q+1)u}}{1 - we^{-u}} du. \tag{18}$$

Proof. In Gamma function integral we change variable, $x = (k+q)u$,

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx = (k+q)^s \int_0^{\infty} e^{-(k+q)u} u^{s-1} du,$$

hence,

$$\Rightarrow (k+q)^{-s} \Gamma(s) = \int_0^{\infty} e^{-ku} e^{-qu} u^{s-1} du. \tag{19}$$

By multiplying both sides of (19) with w^k, summing over $k \in \mathbb{N}$, and using Beppo-Levi's theorem on the right side, we have

$$\Phi^*(s, q, w) \Gamma(s) = \int_0^{\infty} \sum_{k=1}^{\infty} w^k e^{-ku} e^{-qu} u^{s-1} du$$

$$= w \int_0^{\infty} \frac{u^{s-1} e^{-(q+1)u}}{1 - we^{-u}} du.$$

$\square$

THEOREM 4.9. *The function* $q \mapsto \Phi^*(s,q,w)$ *is exponentially convex function on* $(0,\infty)$.

Proof. For a given $n \in \mathbb{N}$, $\xi_m \in \mathbb{R}$, $q_m \in (0,\infty)$ $(m=1,\ldots,n)$, we have, using (18) and

$$\sum_{i,j=1}^{n} \xi_i \xi_j \exp\left(-\left(\frac{q_i+q_j}{2}+1\right)t\right) = \left(\sum_{i=1}^{n} \xi_i \exp\left(-\frac{q_i+1}{2}t\right)\right)^2 \geqslant 0,$$

concluding

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi^*\left(s,\frac{q_i+q_j}{2},w\right) \geqslant 0.$$

$\square$

COROLLARY 4.10. *For $s>1$, the matrix* $\left[\Phi^*\left(s,\frac{q_i+q_j}{2},w\right)\right]_{i,j=1}^{n}$ *is positive semi definite for all $n \in \mathbb{N}$, $q_1,\ldots,q_n$ in $(0,\infty)$.*

COROLLARY 4.11. *For any $s>1$, the function*

$$q \mapsto \Phi^*(s,q,w)$$

is log-convex on $(0,\infty)$.

THEOREM 4.12. *The function* $w \mapsto \frac{\Phi^*(s,q,w)}{w}$ *is exponentially convex on $(0,1)$.*

Proof. From $\frac{1}{w} = \int_0^\infty e^{-wt}\,dt$ we have

$$\frac{1}{1-we^{-u}} = \int_0^\infty e^{t+we^{-u}t}\,dt.$$

If we now rewrite (18)

$$\frac{\Phi^*(s,q,w)}{w} = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-(q+1)u} \int_0^\infty e^{t+we^{-u}t}\,dt\,du,$$

and use Fubini with

$$\sum_{i,j=1}^{n} \xi_i \xi_j e^{t+\frac{w_i+w_j}{2}e^{-u}t} = e^t \left(\sum_{i=1}^{n} e^{\frac{w_i e^{-u}t}{2}}\right)^2 \geqslant 0,$$

our proof is done. $\square$

COROLLARY 4.13. *For any $\alpha>1$ the function* $w \mapsto \frac{\Phi^*(s,q,w)}{w^\alpha}$ *is exponentially convex on $(0,1)$.*

Proof. This follows from the fact that, for $\gamma > 0$, $x \mapsto x^{-\gamma}$ is exponentially convex on $(0,1)$ and that product of exponentially convex function (on the same domain) is again exponentially convex (for details see [5]). $\square$

Let us recall Chebyshev's inequality (see [9, pp. 197]).

THEOREM 4.14. *Let* $(a_1,\ldots,a_N)$ *and* $(b_1,\ldots,b_N)$ *be two N-tuples of real numbers such that*

$$(a_i - a_j)(b_i - b_j) \geqslant 0, \text{ for } i,j = 1,\ldots,N,$$

and $(w_1,\ldots,w_N)$ *be a positive* n*−tuple. Then*

$$\left(\sum_{i=1}^{N} w_i \right) \left(\sum_{i=1}^{N} w_i a_i b i \right) \geqslant \left(\sum_{i=1}^{N} w_i a_i \right) \left(\sum_{i=1}^{N} w_i b_i \right). \tag{20}$$

REMARK 4.15. The previous theorem can be extended to infinite sequences if we impose some obvious convergence

$$\left(\sum_{i=1}^{\infty} w_i \right) \left(\sum_{i=1}^{\infty} w_i a_i b i \right) \geqslant \left(\sum_{i=1}^{\infty} w_i a_i \right) \left(\sum_{i=1}^{\infty} w_i b_i \right). \tag{21}$$

Let us introduce *mean version of Lerch's transcendent*

$$\overline{\Phi^*}(s,q,w) = \frac{1-w}{w} \Phi^*(s,q,w).$$

THEOREM 4.16. *The mean version of Lerch's transcendent is log-subadditive, i.e. for* $s,r > 0$

$$\overline{\Phi^*}(s+r,q,w) \leqslant \overline{\Phi^*}(s,q,w)\overline{\Phi^*}(r,q,w). \tag{22}$$

Proof. We apply Chebyshev's inequality (21) for

$$a_i = \left(\frac{k+q}{i+q} \right)^s, \ b_i = \left(\frac{k+q}{i+q} \right)^r, \ w_i = w^i; \ i \in \mathbb{N}$$

i.e.

$$\frac{w}{1-w} \Phi^*(s+r,q,w) \leqslant \Phi^*(s,q,w)\Phi^*(r,q,w).$$

$\square$

REMARK 4.17. A similar version of the previous theorem can be proved for cut Lerch's transcendent $\overline{\Phi^*}_N(s,q,w)$.

5. Hybrid means

For a fixed $p_i \geqslant 0$, $a_i \geqslant 0$, $i = 1 \ldots, N$ let us define linear functional on $C[a,b]$ with

$$A(f) = \sum_{i=1}^{n} p_i f(a_i),$$

where $a \leqslant \min a_i \leqslant \max a_i \leqslant b$. Then, the next theorem is valid.

THEOREM 5.1. *For a continuous function $g : [a,b] \to \mathbb{R}_+$, the function $t \mapsto A(g^t)$ is exponentially convex on $(0,\infty)$ and for positive $\alpha_1, \alpha_2, \beta_1, \beta_2$; $\alpha_1 \leqslant \alpha_2$, $\beta_1 \leqslant \beta_2$, $\alpha_1 \neq \beta_1$, $\alpha_2 \neq \beta_2$,*

$$\left(\frac{A(g^{\alpha_1})}{A(g^{\alpha_2})}\right)^{\frac{1}{\alpha_1 - \beta_1}} \leqslant \left(\frac{A(g^{\alpha_2})}{A(g^{\beta_2})}\right)^{\frac{1}{\alpha_2 - \beta_2}}. \tag{23}$$

Proof. For a fixed $n \in \mathbb{N}$, $u_i \in \mathbb{R}$, $t_i > 0$, $i = 1, \ldots, n$, we define an auxiliary function

$$\Psi(x) = \sum_{i,j=1}^{n} u_i u_j x^{\frac{t_i + t_j}{2}}.$$

Since $\Psi(x) = \left(\sum_{i=1}^{n} u_i x^{\frac{t_i}{2}}\right)^2 \geqslant 0$, we have $A(\Psi(g)) \geqslant 0$, i.e.

$$\sum_{i,j=1}^{n} u_i u_j A\left(g^{\frac{t_i + t_j}{2}}\right) \geqslant 0$$

concluding exponential convexity of the function $t \mapsto A(g^t)$. Since exponential convexity implies log-convexity (see [5]), (23) follows from [9] pp. 7.   □

REMARK 5.2. For fixed $t > 0$, let $m = \min_{x \in [a,b]} g^t(x)$, $M = \max_{x \in [a,b]} g^t(x)$. Then, from $A(g^t - m) \geqslant 0$ and $A(M - g^t) \geqslant 0$ it follows

$$mA(1) \leqslant A(g^t) \leqslant MA(1).$$

By the mean value theorem it follows that exists $\xi \in [a,b]$ such that

$$g^t(\xi) = \frac{A(g^t)}{A(1)}.$$

Also, for fixed $\alpha, \beta \in (0,\infty), \alpha \neq \beta$, following the very standard technique, we can also prove that there exists $\eta \in [a,b]$ such that

$$g^{\alpha - \beta}(\eta) = \frac{A(g^{\alpha})}{A(g^{\beta})}. \tag{24}$$

Now, if $g(\eta) \in [a,b]$, then the expression

$$\left(\frac{A(g^{\alpha})}{A(g^{\beta})}\right)^{\frac{1}{\alpha-\beta}}. \tag{25}$$

stands for the mean and, as (23) shows, these means have monotonicity property.

For fixed, $k \in \mathbb{N}$, $q \geqslant 0$, let us take $p_i = w^i$, $a_i = \frac{k+q}{i+q}$, $i = 1,\ldots,N$, $g = $ id. Using Remark (5.2) and (8), we can define hybrid means

$$H(\alpha,\beta) = \begin{cases} \left(\frac{f_h(w,N,k,q,\beta)}{f_h(w,N,k,q,\alpha)}\right)^{\frac{1}{\alpha-\beta}}, & \alpha \neq \beta; \\ \exp\left(-\frac{\frac{d}{d\alpha}f_h(w,N,k,q,\alpha)}{f_h(w,N,k,q,\alpha)}\right), & \alpha = \beta. \end{cases} \tag{26}$$

THEOREM 5.3. *For $0 < \alpha_1 \leqslant \alpha_2$, $0 < \beta_1 \leqslant \beta_2$, $\alpha_1 \neq \beta_1$, $\alpha_2 \neq \beta_2$;*

$$H(\alpha_1,\beta_1) \leqslant H(\alpha_2,\beta_2). \tag{27}$$

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