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MONOTONICITY AND INEQUALITIES INVOLVING THE INCOMPLETE GAMMA FUNCTION

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Abstract. In this paper, by observing the monotonicity of three ratios involving the integral function $\int_0^x e^{-t^p}dt$ for $p,x>0$, we offer some new sharp bounds for the incomplete gamma function, which greatly improve and extend some known results. Also, as by-products, we unexpectedly obtain two power series representations for the incomplete gamma function.

1. Introduction

The incomplete gamma function is given by

$$\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt,\ a>0,x>0,$$

while the exponential integral is defined by

$$E_1(x)=\lim_{a\to 0^+}\Gamma(a,x)=\int_x^\infty \frac{e^{-t}}{t}dt,\ x>0.$$

They are related to the integral function $\int_0^x e^{-t^p}dt$ or $\int_x^\infty e^{-t^p}dt$ with $p>0$. Indeed, it is known that

$$\int_x^\infty e^{-t^p}dt=\frac{1}{p}\Gamma\left(\frac{1}{p},x^p\right) \tag{1.1}$$

and

$$\int_0^x e^{-t^p}dt=\frac{1}{p}\Gamma\left(\frac{1}{p}\right)-\frac{1}{p}\Gamma\left(\frac{1}{p},x^p\right). \tag{1.2}$$

In particular, for $p=2$, we know that

$$1-\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},x^2\right)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt=\operatorname{erf}(x)$$

is the error function [1]–[5], while

$$\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},x^2\right)=1-\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

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is the complementary error function.

There are various bounds for the integral functions $\int_0^x e^{-t^p}dt$ and $\int_x^\infty e^{-t^p}dt$ with $p>0$, see for example, Gautschi [6], Gupta and Waknis [7], Elbert and Laforgia [8], Borwein and Chan [9], Komatu [10], Pollak [11], Boyd [12], [13], Neuman [14], Gasull and Utzet [15], Yang and Chu [16], Qi and Huang [17], Qi and Guo [18]. More results for the integrals $\int_0^x e^{-t^p}dt$ and $\int_x^\infty e^{-t^p}dt$, and related special functions can be found in the literatures [19]–[31].

In particular, Elbert and Laforgia [8] presented an interesting lower bound for the function $x^{-1}\int_0^x e^{-t^p}dt$, that is,

$$1-\frac{1}{p+1}\int_0^{x^p}\frac{1-e^{-t}}{t}dt<\frac{1}{x}\int_0^x e^{-t^p}dt \tag{1.3}$$

holds for $0<x<\left[9(3p+1)/(4(2p+1))\right]^{1/p}$ and $p>1$. The inequality (1.3) was proved to be also valid for all $x>0$ and $p>1$ in [32] by Laforgia and Natalini.

Moreover, Qi and Huang in [17] gave a simple lower bound for the integral function $\int_0^x e^{-t^p}dt$:

$$\frac{1-e^{-x^p}}{x^{p-1}}\leqslant\int_0^x e^{-t^p}dt,\ x>0,\ p\geqslant 1. \tag{1.4}$$

Motivated by the inequalities (1.3) and (1.4), the aim of this paper is to investigate the monotonicity of the ratios

$$R_1(x)=\frac{1-v(bx^p)/(b(p+1))}{x^{-1}\int_0^x e^{-t^p}dt}\text{ for }b>0, \tag{1.5}$$

where

$$v(x)=\int_0^x\frac{1-e^{-t}}{t}dt, \tag{1.6}$$

$$R_2(x)=\frac{-\int_0^{x^p}\frac{1-e^{-t}}{t}dt-\sum_{n=1}^m\frac{(-1)^n}{nn!}x^{pn}}{x^{-1}\int_0^x e^{-t^p}dt-\sum_{n=0}^m\frac{(-1)^n}{(np+1)n!}x^{pn}}, \tag{1.7}$$

$$R_3(x)=\frac{x^{-p}\left(1-e^{-x^p}\right)-\sum_{n=0}^{m-1}\frac{(-1)^n x^{np}}{(n+1)!}}{x^{-1}\int_0^x e^{-t^p}dt-\sum_{n=0}^{m-1}\frac{(-1)^n}{(np+1)n!}x^{pn}} \tag{1.8}$$

on $(0,\infty)$. As a consequence, some sharp inequalities for $x^{-1}\int_0^x e^{-t^p}dt$ are established, which refine and generalize Laforgia and Natalini's inequality (1.3) and Qi and Huang' inequality (1.4). It should be emphasized that the range of parameter p is extended from $p>1$ in [8], [32], [17] to $p>0$ in our results.

The rest of this paper is organized as follows. In Section 2, we give some preliminary lemmas. In Section 3, we deal with the monotonicity of the function $R_i(x)$ for $i=1,2,3$; and present some new sharp bounds for the incomplete gamma function. In the final section, the conclusion is drawn to summarize the study.

2. Some lemmas

In order to prove our main results, we need some preliminary lemmas. The following lemma is called L'Hospital Monotone Rule, for short, LMR (see [33]), which is an efficient tool to deal with the monotonicity of the ratio between two functions. A monotonicity rule for ratio of two power series was presented in [34]. A similar monotonicity rule for the ratio of two Laplace transforms was established in [35, Lemma 4] (see also [36]).

LEMMA 2.1. ([33, Theorem 2]) *For* $-\infty < a < b < \infty$, *let* $f,g : [a,b] \to \mathbb{R}$ *be continuous functions that are differentiable on* (a,b), *with* $f(a) = g(a) = 0$ *or* $f(b) = g(b) = 0$. *If* f'/g' *with* $g'(x) \neq 0$ *for each* x *in* (a,b) *is increasing (decreasing) on* (a,b), *then so is* f/g.

To state the final lemma, we need to introduce a useful auxiliary function $H_{f,g}$. For $-\infty \leqslant a < b \leqslant \infty$, let f and g be differentiable on (a,b) and $g' \neq 0$ on (a,b). Then the function $H_{f,g}$ is defined by

$$H_{f,g} := \frac{f'}{g'} g - f. \tag{2.9}$$

The function $H_{f,g}$ has some well properties [37, Property 1] and plays an important role in the proof of a monotonicity criterion for the quotient of power series, also see [38]. Let us recall the following lemma, which is called L'Hospital Piecewise Monotone Rule, for short, LPMR.

LEMMA 2.2. ([37, Theorem 7]) *Let* $-\infty \leqslant a < b \leqslant \infty$. *Suppose that (i)* f *and* g *are differentiable functions on* (a,b); *(ii)* $g' \neq 0$ *on* (a,b); *(iii)* $f(a^+) = g(a^+) = 0$; *(iv) there is a* $c \in (a,b)$ *such that* f'/g' *is increasing (decreasing) on* (a,c) *and decreasing (increasing) on* (c,b). *Then*

(i) when $\operatorname{sgn} g' \operatorname{sgn} H_{f,g}(b^-) \geqslant (\leqslant) 0$, f/g *is increasing (decreasing) on* (a,b);

(ii) when $\operatorname{sgn} g' \operatorname{sgn} H_{f,g}(b^-) < (>) 0$, *there is a unique number* $x_0 \in (a,b)$ *such that* f/g *is increasing (decreasing) on* (a,x_0) *and decreasing (increasing) on* (x_0,b).

The following lemma offers a simple but efficient criterion to determine the sign of a kind of special series, so we call it as "*sign rule of a kind of special series*".

LEMMA 2.3. ([39, Lemma 2]) *Let* $\{a_k\}_{k\geqslant 0}$ *be a nonnegative real sequence with* $a_m > 0$ *and* $\sum_{k=m+1}^{\infty} a_k > 0$ *and let*

$$S(t) = -\sum_{k=0}^{m} a_k t^k + \sum_{k=m+1}^{\infty} a_k t^k$$

be a convergent power series on the interval $(0,r)$ $(r > 0)$. *(i) If* $S(r^-) \leqslant 0$ *then* $S(t) < 0$ *for all* $t \in (0,r)$. *(ii) If* $S(r^-) > 0$ *then there is a unique* $t_0 \in (0,r)$ *such that* $S(t) < 0$ *for* $t \in (0,t_0)$ *and* $S(t) > 0$ *for* $t \in (t_0,r)$.

REMARK 2.4. *From Lemma 2.3, it is easy to see that if there is a $t_1 \in (0,r)$ such that $S(t_1) > 0$, then we have $S(t) > 0$ for all $t \in [t_1, r)$.*

REMARK 2.5. *If $r = \infty$, then Lemma 2.3 is reduced to* [40, Lemma 6.3] *(see also* [41, Lemma 2], [42, Lemma 2.1]*). Furthermore, if we put $a_k = 0$ for all $k \geqslant n > m+1$, then Lemma 2.3 yields* [43, Lemma 7] *(see also* [44]*).*

3. Main results

3.1. Monotonicity of $R_1(x)$

Our first result is the following theorem.

THEOREM 3.1. *For $b, p > 0$, let $v(x)$ be defined on $(0,\infty)$ by (1.6). Then the ratio $R_1(x)$ defined by (1.5) is decreasing on $(0,\infty)$ if $b \leqslant b_0 = 2(p+1)/(2p+1)$. And therefore, the inequality*

$$1 - \frac{1}{b(p+1)} \int_0^{bx^p} \frac{1-e^{-t}}{t} dt < \frac{1}{x} \int_0^x e^{-t^p} dt \tag{3.10}$$

holds for $x > 0$ if $b \leqslant b_0$. While $b > b_0$, there is a unique number $x_0 \in (0,\infty)$ such that $R_1(x)$ is increasing on $(0, x_0)$ and decreasing on (x_0, ∞).

Proof. Let

$$f_1(x) = x - \frac{x}{b(p+1)} v(bx^p) = x - \frac{x}{b(p+1)} \int_0^{bx^p} \frac{1-e^{-t}}{t} dt,$$
$$g_1(x) = \int_0^x e^{-t^p} dt.$$

Then $R_1(x) = f_1(x)/g_1(x)$ with $f_1(0^+) = g_1(0^+) = 0$. Differentiation gives

$$\begin{aligned} \frac{f_1'(x)}{g_1'(x)} &= \frac{e^{x^p}}{b(p+1)} \left(b - p + bp + pe^{-bx^p} - \int_0^{bx^p} \frac{1-e^{-t}}{t} dt \right) \\ &\overset{x^p=y}{=\!=\!=} \frac{e^y}{b(p+1)} \left(b - p + bp + pe^{-by} - \int_0^{by} \frac{1-e^{-t}}{t} dt \right) \end{aligned}$$

and

$$\begin{aligned} \left(\frac{f_1'(x)}{g_1'(x)} \right)' &= \frac{d}{dy} \left[\frac{e^y}{b(p+1)} \left(b - p + bp + pe^{-by} + \int_0^{by} \frac{e^{-t}-1}{t} dt \right) \right] \frac{dy}{dx} \\ &= \frac{px^{p-1}e^y}{b(p+1)} h_1(y), \end{aligned}$$

where

$$h_1(y) = (1-b)pe^{-by} - \frac{1-e^{-by}}{y} - \int_0^{by} \frac{1-e^{-t}}{t} dt + b - p + bp$$

with $h_1(0^+) = 0$.

Differentiation again yields

$$by^2 e^{by} h_1'(y) = be^{by} - bye^{by} - (b-1)by + (b-1)p(by)^2 - b := h_3(y).$$

Expanding in power series leads to

$$\begin{aligned} by^2 e^{by} h_1'(y) &= b\sum_{n=0}^{\infty} \frac{(by)^n}{n!} - \sum_{n=1}^{\infty} \frac{(by)^n}{(n-1)!} - (b-1)(by) + (b-1)p(by)^2 - b \\ &= \sum_{n=1}^{\infty} \frac{(b-n)(by)^n}{n!} - (b-1)(by) + (b-1)p(by)^2 := \sum_{n=2}^{\infty} w_n (by)^n, \end{aligned}$$

where

$$w_2 = \frac{1+2p}{2}\left(b - \frac{2p+2}{2p+1}\right) \text{ and } w_n = \frac{b-n}{n!} \text{ for } n \geqslant 3.$$

Now, if $b \leqslant (2p+2)/(2p+1) = b_0$, then $w_2 \leqslant 0$ and $w_n < 0$ for $n \geqslant 3$, and then $h_1'(y) < 0$. This indicates that $h_1(y) < h_1(0^+) = 0$, which implies that $(f_1'(x)/g_1'(x))' < 0$ for $x \in (0,\infty)$. By Lemma 2.1, the first assertion of this theorem follows.

For $b > b_0 = (2p+2)/(2p+1)$, we distinguish two cases to confirm the sign of w_n for $n \geqslant 3$, and then determine the sign of $h_1(y)$.

Case 1: $(2p+2)/(2p+1) < b \leqslant 3$. It is seen that

$$w_2 > 0,\ w_3 = b-3 \leqslant 0,\ w_n < 0 \text{ for } n \geqslant 4.$$

Case 2: $b > 3$. We have $w_2 > 0$. Since the sequence $\{n!w_n\}_{n\geqslant 3}$ is decreasing, and $3!w_3 = b-3 > 0$ and $\lim_{n\to\infty}(n!w_n) = -\infty$, there is a $n_0 \geqslant 4$ such that $n!w_n > 0$ for $3 \leqslant n \leqslant n_0$ and $n!w_n < 0$ for $n > n_0$. That is to say, $w_n > 0$ for $2 \leqslant n \leqslant n_0$ and $w_n < 0$ for $n > n_0$.

In both cases, by Lemma 2.3, there is a $by_0 \in (0,\infty)$ such that $h_1'(y) > 0$ for $by \in (0, by_0)$ and $h_1'(y) < 0$ for $by \in (by_0, \infty)$. Hence, $h_1(y) > h_1(0^+) = 0$ for $y \in (0, y_0)$, but $h_1(\infty) = -\infty$. It is deduced that there is a $y_1 \in (y_0, \infty)$ such that $h_1(y) > 0$ for $y \in (0, y_1)$ and $h_1(y) < 0$ for $y \in (y_1, \infty)$. This reveals that f_1'/g_1' is increasing on $(0, x_1)$ and decreasing on (x_1, ∞), where $x_1 = y_1^{1/p}$.

Due to $g_1'(x) > 0$ and

$$\begin{aligned} H_{f_1,g_1}(x) &= \frac{f_1'(x)}{g_1'(x)} g_1(x) - f_1(x) \\ &= \frac{e^{x^p}}{b(p+1)}\left(b - p + bp + pe^{-bx^p} - \int_0^{bx^p} \frac{1-e^{-t}}{t} dt\right) \\ &\quad \times \int_0^x e^{-t^p} dt - \left(x - \frac{x}{b(p+1)} \int_0^{bx^p} \frac{1-e^{-t}}{t} dt\right) \\ &= \frac{e^{x^p} \int_0^{bx^p} t^{-1}(1-e^{-t})\, dt}{b(p+1)} \left[\frac{(b-p+bp)\int_0^x e^{-t^p} dt}{\int_0^{bx^p} t^{-1}(1-e^{-t})\, dt} + \frac{pe^{-bx^p}}{\int_0^{bx^p} t^{-1}(1-e^{-t})\, dt}\right. \\ &\quad \left. - \int_0^x e^{-t^p} dt - \frac{b(p+1)x}{e^{x^p}\int_0^{bx^p} t^{-1}(1-e^{-t})\, dt} + \frac{x}{e^{x^p}}\right] \to -\infty, \text{ as } x \to \infty, \end{aligned}$$

where the first, second, fourth and fifth items in the square brackets tend to zero, while the third one tends to $-\Gamma(1+1/p)$, by (ii) of Lemma 2.2, there is a unique number $x_0 \in (0,\infty)$ such that f_1/g_1 is increasing on $(0,x_0)$ and decreasing on (x_0,∞).

This completes the proof. □

PROPOSITION 3.2. *For $b,p>0$, the inequality*

$$1-\frac{1}{b(p+1)}\int_0^{bx^p}\frac{1-e^{-t}}{t}dt<\frac{1}{x}\int_0^x e^{-t^p}dt \tag{3.11}$$

holds for $x>0$ if and only if $b\leqslant b_0=(2p+2)/(2p+1)$.

Proof. The necessity is deduced from the limit relation

$$\lim_{x\to 0^+}\frac{x^{-1}\int_0^x e^{-t^p}dt-1+v(bx^p)/(b(p+1))}{x^{2p}}\geqslant 0.$$

Indeed, expanding in power series gives

$$\begin{aligned}
&\frac{1}{x}\int_0^x e^{-t^p}dt-1+\frac{1}{b(p+1)}\int_0^{bx^p}\frac{1-e^{-t}}{t}dt\\
&=\frac{1}{x}\int_0^x\sum_{n=0}^{\infty}\frac{(-1)^n t^{pn}}{n!}dt-1-\frac{1}{b(p+1)}\int_0^{bx^p}\sum_{n=1}^{\infty}\frac{(-1)^n t^{n-1}}{n!}dt\\
&=\sum_{n=2}^{\infty}\left(\frac{1}{np+1}-\frac{b^{n-1}}{n(p+1)}\right)\frac{(-1)^n x^{pn}}{n!},
\end{aligned}$$

which yields

$$\lim_{x\to 0^+}\frac{x^{-1}\int_0^x e^{-t^p}dt-1+v(bx^p)/(b(p+1))}{x^{2p}}=\frac{1}{2p+1}-\frac{1}{2}\frac{b}{p+1}.$$

Hence, we get that the necessary condition is $b\leqslant(2p+2)/(2p+1)$. The sufficiency easily follows by Theorem 3.1. □

It is easy to check that $b\mapsto v(bx)/b$ is decreasing on $(0,\infty)$. Then taking $b=b_0$, 1, 0^+ in Proposition 3.2 gives the following corollary.

COROLLARY 3.3. *We have*

$$1-\frac{x^p}{p+1}<1-\frac{1}{p+1}\int_0^{x^p}\frac{1-e^{-t}}{t}dt<1-\frac{1}{b_0(p+1)}\int_0^{b_0x^p}\frac{1-e^{-t}}{t}dt<\frac{1}{x}\int_0^x e^{-t^p}dt \tag{3.12}$$

for $p,x>0$, where $b_0=2(p+1)/(2p+1)$.

REMARK 3.4. *Corollary 3.3 shows that the third inequality of (3.12) is a refinement of Laforgia and Natalini's inequality (1.3).*

By utilizing the relation (1.2) and making changes of variables $q=1/p$ and $u=x^p\in(0,\infty)$, we have

$$\int_0^x e^{-t^p}dt=\frac{1}{p}\Gamma\left(\frac{1}{p}\right)-\frac{1}{p}\Gamma\left(\frac{1}{p},x^p\right)=q\Gamma(q)-q\Gamma(q,u). \tag{3.13}$$

Then Proposition 3.2 can be equivalently stated as follows.

PROPOSITION 3.5. *For $u,q>0$, the inequality*

$$\Gamma(q,u)<\frac{\Gamma(q+1)-u^q}{q}+\frac{u^q}{b(q+1)}\int_0^{bu}\frac{1-e^{-t}}{t}dt \tag{3.14}$$

holds if and only if $b\leqslant(2q+2)/(2+q)$.

Since $\Gamma(0,u)=E_1(u)$ and

$$\lim_{q\to 0^+}\frac{\Gamma(q+1)-u^q}{q}=-\gamma-\ln u,$$

where $\gamma=0.57722...$ is the Euler constant, by Proposition 3.5 we have

COROLLARY 3.6. *The inequality*

$$E_1(u)=\int_u^\infty\frac{e^{-t}}{t}dt<-\gamma-\ln u+\frac{1}{b}\int_0^{bu}\frac{1-e^{-t}}{t}dt$$

holds for $u>0$ if and only if $b\leqslant 1$.

3.2. Monotonicity of $R_2(x)$

THEOREM 3.7. *For $p>0$ and $m\in\mathbb{N}$, the function*

$$R_2(x)=\frac{-\int_0^{x^p}\frac{1-e^{-t}}{t}dt-\sum_{n=1}^m\frac{(-1)^n}{nn!}x^{pn}}{\frac{1}{x}\int_0^x e^{-t^p}dt-\sum_{n=0}^m\frac{(-1)^n}{(np+1)n!}x^{pn}}$$

is strictly increasing on $(0,\infty)$ with

$$R_2(0^+)=p+\frac{1}{m+1}\quad\text{and}\quad R_2(\infty)=p+\frac{1}{m}.$$

Consequently, the double inequality

$$(-1)^m L_{m,\alpha_2}(x^p)<\frac{(-1)^m}{x}\int_0^x e^{-t^p}dt<(-1)^m L_{m,\beta_2}(x^p) \tag{3.15}$$

holds for $p,x>0$ with the best constants $\alpha_2=p+1/(m+1)$ and $\beta_2=p+1/m$, where

$$L_{m,\lambda}(x)=1+\sum_{n=1}^m(-1)^n\left(\frac{n}{np+1}-\frac{1}{\lambda}\right)\frac{x^n}{nn!}-\frac{1}{\lambda}\int_0^x\frac{1-e^{-t}}{t}dt.$$

Proof. Let

$$f_2(x) = -x\int_0^{x^p} \frac{1-e^{-t}}{t}dt - \sum_{n=1}^{m} \frac{(-1)^n}{nn!}x^{pn+1}, \tag{3.16}$$

$$g_2(x) = \int_0^{x} e^{-t^p}dt - \sum_{n=0}^{m} \frac{(-1)^n}{(np+1)n!}x^{pn+1}. \tag{3.17}$$

Then $R_2(x) = f_2(x)/g_2(x)$ with $f_2(0^+) = g_2(0^+) = 0$. Differentiation yields

$$f_2'(x) = -\int_0^{x^p} \frac{1-e^{-t}}{t}dt - p\left(1-e^{-x^p}\right) - \sum_{n=1}^{m} \frac{(-1)^n(pn+1)}{nn!}x^{pn} := f_{21}(x^p),$$

$$g_2'(x) = e^{-x^p} - \sum_{n=0}^{m} \frac{(-1)^n}{n!}x^{pn} := g_{21}(x^p).$$

where

$$f_{21}(y) = -\int_0^{y} \frac{1-e^{-t}}{t}dt - p\left(1-e^{-y}\right) - \sum_{n=1}^{m} \frac{(-1)^n(pn+1)}{nn!}y^n,$$

$$g_{21}(y) = e^{-y} - \sum_{n=0}^{m} \frac{(-1)^n}{n!}y^n$$

with $y = x^p$.

Since $\sum_{n=0}^{m}\frac{(-1)^n}{n!}y^n$ is the m-order Taylor polynomial of the function e^{-y} at $y=0$, we have $g_{21}^{(k)}(0) = 0$ for $0 \leqslant k \leqslant m$. This in combination with

$$(-1)^{m+1}g_{21}^{(m+1)}(y) = e^{-y} > 0$$

gives $(-1)^{m+1}g_{21}(y) > 0$ for $y > 0$, which in turn implies that $(-1)^{m+1}g_2'(x) > 0$, and so

$$(-1)^{m+1}g_2(x) > (-1)^{m+1}g_2(0) = 0 \text{ for } x > 0.$$

Also, $f_{21}(0) = 0$. Differentiation again yields

$$-yf_{21}'(y) = pye^{-y} - e^{-y} + 1 + \sum_{n=1}^{m} \frac{(-1)^n(np+1)}{n!}y^n := f_{22}(y),$$

$$-yg_{21}'(y) = ye^{-y} + \sum_{n=1}^{m} \frac{(-1)^n}{(n-1)!}y^n := g_{22}(y). \tag{3.18}$$

Likewise, it is easy to check that $-\sum_{n=1}^{m}\frac{(-1)^n}{(n-1)!}y^n$ is the m-order Taylor polynomial of the function ye^{-y} at $y=0$, which leads to $g_{22}^{(k)}(0^+) = 0$ for $0 \leqslant k \leqslant m$. Also, since

$$\begin{aligned}(-1)^m g_{22}^{(m)}(y) &= (-1)^m\left[(-1)^m ye^{-y} + (-1)^{m-1}me^{-y} + (-1)^m m\right]\\ &= ye^{-y} + m\left(1-e^{-y}\right) > 0,\end{aligned}$$

it is deduced that $(-1)^m g_{22}^{(k)}(y) > (-1)^m g_{22}^{(k)}(0^+) = 0$ for $0 \leqslant k \leqslant m-1$. Moreover, $-1-\sum_{n=1}^{m} \frac{(-1)^n (np+1)}{n!} y^n$ is the m-order Taylor polynomial of the function $pye^{-y} - e^{-y}$, so $f_{22}^{(k)}(0^+) = 0$ for $0 \leqslant k \leqslant m$.

Thus, if we prove the ratio $f_{22}^{(m)}/g_{22}^{(m)}$ is increasing on $(0,\infty)$, then by Lemma 2.1, so is $f_{22}^{(k)}/g_{22}^{(k)}$ for $0 \leqslant k \leqslant m$, and so are $f_{21}'/g_{21}' = f_{22}/g_{22}$ and f_{21}/g_{21}. Due to

$$\left(\frac{f_2'(x)}{g_2'(x)}\right)' = \frac{d}{dy}\left(\frac{f_{21}(y)}{g_{21}(y)}\right) \times \frac{dy}{dx} = px^{p-1}\left(\frac{f_{21}(y)}{g_{21}(y)}\right)' > 0,$$

by Lemma 2.1 again, it then follows that f_2/g_2 is strictly increasing on $(0,\infty)$. Then the proof is done except $R_2(0^+)$ and $R_2(\infty)$.

In fact, we have

$$\begin{aligned}\frac{f_{22}^{(m)}(y)}{g_{22}^{(m)}(y)} &= \frac{p\left((-1)^m ye^{-y} + (-1)^{m-1} me^{-y}\right) - (-1)^m e^{-y} + (-1)^m (mp+1)}{(-1)^m (ye^{-y} + m(1-e^{-y}))} \\ &= \frac{pye^{-y} - (pm+1)e^{-y} + mp + 1}{ye^{-y} + m(1-e^{-y})} = \frac{(mp+1)e^y + py - mp - 1}{me^y + y - m},\end{aligned}$$

$$\left(\frac{f_{22}^{(m)}(y)}{g_{22}^{(m)}(y)}\right)' = \frac{ye^y - e^y + 1}{(me^y + y - m)^2} > 0 \text{ for } y > 0.$$

Finally, we compute $R_2(0^+)$ and $R_2(\infty)$. To obtain $R_2(0^+)$, we write $R_2(x)$ in the form of ratio of two power series

$$R_2(x) = \frac{\sum_{n=1}^{\infty} \frac{(-1)^n}{nn!} x^{pn} - \sum_{n=1}^{m} \frac{(-1)^n}{nn!} x^{pn}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(np+1)n!} x^{pn} - \sum_{n=0}^{m} \frac{(-1)^n}{(np+1)n!} x^{pn}} = \frac{\sum_{n=m+1}^{\infty} \frac{(-1)^n}{nn!} x^{pn}}{\sum_{n=m+1}^{\infty} \frac{(-1)^n}{(np+1)n!} x^{pn}},$$

which yields

$$\lim_{x\to 0^+} R_2(x) = \frac{(-1)^{m+1}/((m+1)(m+1)!)}{(-1)^{m+1}/(((m+1)p+1)(m+1)!)} = p + \frac{1}{m+1}.$$

To compute $R_2(\infty)$, we divide the numerator and denominator in $R_2(x)$ by x^{pm} and note that

$$\lim_{x\to\infty} \frac{1}{x^{pm}} \int_0^{x^p} \frac{1-e^{-t}}{t} dt = \lim_{x\to\infty} \frac{1}{x^{pm+1}} \int_0^x e^{-t^p} dt = 0,$$

to get

$$\lim_{x\to\infty} R_2(x) = \frac{-(-1)^m/(mm!)}{-(-1)^m/((mp+1)m!)} = p + \frac{1}{m}.$$

Then the desired double inequality follows, which ends the proof. □

Taking $m = 1$ in Theorem 3.7, we immediately get the following corollary.

COROLLARY 3.8. *For $p>0$, the double inequality*

$$1-\frac{1}{p+1}\int_0^{x^p}\frac{1-e^{-t}}{t}dt<\frac{1}{x}\int_0^x e^{-t^p}dt<1+\frac{x^p}{(p+1)(2p+1)}-\frac{2}{2p+1}\int_0^{x^p}\frac{1-e^{-t}}{t}dt \tag{3.19}$$

holds for $x>0$. The lower and upper bounds are sharp.

REMARK 3.9. *Corollary 3.8 obviously gives a reverse inequality of Laforgia and Natalini's inequality (1.3).*

Making changes of variables $q=1/p$ and $u=x^p\in(0,\infty)$ and using the relation (3.13), Theorem 3.7 can be equivalently stated as follows.

THEOREM 3.10. *For $q>0$ and $m\in\mathbb{N}$, the function*

$$R_2^*(u)=\frac{-\int_0^u t^{-1}(1-e^{-t})dt-\sum_{n=1}^m\frac{(-1)^n}{nn!}u^n}{u^{-q}(\Gamma(q)-\Gamma(q,u))-\sum_{n=0}^m\frac{(-1)^n}{(n+q)n!}u^n}$$

is strictly increasing on $(0,\infty)$ with

$$R_2^*(0^+)=1+\frac{q}{m+1}\ \text{and}\ R_2^*(\infty)=1+\frac{q}{m}.$$

Consequently, the double inequality

$$(-1)^{m+1}\mathscr{L}_{m,\beta_2^*}(u)<(-1)^m\left(\Gamma(q,u)-\frac{\Gamma(q+1)-u^q}{q}\right)<(-1)^{m+1}\mathscr{L}_{m,\alpha_2^*}(u) \tag{3.20}$$

holds for $p,x>0$ with the best constants $\alpha_2^*=1+q/(m+1)$ and $\beta_2^*=1+q/m$, where

$$\mathscr{L}_{m,\lambda}(u)=\sum_{n=1}^m(-1)^n\left(\frac{n}{n+q}-\frac{1}{\lambda}\right)\frac{u^{n+q}}{nn!}-\frac{u^q}{\lambda}\int_0^u\frac{1-e^{-t}}{t}dt.$$

Letting $m\to\infty$ in Theorem 3.10 yields $\alpha_2^*,\beta_2^*\to1$ and

$$\mathscr{L}_{m,\alpha_2^*}(u),\mathscr{L}_{m,\beta_2^*}(u)\to-qu^q\sum_{n=1}^\infty(-1)^n\frac{u^n}{n(n+q)n!}-u^q\int_0^u\frac{1-e^{-t}}{t}dt.$$

Then inequalities (3.20) imply a series expansion for $\Gamma(q,u)$, which is clearly absolutely convergent.

COROLLARY 3.11. *For $q,u>0$, we have*

$$\Gamma(q,u)=\frac{\Gamma(q+1)-u^q}{q}+u^q\int_0^u\frac{1-e^{-t}}{t}dt+q\sum_{n=1}^\infty(-1)^n\frac{u^{n+q}}{n(n+q)n!}. \tag{3.21}$$

In particular, putting $q\to0^+$, the following identity holds true:

$$E_1(u)+\gamma+\ln u=\int_0^u\frac{1-e^{-t}}{t}dt. \tag{3.22}$$

REMARK 3.12. *The expansion (3.21) seems to be new, and the identity (3.22) appeared in* [45, P. 230, (5.139)].

3.3. Monotonicity of $R_3(x)$

THEOREM 3.13. *For $p > 0$ and $m \in \mathbb{N}$, the function*

$$R_3(x) = \frac{x^{-p}\left(1 - e^{-x^p}\right) - \sum_{n=0}^{m-1} \frac{(-1)^n x^{np}}{(n+1)!}}{\frac{1}{x}\int_0^x e^{-t^p}dt - \sum_{n=0}^{m-1} \frac{(-1)^n}{(np+1)n!}x^{pn}}$$

is increasing (decreasing) on $(0,\infty)$ if $0 < p < (>)\, 1$ with

$$R_3\left(0^+\right) = \frac{mp+1}{m+1} \text{ and } R_3(\infty) = \frac{mp-p+1}{m}.$$

Therefore, the double inequality

$$(-1)^m Q_{m,\alpha_3}(x^p) < \frac{(-1)^m}{x}\int_0^x e^{-t^p}dt < (-1)^m Q_{m,\beta_3}(x^p) \tag{3.23}$$

holds for $p,x > 0$ with the best constants

$$\alpha_3 = \min\left(\frac{m+1}{mp+1}, \frac{m}{mp-p+1}\right) \text{ and } \beta_3 = \max\left(\frac{m+1}{mp+1}, \frac{m}{mp-p+1}\right),$$

where

$$Q_{m,\lambda}(x) = \lambda\frac{1-e^{-x}}{x} + \sum_{n=0}^{m-1}(-1)^n\left(\frac{n+1}{np+1} - \lambda\right)\frac{x^n}{(n+1)!}.$$

Proof. Let

$$f_3(x) = x^{1-p}\left(1 - e^{-x^p}\right) - \sum_{n=0}^{m-1}\frac{(-1)^n x^{np+1}}{(n+1)!},$$

$$g_3(x) = \int_0^x e^{-t^p}dt - \sum_{n=0}^{m-1}\frac{(-1)^n}{(np+1)n!}x^{pn+1}.$$

Then $R_3(x) = f_3(x)/g_3(x)$ with $f_3(0^+) = g_3(0^+) = 0$. Differentiation yields

$$f_3'(x) = pe^{-x^p} - \frac{p-1}{x^p}\left(1 - e^{-x^p}\right) - \sum_{n=0}^{m-1}(-1)^n\frac{(np+1)}{(n+1)!}x^{np} := f_{31}(x^p),$$

$$g_3'(x) = e^{-x^p} - \sum_{n=0}^{m-1}\frac{(-1)^n}{n!}x^{pn} := g_{31}(x^p),$$

where

$$f_{31}(y) = pe^{-y} - \frac{p-1}{y}\left(1 - e^{-y}\right) - \sum_{n=0}^{m-1}\frac{(-1)^n(np+1)}{(n+1)!}y^n,$$

$$g_{31}(y) = e^{-y} - \sum_{n=0}^{m-1}\frac{(-1)^n}{n!}y^n$$

with $y=x^p$.

Similar to show that $(-1)^{m+1}g_2'(x)$, $(-1)^{m+1}g_2(x)>0$ for $x>0$ in the proof of Theorem 3.7, we have that $(-1)^m g_3'(x)$, $(-1)^m g_3(x)>0$ for $x>0$ and $(-1)^m g_{31}(y)>0$ for $y>0$.

Set

$$f_{32}(y) = yf_{31}(y) = pye^{-y}+(p-1)e^{-y}-(p-1)-\sum_{n=0}^{m-1}\frac{(-1)^n(np+1)}{(n+1)!}y^{n+1},$$

$$g_{32}(y) = yg_{31}(y) = ye^{-y}-\sum_{n=0}^{m-1}\frac{(-1)^n}{n!}y^{n+1}.$$

Evidently, $g_{32}(y)\equiv g_{22}(y)$. As shown in the proof of Theorem 3.7, $(-1)^m g_{32}^{(k)}(y)>(-1)^m g_{32}^{(k)}(0^+)=0$ for $0\leqslant k\leqslant m$. Also, we easily see that $p-1+\sum_{n=0}^{m-1}\frac{(-1)^n(np+1)}{(n+1)!}y^{n+1}$ is the m-order Taylor polynomial of the function $pye^{-y}+(p-1)e^{-y}$, so $f_{32}^{(k)}(0^+)=0$ for $0\leqslant k\leqslant m$. Thus, if we prove the ratio $f_{32}^{(m)}/g_{32}^{(m)}$ is increasing (decreasing) on $(0,\infty)$ for $0<p<(>)1$, then by Lemma 2.1, so is $f_{32}^{(k)}/g_{32}^{(k)}$ for $0\leqslant k\leqslant m$, and so are $f_{31}/g_{31}=f_{32}/g_{32}$ and f_{21}/g_{21}. In view of

$$\left(\frac{f_3'(x)}{g_3'(x)}\right)' = \frac{d}{dy}\left(\frac{f_{31}(y)}{g_{31}(y)}\right)\times\frac{dy}{dx} = px^{p-1}\left(\frac{f_{31}(y)}{g_{31}(y)}\right)'>0,$$

by Lemma 2.1 again, it then follows that f_3/g_3 is strictly increasing on $(0,\infty)$.

Indeed, we have

$$\begin{aligned}
&\frac{f_{32}^{(m)}(y)}{g_{32}^{(m)}(y)}\\
&=\frac{p\left((-1)^m ye^{-y}+(-1)^{m-1}me^{-y}\right)+(-1)^m(p-1)e^{-y}-(-1)^{m-1}((m-1)p+1)}{(-1)^m(ye^{-y}+m(1-e^{-y}))}\\
&=\frac{p(ye^{-y}-me^{-y})+(p-1)e^{-y}+(m-1)p+1}{ye^{-y}+m(1-e^{-y})}\\
&=\frac{(mp-p+1)e^y+py-mp+p-1}{me^y+y-m},\\
&\left(\frac{f_{32}^{(m)}(y)}{g_{32}^{(m)}(y)}\right)'=-(p-1)\frac{ye^y-e^y+1}{(me^y+y-m)^2}\begin{cases}>0 \text{ if } p\in(0,1),\\ <0 \text{ if } p\in(1,\infty)\end{cases}
\end{aligned}$$

for $y>0$.

Finally, we compute the limits $R_3(0^+)$ and $R_3(\infty)$. We write $R_3(x)$ in the form of ratio of two power series

$$R_3(x)=\frac{\sum_{n=0}^{\infty}\frac{(-1)^n x^{np}}{(n+1)!}-\sum_{n=0}^{m-1}\frac{(-1)^n x^{np}}{(n+1)!}}{\sum_{n=0}^{\infty}\frac{(-1)^n}{(np+1)n!}x^{pn}-\sum_{n=0}^{m-1}\frac{(-1)^n}{(np+1)n!}x^{pn}}=\frac{\sum_{n=m}^{\infty}\frac{(-1)^n x^{np}}{(n+1)!}}{\sum_{n=m}^{\infty}\frac{(-1)^n}{(np+1)n!}x^{pn}},$$

which gives

$$\lim_{x\to 0^+} R_3(x) = \frac{(-1)^m/(m+1)!}{(-1)^m/((mp+1)m!)} = \frac{mp+1}{m+1}.$$

To compute $R_3(\infty)$, we divide the numerator and denominator in $R_3(x)$ by $x^{p(m-1)}$ to obtain

$$\lim_{x\to\infty} R_3(x) = \frac{-(-1)^{m-1}/m!}{-(-1)^m/((mp-p+1)(m-1)!)} = \frac{mp-p+1}{m}.$$

Then the desired double inequality follows, which completes the proof. □

Taking $m = 1,2$ in Theorem 3.13, the following corollary is immediate.

COROLLARY 3.14. *If $p > 1$, then the inequalities*

$$\frac{1-e^{-x^p}}{x^p} < \frac{1}{x}\int_0^x e^{-t^p}dt < \frac{p-1}{p+1} + \frac{2}{p+1}\frac{1-e^{-x^p}}{x^p},$$

$$2\frac{p-1}{2p+1} - \frac{p-1}{2(p+1)(2p+1)}x^p + \frac{3}{2p+1}\frac{1-e^{-x^p}}{x^p}$$

$$< \frac{1}{x}\int_0^x e^{-t^p}dt < \frac{p-1}{p+1} + \frac{2}{p+1}\frac{1-e^{-x^p}}{x^p}$$

hold for $x > 0$. They are reversed if $0 < p < 1$.

REMARK 3.15. *Corollary 3.14 indicates that our double inequality (3.23) greatly generalizes and extends Qi and Huang's inequality (1.4).*

By the changes of variables $q = 1/p$, $u = x^p \in (0,\infty)$ and the relation (3.13), we can rewrite Theorem 3.13 as follows.

THEOREM 3.16. *For $q > 0$ and $m \in \mathbb{N}$, the function*

$$R_3^*(u) = \frac{u^{-1}(1-e^{-u}) - \sum_{n=0}^{m-1}\frac{(-1)^n}{(n+1)!}u^n}{u^{-q}(\Gamma(q) - \Gamma(q,u)) - \sum_{n=0}^{m-1}\frac{(-1)^n}{(n+q)n!}u^n}$$

is increasing (decreasing) on $(0,\infty)$ if $0 < q > (<)\, 1$ with

$$R_3^*(0^+) = \frac{m+q}{m+1} \text{ and } R_3^*(\infty) = \frac{m-1+q}{m}.$$

Therefore, the double inequality

$$(-1)^{m+1}\mathcal{Q}_{m,\beta_3^*}(u) < (-1)^m\Gamma(q,u) - (-1)^m\frac{\Gamma(q+1)-u^q}{q} < (-1)^{m+1}\mathcal{Q}_{m,\alpha_3^*}(u) \tag{3.24}$$

holds for $q, x > 0$ with the best constants

$$\alpha_3^* = \min\left(\frac{m+1}{m+q}, \frac{m}{m-1+q}\right) \text{ and } \beta_3^* = \max\left(\frac{m+1}{m+q}, \frac{m}{m-1+q}\right),$$

where

$$\mathcal{Q}_{m,\lambda}(u) = \lambda u^{q-1}\left(1-u-e^{-u}\right) + \sum_{n=1}^{m-1}(-1)^n\left(\frac{n+1}{n+q}-\lambda\right)\frac{u^{n+q}}{(n+1)!}.$$

Letting $q \to 0^+$ in Theorem 3.16, we have

COROLLARY 3.17. *For $m \in \mathbb{N}$ with $m \geqslant 2$, the function*

$$R_3^{**}(u) = \frac{u^{-1}\left(1-e^{-u}\right) - \sum_{n=0}^{m-1}\frac{(-1)^n}{(n+1)!}u^n}{-E_1(u) - \gamma - \ln u - \sum_{n=1}^{m-1}\frac{(-1)^n}{nn!}u^n}$$

is decreasing on $(0,\infty)$ with

$$R_3^{**}\left(0^+\right) = \frac{m}{m+1} \text{ and } R_3^{**}(\infty) = \frac{m-1}{m}.$$

Therefore, the double inequality

$$(-1)^{m+1} sc\mathcal{Q}_{m,\beta_3^{**}}(u) < (-1)^m\left(\mathrm{Ei}(u)+\gamma+\ln u\right) < (-1)^{m+1} sc\mathcal{Q}_{m,\alpha_3^{**}}(u) \tag{3.25}$$

holds for $x > 0$ with the best constants

$$\alpha_3^{**} = \frac{m+1}{m} \text{ and } \beta_3^{**} = \frac{m}{m-1},$$

where

$$sc\mathcal{Q}_{m,\lambda}(u) = \lambda\frac{1-u-e^{-u}}{u} + \sum_{n=1}^{m-1}(-1)^n\left(\frac{n+1}{n}-\lambda\right)\frac{u^n}{(n+1)!}.$$

In particular, putting $m = 2$, we have

$$2\frac{e^{-u}+u-1}{u} < E_1(u) + \gamma + \ln u < \frac{1}{4}\frac{6e^{-u}+u^2+6u-6}{u}$$

for $u > 0$.

Letting $m \to \infty$ in Theorem 3.16, we obtain another series expansion for $\Gamma(q,u)$, which is also absolutely convergent.

COROLLARY 3.18. *For $q,u > 0$, we have*

$$\Gamma(q,u) = \frac{\Gamma(q+1)-u^q}{q} + u^{q-1}\left(e^{-u}+u-1\right) + (q-1)\sum_{n=1}^{\infty}\frac{(-1)^n u^{n+q}}{(n+q)(n+1)!}. \tag{3.26}$$

In particular, taking $q \to 0^+$, we have

$$E_1(u) + \gamma + \ln u = \frac{e^{-u}+u-1}{u} - \sum_{n=1}^{\infty}\frac{(-1)^n u^n}{n(n+1)!}. \tag{3.27}$$

4. Conclusions

We proved in this paper the monotonicity of the three ratios $R_i(x)$ $(i=1,2,3)$ given by (1.5), (1.7) and (1.8) on $(0,\infty)$. These monotonicity yield several best bounds for $x^{-1}\int_0^x e^{-t^p}dt$, for example, Corollary 3.3 refines Laforgia and Natalini's inequality (1.3), and inequalities (3.15) and Corollary 3.8 give an improvement and a generalization of inequality (1.3), while inequality (3.23) greatly generalizes and extends Qi and Huang's inequality (1.4). Moreover, inequalities (3.15) and (3.23) imply corresponding ones for the exponential integral $E_1(x)$, which are new and sharp. Interestingly, as by-products, we obtain two power series representations of $\Gamma(q,u)$, that are, (3.21) and (3.26), which seem to be new comers.

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