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ORLICZ–FRACTIONAL MAXIMAL OPERATORS ON WEIGHTED L^p SPACES

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Abstract. Necessary and sufficient conditions for weight norm inequalities on Lebesgue spaces to hold are given in the scale of Orlicz spaces for the fractional Orlicz maximal operators which generalizes the fractional maximal operators. A similar argument for the Orlicz maximal operators is due to Pérez, who generalizes for the Fefferman–Stein inequality. The main result is the Fefferman–Stein inequality for the fractional maximal operators of the Sawyer type and the Hardy–Littlewood–Sobolev type. In this paper, we establish that the L^p-boundedness and the Fefferman–Stein type inequality of Orlicz maximal operator are essentially equivalent to the Saywer type inequality for the fractional Orlicz maximal operators. These inequalities are stronger than the Hardy–Littlewood–Sobolev type inequalities. More generally, we consider several mixed strong type inequalities for the ordinary and generalized fractional Orlicz maximal operators. As an application, we investigate the weight norm inequalities of the commutator $[b, I_\alpha]$, where $b \in \mathrm{BMO}(\mathbb{R}^n)$, and I_α the fractional integral operator.

1. Introduction

This paper concerns the boundedness of the Hardy–Littlewood maximal function M, the fractional maximal function M_α, and the fractional integral operator I_α on weighted Lebesgue spaces. Here by an weight we mean a non-negative measurable function. Here and below for a measurable set E, the symbol χ_E denotes the characteristic function of $E \subset \mathbb{R}^n$ and the symbol $|E|$ denotes the Lebesgue measure of E. By a "cube" we mean a compact cube whose edges are parallel to cooridinate axes. Let us recall their definition:

DEFINITION 1.1. Let f be a measurable function defined on $\mathbb{R}^n$.

(1) Given $0 < \alpha < n$, as long as the definition makes sense, define the fractional integral operator I_α by:

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

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(2) Given $0 \leqslant \alpha < n$, define the fractional maximal operator M_α by:

$$M_\alpha f(x) := \sup_{Q:\text{cube}} \frac{\chi_Q(x)}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)|dy.$$

In particular, $M := M_0$ is called the Hardy–Littlewood maximal operator.

These operators M, M_α, and I_α are fundamental tools in harmonic analysis and potential theory (see [5, 13]).

Our aim in this paper is to obtain various weighted norm inequalities of the fractional Orlicz maximal operators M_B and $M_{B,\alpha}$. We will describe the Young functions, B_p-condition, the operators M_B and $M_{B,\alpha}$. As usual, a function $B : [0,\infty) \to [0,\infty)$ is said to be a Young function if it is continuous, convex and increasing and it satisfies $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. Define the B-average of a measurable function f over a cube Q by means of the Luxemburg norm.

DEFINITION 1.2. Given a Young function B and a cube Q, define the B-average of a measurable function f over a cube Q by

$$\|f\|_{B,Q} := \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f(x)|}{\lambda}\right) dx \leqslant 1\right\}. \tag{1.1}$$

By (1.1), we can define the Orlicz maximal operator and the fractional Orlicz maximal operator.

DEFINITION 1.3. Let B be a Young function, $0 \leqslant \alpha < n$, and $0 < u < \infty$.

(1) Define the fractional Orlicz maximal operators by

$$M_{B,\alpha}(f)(x) = M_{B,\alpha}f(x) := \sup_{Q:\text{cube}} \chi_Q(x)|Q|^{\frac{\alpha}{n}} \|f\|_{B,Q}, \tag{1.2}$$

where Q ranges over all cubes. If $\alpha = 0$, then abbreviate $M_{B,\alpha}$ to M_B.

(2) Let $0 < u < \infty$. Define the powered fractional Orlicz maximal operators by

$$M^{(u)}_{B,\alpha}(f)(x) := (M_{B,\alpha}(|f|^u)(x))^{\frac{1}{u}} \quad (x \in \mathbb{R}^n). \tag{1.3}$$

If $\alpha = 0$, then abbreviate $M^{(u)}_{B,\alpha}$ to $M^{(u)}_B$. Furthermore, if $B(t) \equiv t$ for $t \geqslant 0$, we write $M^{(u)}$ instead of $M^{(u)}_B$.

Let $0 \leqslant \alpha < n$, and let B be a Young function. It is easy to check the following inequality (see [1, p.108]): for a measurable function f

$$M_\alpha f(x) \leqslant C M_{B,\alpha} f(x). \tag{1.4}$$

So, $M_{B,\alpha}$ dominates M_α.

We study the weak $(1,1)$ of the Fefferman–Stein type inequality for M_B: The following theorem is a starting point of this paper:

THEOREM 1.4. *Let w be a weight and B be a Young function. Then for every $\lambda > 0$*

$$\int_{\{x\in\mathbb{R}^n : M_B f(x)>\lambda\}} w(x)dx \leqslant 3^n \int_{\mathbb{R}^n} B\left(\frac{4^n f(x)}{\lambda}\right) Mw(x)dx. \tag{1.5}$$

One of our aims in this paper is to extend Theorem 1.4.

Write $\log^+ a = \log(\max(1,a))$ for $a \geqslant 0$.

THEOREM 1.5. *Let B be a Young function, Q_0 be a cube and f be a measurable function supported on Q_0.*

(1) There exists $C > 0$ independent of f and Q such that

$$\frac{1}{|Q_0|}\int_{Q_0} M_B f(x)dx \leqslant C + \frac{C}{|Q_0|}\int_{Q_0} B(|f(x)|)\log^+ \frac{|f(x)|}{B^{-1}(1)}dx. \tag{1.6}$$

(2) Let $D(t) = B(t)\log^+ t$ for $t \geqslant 0$. Then

$$\frac{1}{|Q_0|}\int_{Q_0} M_B f(x)dx \leqslant C\|f\|_{D,Q_0}. \tag{1.7}$$

In particular, $M \circ M_B f \leqslant CM_D f$.

DEFINITION 1.6. A Young function B is said to satisfy the B_p-condition with $1 < p < \infty$, if

$$\int_1^\infty \frac{B(t)}{t^{p+1}}dt < \infty.$$

The set B_p collects all Young functions satisfying the B_p-condition.

The classes $\{B_p\}_{p\in(1,\infty)}$ of the set of functions are nested:

$$B_p \subsetneqq B_q \quad (1 < p < q < \infty). \tag{1.8}$$

Let $1 < p < \infty$ and B be a Young function. Perez showed that M_B is bounded on L^p if and only if $B \in B_p$; see [11, p.139].

The complementary Young function $\bar{B}$ of a Young function B is defined by

$$\bar{B}(t) := \sup_{s>0}(st - B(s)) \ (t > 0). \tag{1.9}$$

We next point out the following characterization of this class.

THEOREM 1.7. *Let $0 \leqslant \alpha < n$ and $1 < p < \frac{n}{\alpha}$. A Young function B belongs to B_p if and only if B satisfies one of the following equivalent conditions*:

(1) There is a constant C such that

$$\int_{\mathbb{R}^n} M_\alpha f(y)^p \frac{w(y)}{M_{\bar{B},\alpha}^{(\frac{1}{p})}(u)(y)} dy \leqslant C \int_{\mathbb{R}^n} |f(y)|^p \frac{Mw(y)}{u(y)} dy$$

for all measurable functions f, weights w and u such that u is positive almost everywhere.

(2) There is a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} M_{B,\alpha} f(x)^p w(x) dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p M_{\alpha p} w(x) dx$$

for all measurable functions f and weights w.

(3) There is a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} M_\alpha f(y)^p \frac{w(y)}{M_{\bar{B}}^{\left(\frac{1}{p}\right)}(u)(y)} dy \leqslant C \int_{\mathbb{R}^n} |f(y)|^p \frac{M_{\alpha p} w(y)}{u(y)} dy$$

for all measurable functions f, weights w and u such that u is positive almost everywhere.

Theorem 1.7 reinforces the following characterization by Perez [11, p.139].

PROPOSITION 1.8. *Let $1 < p < \infty$. A Young function B belongs to B_p if and only if B satisfies one of the following equivalent conditions*:

(4) There is a constant C such that

$$\int_{\mathbb{R}^n} M_B f(y)^p dy \leqslant C \int_{\mathbb{R}^n} |f(y)|^p dy$$

for all measurable functions f.

(5) There is a constant C such that

$$\int_{\mathbb{R}^n} M_B f(y)^p w(y) dy \leqslant C \int_{\mathbb{R}^n} |f(y)|^p Mw(y) dy$$

for all measurable functions f and all weights w.

(6) There is a constant C such that

$$\int_{\mathbb{R}^n} Mf(y)^p \frac{w(y)}{M_{\bar{B}}^{\left(\frac{1}{p}\right)}(u)(y)} dy \leqslant C \int_{\mathbb{R}^n} |f(y)|^p \frac{Mw(y)}{u(y)} dy$$

for all measurable functions f, and all weights w and u such that u is positive almost everywhere.

Remark that Theorem 1.7 (1), (2) and (3) with $\alpha = 0$ recapture Proposition 1.8 (4), (5) and (6), respectively.

For each $1 \leqslant p \leqslant \infty$, p' will denote the dual exponent of p, i.e., $p' = \frac{p}{p-1}$ with the usual modifications $1' = \infty$ and $\infty' = 1$. We remark that Proposition 1.8 gives the following estimate on the $[p'+1]$-fold iterated maximal operator $M^{[p'+1]}$ of M. As usual $[u]$ denotes the integer part of $u \in \mathbb{R}$.

COROLLARY 1.9. [11, p.139] *Let $1 < p < \infty$. Then for all measurable functions f and a measurable function u which is positive almost everywhere,*

$$\int_{\mathbb{R}^n} Mf(x)^p \left(M^{[p'+1]}u(x)\right)^{1-p} dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p u(x)^{1-p} dx.$$

As is pointed out in [11], the example of $f = u = \chi_{(0,1)^n}$ shows that Corollary 1.9 is sharp in the sense that we can not replace $M^{[p'+1]}$ by $M^{[p']}$.

Cruz-Uribe, Martell and Pérez obtained a necceary and sufficient condition of the weak boundedness; see [2, p.100, Proposition 5.6].

PROPOSITION 1.10. *Let $1 < p < \infty$. For a Young function B, the following are eqivalent.*

(W-1) There exists a constant $C > 0$ such that for sufficiently large $t > 0$, the growth condition

$$B(t) \leqslant Ct^p : \tag{1.10}$$

is satisfied.

(W-2) The maximal operator M_B is weak L^p-bounded, that is, there exists a constant $C > 0$ such that for all measurable functions f and $\lambda > 0$

$$|\{x \in \mathbb{R}^n : M_B f(x) > \lambda\}| \leqslant \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p dx.$$

According to [1], if $B \in B_p$, then B satisfies condition (1.10).

Recall that a function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing if there exists a constant $C > 0$ such that $C\phi(s) \geqslant \phi(t)$ for all $0 < t < s < \infty$. In terms of weights, we can further characterize the conditions of the weak boundedness of the fractional Orlicz maximal operators. Here and below, given a weight w and a measurable set E, let $w(E) := \int_E w(x)\,dx$.

THEOREM 1.11. *Let $0 \leqslant \alpha < n$, and let $1 < p \leqslant q < \frac{n}{\alpha}$ satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. For a Young function B, assume that*

(1.11)

$$t \in (0,\infty) \mapsto t^{-n/\alpha} B(t) \in (0,\infty) \text{ is almost decreasing and that } t^{-n/\alpha} B(t) \to 0 \ (t \to \infty).$$

Then a Young function B satisfies the growth condition 1 if and only if B satisfies one of the following equivalent conditions for all measurable functions f and weights w and u such that u does not vanish almost everywhere:

(W-3) There is a constant $C>0$ independent of f and w such that for every $\lambda>0$,

$$w(\{x\in\mathbb{R}^n : M_{B,\alpha}f(x)>\lambda\}) \leqslant \frac{C}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^p M_{\alpha p}w(x)dx.$$

(W-4) There is a constant $C>0$ independent of f, w and u such that for every $\lambda>0$,

$$w\left(\left\{x\in\mathbb{R}^n : M_\alpha f(x)>\lambda M_{\overline{B}}\left(u^{\frac{1}{p}}\right)(x)\right\}\right) \leqslant \frac{C}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^p \frac{M_{\alpha p}w(x)}{u(x)}dx.$$

(W-5) There is a constant $C>0$ independent of f such that for every $\lambda>0$,

$$|\{x\in\mathbb{R}^n : M_{B,\alpha}f(x)>\lambda\}| \leqslant \left(\frac{C}{\lambda}\left(\int_{\mathbb{R}^n}|f(x)|^p dx\right)^{\frac{1}{p}}\right)^q.$$

(W-6) There is a constant $C>0$ independent of f and w such that for every $\lambda>0$,

$$w(\{x\in\mathbb{R}^n : M_{B,\alpha}f(x)>\lambda\}) \leqslant \left(\frac{C}{\lambda}\left(\int_{\mathbb{R}^n}|f(x)|^p Mw(x)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}\right)^q.$$

(W-7) There is a constant $C>0$ independent of f, w and u such that for every $\lambda>0$,

$$\begin{aligned} &w\left(\left\{x\in\mathbb{R}^n : M_\alpha f(x)>\lambda M_{\overline{B}}\left(u^{\frac{1}{q}}\right)(x)\right\}\right) \\ &\leqslant \left(\frac{C}{\lambda}\left(\int_{\mathbb{R}^n}|f(x)|^p \left(\frac{Mw(x)}{u(x)}\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}\right)^q. \end{aligned}$$

(W-8) There is a constant $C>0$ independent of f, w and u such that for every $\lambda>0$,

$$w\left(\left\{x\in\mathbb{R}^n : M_\alpha f(x)>\lambda M_{\overline{B},\alpha}\left(u^{\frac{1}{p}}\right)(x)\right\}\right) \leqslant \frac{C}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^p \frac{Mw(x)}{u(x)}dx.$$

In particular, a Young function B belongs to B_p if and only if B satisfies the following condition for all measurable functions f and all weights w:

(W-9) There is a constant $C>0$ independent of the measurable functions f and w such that for every $\lambda>0$,

$$w(\{x\in\mathbb{R}^n : M_B f(x)>\lambda\}) \leqslant \frac{C}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^p Mw(x)dx.$$

We discuss relation between these weighted inequalities and the existing result.

We start with the Hardy–Littlewood–Sobolev theorem for the fractional maximal operator M_α (see [5, p.89]).

PROPOSITION 1.12. *Let $0 \leqslant \alpha < n$, and let $1 \leqslant p \leqslant q < \infty$ satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.*

(A) *If $p > 1$, then there exists a constant $C = C(n,p)$ such that for all measurable functions f,*

$$\|M_\alpha f\|_{L^q(\mathbb{R}^n)} \leqslant C \|f\|_{L^p(\mathbb{R}^n)}.$$

(B) *If $p = 1$, then there exists a constant $C = C(n,1)$ such that for $\lambda > 0$ and all measurable functions f,*

$$|\{x \in \mathbb{R}^n : M_\alpha f(x) > \lambda\}|^{\frac{1}{q}} \leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, dx.$$

We next recall the Fefferman–Stein dual inequality (see [5, p.37]):

PROPOSITION 1.13.

(A) *If $1 < p < \infty$, then there exists a constant $C = C(n,p) > 0$ such that for all measurable functions f and all weights w, $\|Mf\|_{L^p(w)} \leqslant C \|f\|_{L^p(Mw)}$.*

(B) *There exists a constant $C = C(n,1) > 0$ such that for all $\lambda > 0$ and for all measurable functions f and all weights w, $w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leqslant \dfrac{C}{\lambda}$ $\displaystyle\int_{\mathbb{R}^n} |f(x)| Mw(x) dx$.*

We can mix Propositions 1.12 and 1.13. We add a statement with respect to a Young function B:

PROPOSITION 1.14. *Let $0 \leqslant \alpha < n$, and let $1 < p < \frac{n}{\alpha}$.*

(1) [12] *There exists a constant $C = C(n,p)$ such that for all non-negative measurable functions f and w*

$$\|M_\alpha f\|_{L^p(w)} \leqslant C \|f\|_{L^p(M_{\alpha p} w)}.$$

(2) [2, p.115] *There exists a constant $C = C(n,1)$ such that for all non-negative measurable functions f and w and $\lambda > 0$,*

$$w(\{x \in \mathbb{R}^n : M_\alpha f(x) > \lambda\}) \leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_\alpha w(x) dx.$$

(3) *Let B be a Young function. Define $q \in (p, \infty)$ by $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Assume in addition that $B(t) \leqslant C t^u$ for some $1 \leqslant u < q$. Then there is a constant $C > 0$ such that*

$$\left(\int_{\mathbb{R}^n} M_{B,\alpha} f(x)^q dx \right)^{\frac{1}{q}} \leqslant C \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}. \tag{1.12}$$

for all measurable functions f.

The proof of (3) is a consequence of the Hardy–Littlewood–Sobolev inequality. We invoke the following proposition from [9], which extends Proposition 1.12 in that the operator M_α or $M_{B,\alpha}$ in Proposition 1.12 is replaced by a more general operator of $M_{B,\alpha}$ (see [1, pp.115–116]).

PROPOSITION 1.15. *Let $0 \leqslant \alpha < n$, $1 < p < \frac{n}{\alpha}$. Suppose that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then conclusion (1.12) in Proposition 1.14 holds if $B \in B_p$.*

REMARK 1.

(1) Suppose that $M_{B,\alpha} : L^p \to L^q$ is bounded. Then the dilation transform forces q to satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, which justifies the assumption $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.
(2) In Proposition 1.15, the condition $B \in B_p$ is best possible: Let $\varepsilon > 0$. Then as the example of $B(t) = t^{p+\varepsilon-\delta}$ shows, there exists a Young function B such that $B \in B_{p+\varepsilon}$ but that $M_{B,\alpha}(\chi_{Q(0,1)}) \notin L^q(\mathbb{R}^n)$.

It might be interesting to ask ourselves whether the condition $B \in B_p$ is the necessary condition of $M_{B,\alpha} : L^p \to L^q$ based on this remark. In this paper, we also investigate the problem.

Going through an argument similar to Theorem 1.7, we will develop Proposition 1.15.

THEOREM 1.16. *Let $0 \leqslant \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $1 < q < \infty$. Consider three conditions with respect to a Young function B,*

(8) There is a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} M_{B,\alpha} f(x)^q w(x) dx\right)^{\frac{1}{q}} \leqslant C \left(\int_{\mathbb{R}^n} |f(x)|^p (Mw(x))^{\frac{p}{q}} dx\right)^{\frac{1}{p}}.$$

for all measurable functions f and w.

(9) There is a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} M_\alpha f(x)^q \frac{w(x)}{M_{\bar{B}}^{\left(\frac{1}{q}\right)} u(x)} dx\right)^{\frac{1}{q}} \leqslant C \left(\int_{\mathbb{R}^n} |f(x)|^p \left(\frac{Mw(x)}{u(x)}\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

for all measurable functions f and weights w and u such that u is positive almost everywhere.

(10) $B^{\frac{\alpha q}{n}+1} \in B_q$

Then (3) $\Leftrightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10) hold.

Having clarified the characterization of the class B_p which we are going to prove, we will obtain general inequalities related to $M_{B,\alpha}$. To proceed further we recall the A_p-condition and the A_∞-condition. Here and below for a measurable function F defined on Q, we write

$$\fint_Q F(x)dx := \frac{1}{|Q|}\int_Q F(x)dx.$$

DEFINITION 1.17. Let $1 < p < \infty$. One says that a weight w satisfies the A_p-condition if

$$[w]_{A_p} := \sup_{Q:\text{cube}} \left(\fint_Q w(x)dx\right)\left(\fint_Q w(x)^{-\frac{p'}{p}}dx\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$. Write A_p for the set of all weights satisfying the A_p-condition. Finally, define $A_\infty(\mathbb{R}^n) := \bigcup_{p>1} A_p(\mathbb{R}^n)$.

It is well known that the class A_p emerged from the following celebrated theorem:

THEOREM 1.18. ([8]) *Let $1 < p < \infty$. Then the following statements are equivalent:*

- *(A) $w \in A_p$.*
- *(B) There exists a constant $C > 0$ such that $\|Mf\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)}$ for all measurable functions f.*

For the proof of Theorem 1.18 we also refer to the textbooks [5, 6].

Going back to the general estimates of $M_{B,\alpha}$, we note that estimates in Theorem 1.16 are useful since they will give us further estimates.

THEOREM 1.19. *Let $0 \leqslant \alpha < n$, $1 < p < \frac{n}{\alpha}$, A, B and C be Young functions such that*

$$A^{-1}(t)C^{-1}(t) \leqslant B^{-1}(t). \tag{1.13}$$

Moreover we assume that $V \in A_\infty(\mathbb{R}^n)$,

$$K_1 := \sup_{Q:\text{cube}} V(Q)^{\frac{\alpha}{n}}\left(\frac{1}{V(Q)}\int_Q v(x)V(x)dx\right)^{\frac{1}{p}}\left\|w^{-\frac{1}{p}}\right\|_{A,Q} < \infty \tag{1.14}$$

and

$$M_C : L^p(W) \to L^p(V). \tag{1.15}$$

Then

$$\int_{\mathbb{R}^n} M_{V,B,\alpha}f(x)^p v(x)V(x)dx \leqslant C\int_{\mathbb{R}^n} |f(x)|^p w(x)W(x)dx, \tag{1.16}$$

where $M_{V,B,\alpha}f(x) := \sup_{Q:\text{cube}} \chi_Q(x)V(Q)^{\frac{\alpha}{n}}\|f\|_{B,Q}$ for $x \in \mathbb{R}^n$.

As a special case in Theorem 1.19, we get the following general weight estimate:

COROLLARY 1.20. *Let* $0 \leqslant \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* B *be a Young function. Moreover we assume that* $V \in A_\infty(\mathbb{R}^n)$ *and that*

$$M_B : L^p(W) \to L^p(V). \tag{1.17}$$

Then

$$\left(\int_{\mathbb{R}^n} M_{V,t,\alpha} f(x)^p |g(x)| V(x) dx\right)^{\frac{1}{p}} \leqslant C \left(\int_{\mathbb{R}^n} |f(x)|^p M_{V,\alpha p} g(x) W(x) dx\right)^{\frac{1}{p}}, \tag{1.18}$$

where $M_{V,\alpha p} g(x) := \sup_{Q:\text{cube}} \chi_Q(x) V(Q)^{\frac{\alpha p}{n}-1} \int_Q |g(y)| V(y) dy$ *for* $x \in \mathbb{R}^n$.

To prove Corollary 1.20, in Theorem 1.19 we simply take $v \equiv |g|$, $w \equiv M_{V,\alpha p} g$, and we let $A^{-1} = \dfrac{\bar{B}^{-1}}{2}$ and $C = B$, so that $A^{-1}(t) C^{-1}(t) = \dfrac{\bar{B}^{-1}(t) B^{-1}(t)}{2} \leqslant t$.

Moreover, taking $V \equiv W \equiv 1$, we learn that Corollary 1.20 boils down to Proposition 1.14.

Furthermore, we obtain the following result:

THEOREM 1.21. *Let* $0 \leqslant \alpha < n$, $1 < p \leqslant q < \frac{n}{\alpha}$, *and let* v, w *be weights such that* w *is positive almost everywhere. Suppose that* A, B *and* C *are Young functions which satisfy* (1.13). *Moreover we assume that* $V \in A_\infty(\mathbb{R}^n)$, *that*

$$K_2 := \sup_{Q:\text{cube}} \left(\frac{1}{V(Q)} \int_Q v(x) V(x) dx\right)^{\frac{1}{q}} \left\| w^{-\frac{1}{p}} \right\|_{A,Q} < \infty \tag{1.19}$$

and that

$$M_{C,\alpha} : L^p(W) \to L^q(V). \tag{1.20}$$

Then for all measurable functions f

$$\left(\int_{\mathbb{R}^n} M_{B,\alpha} f(x)^q v(x) V(x) dx\right)^{\frac{1}{q}} \leqslant C \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) W(x) dx\right)^{\frac{1}{p}}. \tag{1.21}$$

As a special case in Theorem 1.21, we get Corollary 1.22 below by taking $v = |g|$, $w = (M_V g)^{\frac{p}{q}}$, and let $A^{-1} = \dfrac{\bar{B}^{-1}}{2}$ and $C = B$. Note that $A^{-1}(t) C^{-1}(t) = \dfrac{\bar{B}^{-1}(t) B^{-1}(t)}{2}$ $\leqslant t$.

COROLLARY 1.22. *Let* $0 \leqslant \alpha < n$, $1 < p \leqslant q < \frac{n}{\alpha}$ *and* B *be a Young function. Let* V *and* W *be weights. Moreover we assume that* $V \in A_\infty(\mathbb{R}^n)$ *and*

$$M_{B,\alpha} : L^p(W) \to L^q(V). \tag{1.22}$$

Then for all measurable functions f *and* g

$$\left(\int_{\mathbb{R}^n} M_\alpha f(x)^q |g(x)| V(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n} |f(x)|^p M_V(g)(x)^{\frac{p}{q}} W(x)dx\right)^{\frac{1}{p}}. \tag{1.23}$$

Moreover, taking $V \equiv W \equiv 1$, $W = |g|$ and

$$\alpha = \frac{n}{p} - \frac{n}{q},$$

we see that Corollary 1.22 reduces $\|M_\alpha f\|_{L^q(w)} \leqslant C\|f\|_{L^p\left((Mw)^{\frac{p}{q}}\right)}$. Meanwhile, as the other case in Theorem 1.21, we can recover (8) $\Rightarrow$ (9). In fact, replacing weight functions V by w, v by $\left(M_{\overline{B}}^{(\frac{1}{q})} u\right)^{-1}$, W by $(Mw)^{\frac{p}{q}}$, w by u^{-1}, A by $\overline{B}$ and C by B in Theorem 1.21, respectively, then, (1.20) reduces matters to (8) and the estimate of

$$K_2 = \sup_{Q:\text{cube}} \left(\frac{1}{w(Q)} \int_Q \frac{w(x)}{M_{\overline{B}}^{(\frac{1}{q})} u(x)} dx\right)^{\frac{1}{q}} \left\|u^{\frac{1}{q}}\right\|_{\overline{B},Q}.$$

Since $M_{\overline{B}}^{(\frac{1}{q})} u(x) \geqslant \left\|u^{\frac{1}{q}}\right\|_{\overline{B},Q}^q$ for every cube Q and $x \in Q$, we obtain

$$\sup_{Q:\text{cube}} \left(\frac{1}{w(Q)} \int_Q \frac{w(x)}{M_{\overline{B}}^{(\frac{1}{q})} u(x)} dx\right)^{\frac{1}{q}} \left\|u^{\frac{1}{q}}\right\|_{\overline{B},Q} \leqslant 1.$$

Applying Theorem 1.21, we get estimate (9).

For the sufficient condition, the following result can be found in [3, Theorem 3.3 in p.428].

PROPOSITION 1.23. *Given* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and define* q *by:* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *let* B *be a Young function satisfying* (1.11) *and* $B^{\frac{q}{p}} \in B_q$. *Then for all measurable functions* f

$$\|M_{B,\alpha} f\|_{L^q} \leqslant C\|f\|_{L^p}.$$

REMARK 2. It seems that [3, Theorem 3.3 in p.428] does not seem to contain condition (1.11). However, Proposition 1.23 seems to need condition (1.11). We prove Proposition 1.23 in Section 5.

Meanwhile, the weighted version of Proposition 1.15 is in [1, p.115].

PROPOSITION 1.24. *Let $0 \leqslant \alpha < n$, $1 < p \leqslant q < \frac{n}{\alpha}$. Let A, B and C be Young functions such that (1.13) holds, and that C is doubling and satisfies the B_p condition. If (u,v) is a pair of weights such that for every cube Q,*

$$|Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\fint_Q u(x)^q dx \right)^{\frac{1}{q}} \left\| v^{-1} \right\|_{A,Q} < \infty, \tag{1.24}$$

then, for all $f \in L^p(v^p)$,

$$\left(\int_{\mathbb{R}^n} M_{B,\alpha} f(x)^q u(x)^q dx \right)^{\frac{1}{q}} \leqslant C \|f\|_{L^p(v^p)}. \tag{1.25}$$

According to [2, p.115], the following proposition does not seem to have been stated explicitly in the literature, but the proof is almost identical to that for the analogolous result for the Hardy–Littlewood maximal operator.

PROPOSITION 1.25. *Let $0 \leqslant \alpha < n$ and $1 \leqslant p \leqslant q < \infty$. For a pair of weights (u,v), the following are equivalent:*

(1)

$$\sup_{Q:\text{cube}} |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\fint_Q u(x)^q dx \right)^{\frac{1}{q}} \left(\fint_Q v(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty. \tag{1.26}$$

When $p = 1$, $\left(\fint_Q v(x)^{-p'} dx \right)^{\frac{1}{p'}}$ is understood as $\operatorname{ess\,sup}_{x \in Q} \frac{1}{v(x)}$.

(2) For all $\lambda > 0$ and measurable functions f defined on $\mathbb{R}^n$,

$$u^q\left(\{x \in \mathbb{R}^n : M_\alpha f(x) > \lambda\}\right) \leqslant C \left(\frac{1}{\lambda} \|f\|_{L^p(v^p)} \right)^q. \tag{1.27}$$

For $p > 1$, the fractional Orlicz maximal function $M_{B,\alpha}$ is also of weak type (p,q) provided that B satifies condition (1.10):

Let $\alpha = \frac{n}{p} - \frac{n}{q}$. As a spacial case of Proposition 1.23, we obtain the following:

REMARK 3. Let $1 < p \leqslant q < \infty$ and B be a Young function. If $B^{\frac{q}{p}} \in B_q$, then $M_{B,\frac{n}{p}-\frac{n}{q}} : L^p \to L^q$.

By Remark 3, we can improve Proposition 1.24:

THEOREM 1.26. *Let $0 \leqslant \alpha < n$, $1 < p \leqslant q < \frac{n}{\alpha}$. Let A, B and C be Young functions such that $A^{-1}(t)C^{-1}(t) \leqslant B^{-1}(t)$, and that $C^{\frac{q}{p}} \in B_q$. If a pair of weights (u,v) satisfies (1.24). Then for all $f \in L^p(v^p)$, inequality (1.25) holds.*

Since we need only modify slightly the proof of Proposition 1.24 to prove Theorem 1.26, we omit the proof.

The following notation is used: The letter C always denotes a positive constant, which is independent of the essential parameters, but is not necessary the same at each occurrence. We will use the following observation on the class A_∞: Assume that $V \in A_\infty(\mathbb{R}^n)$. Then V satisfies doubling condition:

$$V(2Q) \leqslant CV(Q) \quad (Q : \text{cube}) \tag{1.28}$$

The rest of this paper is organized as follows. In Section 2, we list some lemmas needed in this paper. Section 3 is devoted to the proof of the main results. In Section 4, we consider the applications: We prove the boundedness of the commutator $[b, I_\alpha]$, where $b \in \mathrm{BMO}(\mathbb{R}^n)$. Section 5 is an appendix: We prove Proposition 1.23 in Section 5.

2. Some lemmas

We invoke the properties of Young functions to prove main results.

LEMMA 2.1. *Write*

$$\overline{D}^+ B(t) = \limsup_{h \to 0^+} \frac{B(t+h) - B(t)}{h}.$$

The Young function B satisfies that

$$B(t) \cong t\overline{D}^+ B(t) \quad (t > 0). \tag{2.1}$$

and

$$aB(t) \leqslant B(at) \quad \text{and} \quad B\left(\frac{t}{a}\right) \leqslant \frac{B(t)}{a} \quad (a > 1). \tag{2.2}$$

Inequalities (2.2) *entail*

$$\frac{B(t)}{t} \leqslant \frac{B(s)}{s} \quad (0 < t < s). \tag{2.3}$$

The functions B and $\bar{B}$ satisfy the following inequality:

$$t \leqslant B^{-1}(t) \cdot \bar{B}^{-1}(t) \leqslant 2t \quad (t > 0). \tag{2.4}$$

It is also well known that generalized Hölder's inequality holds:

$$⨍_Q |f(y)g(y)|\,dy \leqslant 2\,\|f\|_{B,Q}\,\|g\|_{\bar{B},Q}. \tag{2.5}$$

More generally, if A, B and C are Young functions such that for all $t > 0$, $A^{-1}(t)C^{-1}(t) \leqslant B^{-1}(t)$, then

$$\|fg\|_{B,Q} \leqslant 2\,\|f\|_{A,Q}\,\|g\|_{C,Q}. \tag{2.6}$$

We need the following two-sided estimate to prove Theorem 1.7.

LEMMA 2.2. *Let* $0 \leqslant \alpha < n$. *Given a Young function* B, *Whenever* $|x| > 1$,

$$M_{\bar{B},\alpha}(\chi_{Q(0,1)})(x) \cong |x|^{\alpha}\,\bar{B}^{-1}\left(|x|^{n}\right)^{-1}, \tag{2.7}$$

where the center of the cube $Q(0,1)$ *is the origin and the side-length* $\ell(Q(0,1))$ *equals* 1.

Proof. By definition,

$$M_{\bar{B},\alpha}(\chi_{Q(0,1)})(x) = \sup_{Q:\text{cube}} \chi_Q(x)\ell(Q)^{\alpha}\bar{B}^{-1}\left(\frac{|Q|}{|Q \cap Q(0,1)|}\right)^{-1}.$$

We omit further details since (2.7) can be obtained from a geometric observation; see Figure 1.

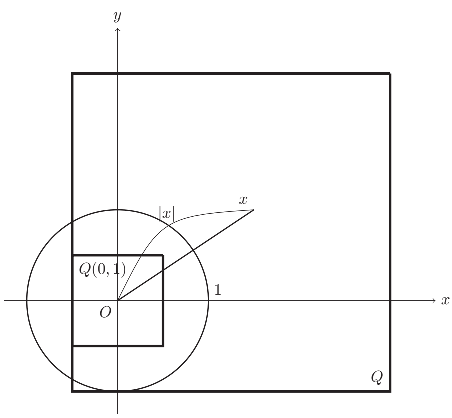

FIGURE 1. (In the case of $n = 2$)

□ □

REMARK 4. In particular taking $\bar{B}(t) = t$ in (2.7), we obtain $M_{\alpha}(\chi_{Q(0,1)})(x) \cong \dfrac{1}{|x|^{n-\alpha}}$ $(|x| > 1)$.

Denote by $\mathscr{D} = \mathscr{D}(\mathbb{R}^n)$ the set of all dyadic cubes. To analyze simply the operator $M_{V,B,\alpha}$, we study the equivalent of the dyadic version:

LEMMA 2.3. *Let* $0 \leqslant \alpha < n$. *Given a Young function* B *and a weight function* V, *the following equivalence holds:*

$$M_{V,B,\alpha,3\mathscr{D}}f(x) \leqslant M_{V,B,\alpha}f(x) \lesssim M_{V,B,\alpha,3\mathscr{D}}f(x),$$

where

$$M_{V,B,\alpha,3\mathscr{D}}f(x) := \sup_{x\in Q\in\mathscr{D}(\mathbb{R}^n)} V(3Q)^{\frac{\alpha}{n}} \|f\|_{B,3Q}.$$

Proof. It suffices to verify $M_{V,B,\alpha}f(x) \leqslant CM_{V,B,\alpha,3\mathscr{D}}f(x)$. Fix a point $x\in\mathbb{R}^n$. For every cube $Q\subset\mathbb{R}^n$ such that $Q\ni x$, there exists a unique integer $k\in\mathbb{Z}$ such that $2^{-(k+1)n}\leqslant|Q|<2^{-kn}$. Then we can choose dyadic cubes J_i $(i=1,2,\dots,2^n)$ such that $|J_i|=2^{-kn}$ and the dyadic cubes J_i $(i=1,2,\dots,2^n)$ cover Q (see Figure 2).

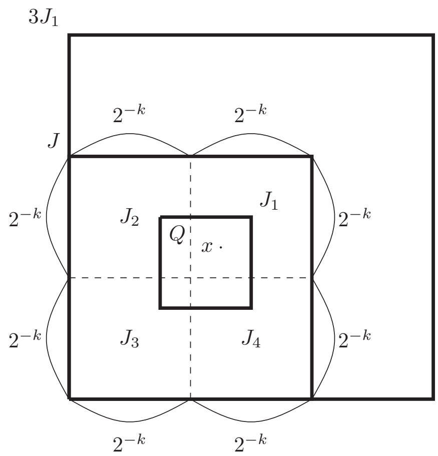

FIGURE 2. (In the case of $n=2$)

That is,

$$Q\subset\bigcup_{i=1}^{2^n}J_i \tag{2.8}$$

and

$$|Q|<|J_i|\leqslant 2^n|Q|. \tag{2.9}$$

Hence,

$$V(Q)^{\frac{\alpha}{n}}\|f\|_{B,Q}=V(Q)^{\frac{\alpha}{n}}\|f\chi_J\|_{B,Q}, \tag{2.10}$$

where $J:=\bigcup_{i=1}^{2^n}J_i$. Obviously, for $i=1,2,\dots,2^n$,

$$|J|=2^n|J_i|. \tag{2.11}$$

By (2.8),

$$V(Q)^{\frac{\alpha}{n}}\|f\chi_J\|_{B,Q}\leqslant V(Q)^{\frac{\alpha}{n}}\sum_{i=1}^{2^n}\|f\chi_{J_i}\|_{B,Q}. \tag{2.12}$$

By (2.9),

$$\|f\chi_{J_i}\|_{B,Q} \leqslant \inf\left\{\lambda > 0 : \frac{2^n}{|J_i|}\int_{J_i} B\left(\frac{f(x)}{\lambda}\right)dx \leqslant 1\right\}. \tag{2.13}$$

Since $J_i \subset 3J_1$,

(2.14)

$$\inf\left\{\lambda > 0 : \frac{2^n}{|J_i|}\int_{J_i} B\left(\frac{f(x)}{\lambda}\right)dx \leqslant 1\right\} \leqslant \inf\left\{\lambda > 0 : \frac{6^n}{|3J_1|}\int_{3J_1} B\left(\frac{f(x)}{\lambda}\right)dx \leqslant 1\right\}.$$

By (2.2),

$$\inf\left\{\lambda > 0 : \frac{6^n}{|3J_1|}\int_{3J_1} B\left(\frac{f(x)}{\lambda}\right)dx \leqslant 1\right\} \leqslant 6^n \|f\|_{B,3J_1}. \tag{2.15}$$

Estimates (2.10)–(2.15) imply

$$V(Q)^{\frac{\alpha}{n}} \|f\|_{B,Q} \leqslant 6^n V(3J_1)^{\frac{\alpha}{n}} \|f\|_{B,3J_1}.$$

Since the cube $J_1 \ni x$ is a dyadic cube, we obtain Theorem 2.3. □ □

We invoke the following Lemma in [11, p.146].

LEMMA 2.4. *Suppose that B is a Young function, and that f is a non-negative bounded function with compact support. For each $\lambda > 0$, let $\Omega_\lambda = \{x \in \mathbb{R}^n : M_B f(x) > \lambda\}$. Then if Ω_λ is not empty, we have*

$$\Omega_\lambda \subset \bigcup_{j=1}^{\infty} 3Q_j, \tag{2.16}$$

where $\{Q_j\}_{j=1}^{\infty}$ is a family of non-overlapping maximal dyadic cubes satisfying

$$\frac{\lambda}{4^n} < \|f\|_{B,Q_j} \leqslant \frac{\lambda}{2^n} \tag{2.17}$$

for each integer j. Furthermore, it follows that

$$|\Omega_\lambda| \leqslant C_0 \int_{\left\{y\in\mathbb{R}^n : f(y) > \frac{\lambda}{2^n}\right\}} B\left(\frac{f(y)}{\lambda}\right)dy. \tag{2.18}$$

Lemma 2.4 gives us the weak $(1,1)$ of the Fefferman–Stein type inequality for $M_B f$ (see Theorem 1.4). Similarly, we use the following lemma without weight.

LEMMA 2.5. *Let $0 \leqslant \alpha < n$. Given a Young function B, suppose f is a non-negative function such that $|3Q|^{\frac{\alpha}{n}} \|f\|_{B,3Q}$ tends to zero as $|Q|$ tends to infinity. Then given $a > \max\{2\cdot 6^n C_0, 2^n\}$, where C_0 is the constant in inequality (2.18), for each $k \in \mathbb{Z}$ there exists a disjoint collection of maximal dyadic cubes $\{Q_{k,j}\}$ such that for each j,*

$$a^k < \left|3Q_{k,j}\right|^{\frac{\alpha}{n}} \|f\|_{B,3Q_{k,j}} \leqslant 2^n a^k. \tag{2.19}$$

Moreover,

$$\left\{x \in \mathbb{R}^n : M_{B,\alpha,3\mathscr{D}} f(x) > a^k\right\} = \bigcup_j Q_{k,j} \tag{2.20}$$

holds. Further, let $D_k = \bigcup_j Q_{k,j}$ *and* $E_{k,j} = Q_{k,j} \setminus D_{k+1}$. *Then the* $E_{k,j}$ *'s are pairwise disjoint for all* j *and* k. *Moreover*

$$|Q_{k,j}| \leqslant 2|E_{k,j}| \tag{2.21}$$

holds.

We use the following calculation to show that our results are sharp.

LEMMA 2.6. *Let* $0 \leqslant \alpha < n$, *and let* $w_0(x) = (1+|x|)^{-\alpha}, x \in \mathbb{R}^n$. *Then*

$$M_\alpha w_0 \in L^\infty$$

Proof. Recall that $\mathscr{M}_q^p(\mathbb{R}^n)$ is the set of all measurable functions f for which

$$\|f\|_{\mathscr{M}_q^p} = \sup_{Q \in \mathscr{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q)}$$

is finite. It is well known that $w_0 \in \mathscr{M}_1^{n/\alpha}(\mathbb{R}^n)$. Since M_α maps $\mathscr{M}_1^{n/\alpha}(\mathbb{R}^n)$ boundedly to $L^\infty(\mathbb{R}^n)$, it follows that $M_\alpha w_0 \in L^\infty$. □ □

We use the Hölder inequality to decompose the fractional maximal operator.

LEMMA 2.7. *Let* $0 \leqslant \alpha < n$, B *be a Young function. Then for all measurable functions* $f : \mathbb{R}^n \to \mathbb{C}$ *and* $u : \mathbb{R}^n \to (0,\infty)$, $M_\alpha f \leqslant 2M_B\left(fu^{-\frac{1}{p}}\right) \cdot M_{\bar{B},\alpha}\left(u^{\frac{1}{p}}\right)$.

Proof. Let $x \in \mathbb{R}^n$ be fixed. Note that

$$M_\alpha f(x) = M_\alpha\left(fu^{-\frac{1}{p}} \cdot u^{\frac{1}{p}}\right)(x) = \sup_{Q:\text{cube}} \chi_Q(x)\ell(Q)^\alpha \fint_Q |f(z)|u(z)^{-\frac{1}{p}}u(z)^{\frac{1}{p}}\,dz.$$

By (2.5),

$$\begin{aligned} M_\alpha f(x) &\leqslant 2 \sup_{Q:\text{cube}} \chi_Q(x) \left\|fu^{-\frac{1}{p}}\right\|_{B,Q} \cdot \ell(Q)^\alpha \left\|u^{\frac{1}{p}}\right\|_{\bar{B},Q} \\ &\leqslant 2M_B\left(fu^{-\frac{1}{p}}\right)(x) \cdot M_{\bar{B},\alpha}\left(u^{\frac{1}{p}}\right)(x). \quad \square \end{aligned}$$

□

3. Proofs of the results

Proof of Theorem 1.4. We use Lemma 2.4. Let $\Omega_\lambda := \{x \in \mathbb{R}^n : M_B f(x) > \lambda\}$. We may assume that Ω_λ is not empty; otherwise the left-hand side of the conclusion in Theorem 1.4 is zero. Assuming Ω_λ is not empty, we are in the position of using Lemma 2.4. By (2.16),

$$\int_{\Omega_\lambda} w(x)dx \leqslant \int_{\bigcup_j 3Q_j} w(x)dx \leqslant \sum_j \int_{3Q_j} w(x)dx.$$

Meanwhile, by (2.17),

$$1 < ⨍_{Q_j} B\left(\frac{4^n f(x)}{\lambda}\right) dx.$$

Therefore, we get

$$\begin{aligned}
\int_{\Omega_\lambda} w(x)dx &\leqslant 3^n \sum_j \left(⨍_{3Q_j} w(x)dx\right)\left(\int_{Q_j} B\left(\frac{4^n f(x)}{\lambda}\right) dx\right) \\
&= \sum_j \left(\int_{Q_j} B\left(\frac{4^n f(x)}{\lambda}\right)\left(⨍_{3Q_j} w(y)dy\right) dx\right) \\
&\leqslant \sum_j \left(\int_{Q_j} B\left(\frac{4^n f(x)}{\lambda}\right) Mw(x)dx\right) \leqslant \left(\int_{\mathbb{R}^n} B\left(\frac{4^n f(x)}{\lambda}\right) Mw(x)dx\right). \quad \square
\end{aligned}$$

Proof of Theorem 1.5.

(1) We estimate $\int_{Q_0} M_B f(x)dx$ by the Layer Cake Formula. We calculate

$$\begin{aligned}
\int_{Q_0} M_B f(x)dx &= 2\int_0^\infty |\{x \in Q_0 : M_B f(x) > 2\lambda\}|d\lambda \\
&= 2\left(\int_0^1 + \int_1^\infty\right) |\{x \in Q_0 : M_B f(x) > 2\lambda\}|d\lambda \\
&\leqslant 2\left(\int_0^1 |Q_0|dx + \int_1^\infty |\{x \in Q_0 : M_B f(x) > 2\lambda\}|d\lambda\right) \\
&= 2\left(|Q_0| + \int_1^\infty |\{x \in Q_0 : M_B f(x) > 2\lambda\}|d\lambda\right).
\end{aligned}$$

We evaluate $|\{x \in Q_0 : M_B f(x) > 2\lambda\}|$ for $\lambda > 0$. Write $f_1 := f \chi_{\{x \in Q_0 : |f(x)| > B^{-1}(1)\lambda\}}$ and $f_2 := f - f_1$, so that we have $M_B f \leqslant M_B f_1 + M_B f_2$. This gives that

$$\begin{aligned}
|\{x \in Q_0 : M_B f(x) > 2\lambda\}| \leqslant\ & |\{x \in Q_0 : M_B f_1(x) > \lambda\}| \\
&+ |\{x \in Q_0 : M_B f_2(x) > \lambda\}|.
\end{aligned}$$

Since $M_B f_2 \leqslant \lambda$, we have $|\{x \in Q_0 : M_B f_2(x) > \lambda\}| = 0$. Hence, we get

$$|\{x \in Q_0 : M_B f(x) > 2\lambda\}| \leqslant |\{x \in Q_0 : M_B f_1(x) > \lambda\}|. \tag{3.1}$$

By (2.18),

$$\begin{aligned} |\{x \in Q_0 : M_B f_1(x) > \lambda\}| &\leqslant C_0 \int_{\mathbb{R}^n} B\left(\frac{f_1(x)}{\lambda}\right) dx \\ &= C_0 \int_{\{x \in Q_0 : f(x) > B^{-1}(1)\lambda\}} B\left(\frac{f(x)}{\lambda}\right) dx. \end{aligned} \tag{3.2}$$

Estimates (3.1) and (3.2) imply that

$$|\{x \in Q_0 : M_B f(x) > 2\lambda\}| \leqslant C_0 \int_{\{x \in Q_0 : |f(x)| > B^{-1}(1)\lambda\}} B\left(\frac{|f(x)|}{\lambda}\right) dx. \tag{3.3}$$

By (3.3),

$$\begin{aligned} &\int_1^\infty |\{x \in Q_0 : M_B f(x) > 2\lambda\}| \, d\lambda \\ \leqslant & C_0 \int_1^\infty \int_{Q_0} B\left(\frac{|f(x)|}{\lambda}\right) \chi_{\{x \in Q_0 : |f(x)| > B^{-1}(1)\lambda\}}(x) dx d\lambda. \end{aligned}$$

Using Fubini's theorem, we calculate

$$\begin{aligned} &\int_1^\infty \int_{Q_0} B\left(\frac{|f(x)|}{\lambda}\right) \chi_{\{x \in Q_0 : |f(x)| > B^{-1}(1)\lambda\}}(x) dx d\lambda \\ =&\int_{Q_0} \int_1^\infty B\left(\frac{|f(x)|}{\lambda}\right) \chi_{\{\lambda > 0 : |f(x)| > B^{-1}(1)\lambda\}}(\lambda) d\lambda dx \\ \leqslant&\int_{Q_0} \int_1^{\max\left\{1, \frac{|f(x)|}{B^{-1}(1)}\right\}} B\left(\frac{|f(x)|}{\lambda}\right) d\lambda dx. \end{aligned}$$

By (2.2), we have

$$\begin{aligned} \int_{Q_0} \int_1^{\max\left\{1, \frac{|f(x)|}{B^{-1}(1)}\right\}} B\left(\frac{|f(x)|}{\lambda}\right) d\lambda dx &\leqslant \int_{Q_0} B(|f(x)|) \int_1^{\max\left\{1, \frac{|f(x)|}{B^{-1}(1)}\right\}} \frac{d\lambda}{\lambda} dx \\ &= \int_{Q_0} B(|f(x)|) \log^+ \frac{|f(x)|}{B^{-1}(1)} dx. \end{aligned}$$

(2) Let $\tilde{B}(t) := B(t) \log^+ \dfrac{t}{B^{-1}(1)}$ $(t \geqslant 0)$. Then we have

$$\int_{Q_0} B(|f(x)|) \log^+ \frac{|f(x)|}{B^{-1}(1)} dx = \int_{Q_0} \tilde{B}(|f(x)|) dx.$$

Taking $F(x) := \dfrac{|f(x)|}{\|f\|_{\tilde{B},Q_0}}$, we apply (1.6):

$$\fint_{Q_0} M_B F(x)dx \leqslant 2\left(1 + C_0 \fint_{Q_0} \tilde{B}(F(x))\,dx\right).$$

By the definition of $\|\cdot\|_{\tilde{B},Q_0}$, we get

$$\fint_{Q_0} \tilde{B}(F(x))\,dx = \fint_{Q_0} \tilde{B}\left(\frac{|f(x)|}{\|f\|_{\tilde{B},Q_0}}\right) dx \leqslant 1.$$

This implies that

$$\fint_{Q_0} M_B F(x)dx \leqslant 2(1+C_0),$$

that is,

$$\fint_{Q_0} M_B f(x)dx \leqslant 2(1+C_0)\,\|f\|_{\tilde{B},Q_0} \cong \|f\|_{B(L)\log L,Q_0}. \qquad \square$$

Proof of Theorem 1.7. We plan to verify the following keeping in mind that (4)–(6) are equivalent to the fact that $B \in B_p$:

- $(B \in B_p \iff)$ (5) $\Rightarrow$ (1) $\Rightarrow$ $B \in B_p$,
- $(B \in B_p \iff)$ (4) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ $B \in B_p$.

(1) We prove (5) $\Rightarrow$ (1). Lemma 2.7 implies that

$$\int_{\mathbb{R}^n} M_\alpha f(y)^p \frac{w(y)}{M_{\bar{B},\alpha}^{\left(\frac{1}{p}\right)} u(y)} dy \leqslant 2^p \int_{\mathbb{R}^n} M_B\left(fu^{-\frac{1}{p}}\right)(y)^p w(y)dy.$$

By (5),

$$\int_{\mathbb{R}^n} M_B\left(fu^{-\frac{1}{p}}\right)(y)^p w(y)dy \leqslant C\int_{\mathbb{R}^n} |f(y)|^p \frac{Mw(y)}{u(y)}dy.$$

Hence (1) holds.

(2) We prove $B \in B_p$ assuming (1). Let $f = \chi_{Q(0,1)}$, $u = f + \varepsilon$ and $w = 1$, where $\varepsilon > 0$ is chosen arbitrarily. By (1) and the monotone convergence theorem,

$$\int_{\mathbb{R}^n} M_\alpha(\chi_{Q(0,1)})(x)^p \frac{dx}{M_{\bar{B},\alpha}\left(\chi_{Q(0,1)}\right)(x)^p} \leqslant C\int_{Q(0,1)} dx = C.$$

We insert $M_{\bar{B},\alpha}(\chi_{Q(0,1)})(x)$, $x \in \mathbb{R}^n$. By (2.7) itself and (2.7) with $B(t) = t$,

$$\int_{\mathbb{R}^n} M_\alpha(\chi_{Q(0,1)})(x)^p \frac{dx}{M_{\bar{B},\alpha}\left(\chi_{Q(0,1)}\right)(x)^p}$$

$$\geqslant C \int_{|y|>1} \frac{1}{|y|^{(n-\alpha)p}} \cdot \frac{1}{|y|^{\alpha p}\left(\bar{B}^{-1}\left(|y|^n\right)^{-1}\right)^p} dy = C \int_{|y|>1} \frac{\left(\bar{B}^{-1}\left(|y|^n\right)\right)^p}{|y|^{np}} dy.$$

Using polar coordinates, we have

$$\int_{|y|>1} \frac{\left(\bar{B}^{-1}\left(|y|^n\right)\right)^p}{|y|^{np}} dy = C \int_1^\infty \frac{r^{n-1} \cdot \left(\bar{B}^{-1}\left(r^n\right)\right)^p}{r^{np}} dr.$$

We change variables $r \mapsto t = r^n$. Then we have

$$\int_1^\infty \frac{r^{n-1} \cdot \left(\bar{B}^{-1}\left(r^n\right)\right)^p}{r^{np}} dr = C \int_1^\infty \frac{\bar{B}^{-1}(t)^p}{t^p} dt.$$

By (2.4),

$$\int_1^\infty \frac{\bar{B}^{-1}(t)^p}{t^p} dt \cong \int_1^\infty \frac{1}{\left(B^{-1}(t)\right)^p}\, dt.$$

Taking $\ell = B^{-1}(t)$, we obtain

$$\int_1^\infty \frac{1}{\left(B^{-1}(t)\right)^p}\, dt = \int_{B^{-1}(1)}^\infty \frac{B'(\ell)}{\ell^p} d\ell.$$

By (2.1),

$$\int_{B^{-1}(1)}^\infty \frac{B'(\ell)}{\ell^p} d\ell \cong \int_{B^{-1}(1)}^\infty \frac{B(\ell)}{\ell^{p+1}} d\ell.$$

If we combine all these observations, we obtain

$$\int_{B^{-1}(1)}^\infty \frac{B(\ell)}{\ell^{p+1}} d\ell \leqslant C \int_{\mathbb{R}^n} M_\alpha\left(\chi_{Q(0,1)}\right)(x)^p \frac{dx}{M_{\bar{B},\alpha}\left(\chi_{Q(0,1)}\right)(x)^p} \leqslant C < \infty.$$

Hence we obtain $B \in B_p$.

(3) We prove that (4) $\Rightarrow$ (2). We may assume that $f \in L^\infty_{\mathrm{c}}$. Let $a > \max\{2 \cdot 6^n C_0, 2^n\}$, and let $\Omega_k = \left\{x \in \mathbb{R}^n : M_{B,\alpha,3\mathscr{D}} f(x) > a^{k+1}\right\}$. We may assume that Ω_k is not empty; otherwise there is nothing to prove. By Lemma 2.5, there exists a disjoint collection of maximal dyadic cubes $\{Q_{k,j}\}$, $\Omega_k = \bigcup_j Q_{k,j}$ holds. Then

$$\begin{aligned}\int_{\mathbb{R}^n} M_{B,\alpha} f(x)^p w(x) dx &= \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} M_{B,\alpha} f(x)^p w(x) dx \\ &= \sum_{k,j} \int_{Q_{k,j} \setminus \Omega_{k+1}} M_{B,\alpha} f(x)^p w(x) dx.\end{aligned}$$

By Lemma 2.3,

$$\sum_{k,j}\int_{Q_{k,j}\setminus\Omega_{k+1}} M_{B,\alpha}f(x)^p w(x)dx \cong \sum_{k,j}\int_{Q_{k,j}\setminus\Omega_{k+1}} M_{B,\alpha,3\mathscr{D}}f(x)^p w(x)dx.$$

By the definition of Ω_{k+1},

$$\sum_{k,j}\int_{Q_{k,j}\setminus\Omega_{k+1}} M_{B,\alpha,3\mathscr{D}}f(x)^p w(x)dx \leqslant \sum_{k,j} a^{(k+1)p} ⨍_{3Q_{k,j}} w(x)dx\cdot|3Q_{k,j}|.$$

By (2.19) and (2.21),

$$\begin{aligned}&\sum_{k,j} a^{(k+1)p} ⨍_{3Q_{k,j}} w(x)dx\cdot|3Q_{k,j}|\\ \leqslant& C\sum_{k,j}|3Q_{k,j}|^{\frac{\alpha p}{n}}\|f\|^p_{B,3Q_{k,j}} ⨍_{3Q_{k,j}} w(x)dx\cdot|E_{k,j}|\\ =&C\sum_{k,j}\int_{E_{k,j}}\left\|f\left(|3Q_{k,j}|^{\frac{\alpha p}{n}} ⨍_{3Q_{k,j}} w(y)dy\right)^{\frac{1}{p}}\right\|^p_{B,3Q_{k,j}} dx.\end{aligned}$$

From the definition of the maximal operator, we have

$$\begin{aligned}&\sum_{k,j}\int_{E_{k,j}}\left\|f\left(|3Q_{k,j}|^{\frac{\alpha p}{n}} ⨍_{3Q_{k,j}} w(y)dy\right)^{\frac{1}{p}}\right\|^p_{B,3Q_{k,j}} dx\\ \leqslant&\sum_{k,j}\int_{E_{k,j}}\left\|f\,(M_{\alpha p}w)^{\frac{1}{p}}\right\|^p_{B,3Q_{k,j}}dx \leqslant \sum_{k,j}\int_{E_{k,j}} M_B\left(f\,(M_{\alpha p}w)^{\frac{1}{p}}\right)(x)^p dx\\ \leqslant&\int_{\mathbb{R}^n} M_B\left(f\,(M_{\alpha p}w)^{\frac{1}{p}}\right)(x)^p dx.\end{aligned}$$

By (4), we have (2).

(4) We prove that (2) $\Rightarrow$ (3). By (2.5), we obtain the point-wise inequality:

$$M_\alpha f(y) \leqslant 2\sup_{Q\ni y}\ell(Q)^\alpha\left\|fu^{-\frac{1}{p}}\right\|_{B,Q}\left\|u^{\frac{1}{p}}\right\|_{\bar{B},Q} \leqslant 2M_{B,\alpha}\left(fu^{-\frac{1}{p}}\right)(y)\cdot M_{\bar{B}}\left(u^{\frac{1}{p}}\right)(y).$$

Hence inserting this pointwise estimate into the left-hand side of (3), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} M_\alpha f(y)^p\frac{w(y)}{M_{\bar{B}}\left(u^{\frac{1}{p}}\right)(y)^p}dy\\ \leqslant&2^p\int_{\mathbb{R}^n} M_{B,\alpha}\left(fu^{-\frac{1}{p}}\right)(y)^p\cdot M_{\bar{B}}\left(u^{\frac{1}{p}}\right)(y)^p\frac{w(y)}{M_{\bar{B}}\left(u^{\frac{1}{p}}\right)(y)^p}dy\\ \leqslant&2^p\int_{\mathbb{R}^n} M_{B,\alpha}\left(fu^{-\frac{1}{p}}\right)(y)^p w(y)dy.\end{aligned}$$

By (2), we obtain (3).

(5) We next prove that (3) if $B \in B_p$. Let $f = u = \chi_{Q(0,1)}$. Then

$$\int_{\mathbb{R}^n} |f(y)|^p \frac{M_{\alpha p} w(y)}{u(y)} dy = \int_{Q(0,1)} M_{\alpha p} w(y) dy.$$

Let $w(x) = \dfrac{1}{(1+|x|)^{\alpha p}}$. Then by Lemma 2.6, $M_{\alpha p} w(x) \leqslant C$. Hence, we have

$$\int_{\mathbb{R}^n} |f(x)|^p \frac{M_{\alpha p} w(x)}{u(x)} dx \leqslant C.$$

Meanwhile,

$$\int_{\mathbb{R}^n} M_\alpha f(x) \frac{w(x)}{M_B\left(u^{\frac{1}{p}}\right)(x)^p} dx \geqslant \int_{|x|>1} M_\alpha(\chi_{Q(0,1)})(x) \frac{1}{M_B\left(\chi_{Q(0,1)}\right)(x)^p} \frac{dx}{(1+|x|)^{\alpha p}}.$$

Note that $\dfrac{1}{1+|x|} \geqslant \dfrac{1}{2|x|}$ whenever $|x| > 1$. Thus,

$$\begin{aligned}&\int_{|x|>1} M_\alpha(\chi_{Q(0,1)})(x) \frac{1}{M_B\left(\chi_{Q(0,1)}\right)(x)^p} \frac{dx}{(1+|x|)^{\alpha p}}\\ \geqslant &C \int_{|x|>1} M_\alpha(\chi_{Q(0,1)})(x) \frac{1}{M_B\left(\chi_{Q(0,1)}\right)(x)^p} \frac{dx}{|x|^{\alpha p}}.\end{aligned}$$

By Lemma 2.7,

$$\int_{|x|>1} M_\alpha(\chi_{Q(0,1)})(x) \frac{1}{M_B\left(\chi_{Q(0,1)}\right)(x)^p} \frac{dx}{|x|^{\alpha p}} \geqslant \int_{|x|>1} \frac{\overline{B}^{-1}(|x|^n)^p}{|x|^{np}} dx.$$

(6) If (3) holds, we can show $B \in B_p$ similar to the proof of (1) $\Rightarrow$ $B \in B_p$. We omit the further details. □

Proof of Theorem 1.16.

Clearly, as a special case of $w = 1$, (8) implies (3). In the rest of the proof, we will verify implications (3) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10). The sufficient condition on which $M_{B,\alpha} : L^p \to L^q$ is bounded is due to [3]: Assume that the function $t \in (0,\infty) \mapsto t^{-n/\alpha} B(t) \in (0,\infty)$ is almost decreasing, that $t^{-n/\alpha} B(t) \to 0$ $(t \to \infty)$ and (10) with the condition $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $M_{B,\alpha} : L^p \to L^q$ is bounded (see Proposition 1.23).

(1) We prove that (3) implies (8). By Lemma 2.3, we have only to analyze the weighted norm of $M_{B,\alpha,3\mathscr{D}} f$. We may assume that $f \in L^\infty_c$ by the monotone convergence theorem to show (8). For $k \in \mathbb{Z}$, let $\Omega_k :=$ $\left\{x \in \mathbb{R}^n : M_{B,\alpha,3\mathscr{D}} f(x) > a^k\right\}$ in Lemma 2.5. Since $f \in L^\infty_c$, we have

$$\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}} \Omega_k \setminus \Omega_{k+1}.$$

Thus,

$$\int_{\mathbb{R}^n} M_{B,\alpha,3\mathscr{D}}f(x)^q w(x)dx = \sum_{k\in\mathbb{Z}} \int_{\Omega_k\setminus\Omega_{k+1}} M_{B,\alpha,3\mathscr{D}}f(x)^q w(x)dx.$$

By the definition of Ω_{k+1},

$$\int_{\Omega_k\setminus\Omega_{k+1}} M_{B,\alpha,3\mathscr{D}}f(x)^q w(x)dx \leqslant a^q \cdot a^{kq} \int_{\Omega_k\setminus\Omega_{k+1}} w(x)dx.$$

Since $\Omega_k \setminus \Omega_{k+1} \subset \bigcup_j Q_{k,j}$,

$$\sum_{k\in\mathbb{Z}} a^{kq} \int_{\Omega_k\setminus\Omega_{k+1}} w(x)dx \leqslant \sum_{k\in\mathbb{Z}} \sum_j \left(\int_{Q_{k,j}} w(x)dx\right) a^{kq}.$$

Since $a^k < |3Q_{k,j}|^{\frac{\alpha}{n}} \|f\|_{B,3Q_{k,j}}$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \sum_j \left(\int_{Q_{k,j}} w(x)dx\right) a^{kq} \\
&\leqslant \sum_{k\in\mathbb{Z}} \sum_j \ell(3Q_{k,j})^{\alpha q} \left(⨍_{Q_{k,j}} w(x)dx\right) \|f\|_{B,3Q_{k,j}}^q |Q_{k,j}| \\
&= \sum_{k\in\mathbb{Z}} \sum_j \ell(3Q_{k,j})^{\alpha q} \left\| f\cdot(Mw)^{\frac{1}{q}}\cdot(Mw)^{-\frac{1}{q}} \right\|_{B,3Q_{k,j}}^q w(Q_{k,j}).
\end{aligned}$$

If $x \in 3Q_{k,j}$, then $Mw(x) \geqslant m_{3Q_{k,j}}(w)$. Thus

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \sum_j \ell(3Q_{k,j})^{\alpha q} \left\| f\cdot(Mw)^{\frac{1}{q}}\cdot(Mw)^{-\frac{1}{q}} \right\|_{B,3Q_{k,j}}^q w(Q_{k,j}) \\
&\leqslant \sum_{k\in\mathbb{Z}} \sum_j \ell(3Q_{k,j})^{\alpha q} \left\| f\cdot(Mw)^{\frac{1}{q}} \right\|_{B,3Q_{k,j}}^q m_{3Q_{k,j}}(w)^{-1} w(Q_{k,j}) \\
&\leqslant 3^n \sum_{k\in\mathbb{Z}} \sum_j \ell(3Q_{k,j})^{\alpha q} \left\| f\cdot(Mw)^{\frac{1}{q}} \right\|_{B,3Q_{k,j}}^q |Q_{k,j}|.
\end{aligned}$$

Since $|Q_{k,j}| \leqslant C|E_{k,j}|$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \sum_j \ell(3Q_{k,j})^{\alpha q} \left\| f\cdot(Mw)^{\frac{1}{q}} \right\|_{B,3Q_{k,j}}^q |Q_{k,j}| \\
&\leqslant C \sum_{k\in\mathbb{Z}} \sum_j \int_{E_{k,j}} \left(\ell(3Q_{k,j})^{\alpha} \left\| f\cdot(Mw)^{\frac{1}{q}} \right\|_{B,3Q_{k,j}}\right)^q dx \\
&\leqslant C \sum_{k\in\mathbb{Z}} \sum_j \int_{E_{k,j}} M_{B,\alpha}\left[f\cdot(Mw)^{\frac{1}{q}}\right](x)^q dx.
\end{aligned}$$

Since the $E_{k,j}$'s are disjoint, we obtain

$$\sum_{k\in\mathbb{Z}} \sum_j \int_{E_{k,j}} \left\{M_{B,\alpha}\left[f\cdot(Mw)^{\frac{1}{q}}\right](x)\right\}^q dx \leqslant \int_{\mathbb{R}^n} M_{B,\alpha}\left[f\cdot(Mw)^{\frac{1}{q}}\right](x)^q dx.$$

By (3),

$$\left(\int_{\mathbb{R}^n} \left\{M_{B,\alpha}\left[f\cdot(Mw)^{\frac{1}{q}}\right](x)\right\}^q dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n} |f(x)|^p Mw(x)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}.$$

This implies that

$$\left(\int_{\mathbb{R}^n} M_{B,\alpha} f(x)^q w(x) dx\right) \leqslant C\left(\int_{\mathbb{R}^n} |f(x)|^p Mw(x)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}.$$

(2) We prove that (8) $\Rightarrow$ (9). By (2.5),

$$M_\alpha f(x) \leqslant 2M_{B,\alpha}\left(fu^{-\frac{1}{q}}\right)(x) M_{\bar{B}}\left(u^{\frac{1}{q}}\right)(x) \quad (x\in\mathbb{R}^n).$$

Hence

$$\left(\int_{\mathbb{R}^n} M_\alpha f(x)^q \cdot \frac{w(x)}{M_{\bar{B}}\left(u^{\frac{1}{q}}\right)(x)^q} dx\right)^{\frac{1}{q}} \leqslant 2\left(\int_{\mathbb{R}^n} \left\{M_{B,\alpha}\left(fu^{-\frac{1}{q}}\right)(x)\right\}^q w(x) dx\right)^{\frac{1}{q}}.$$

By (8),

$$\left(\int_{\mathbb{R}^n} \left\{M_{B,\alpha}\left(fu^{-\frac{1}{q}}\right)(x)\right\}^q w(x) dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n} |f(x)|^p \left(\frac{Mw(x)}{u(x)}\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}.$$

Hence, we obtain (9).

(3) We prove that (9) $\Rightarrow$ (10). Let $f = u = \chi_{Q(0,1)}$ and $w = 1$. Then

$$\left(\int_{\mathbb{R}^n} |f(x)|^p \left(\frac{Mw(x)}{u(x)}\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} = 1.$$

By (2.7),

$$\begin{aligned}\int_{\mathbb{R}^n} M_\alpha f(x)^q \frac{dx}{M_{\bar{B}}\left(\chi_{Q(0,1)}\right)(x)^q} &\geqslant \int_{|y|>1} M_\alpha f(y)^q \frac{dy}{M_{\bar{B}}\left(\chi_{Q(0,1)}\right)(y)^q} \\ &\cong \int_{|y|>1} |y|^{\alpha q} \bar{B}^{-1}\left(|y|^n\right)^q \frac{dy}{|y|^{nq}}.\end{aligned}$$

Using polar coordinates, we have

$$\int_{|y|>1} |y|^{\alpha q} \bar{B}^{-1}\left(|y|^n\right)^q \frac{dy}{|y|^{nq}} = C\int_1^\infty r^{n-1-nq+\alpha q}\cdot\left(\bar{B}^{-1}\left(r^n\right)\right)^q dr.$$

Taking $t = r^n$, we obtain

$$\int_1^\infty r^{n-1-nq+\alpha q} \cdot \left(\overline{B}^{-1}(r^n)\right)^q dr = C\int_1^\infty t^{\frac{\alpha}{n}q}\left(\frac{\overline{B}^{-1}(t)}{t}\right)^q dt.$$

By (2.4),

$$\int_1^\infty t^{\frac{\alpha}{n}q}\left(\frac{\overline{B}^{-1}(t)}{t}\right)^q dt \cong \int_1^\infty t^{\frac{\alpha}{n}q}\frac{1}{B^{-1}(t)^q}dt.$$

Taking $\ell = B^{-1}(t)$, we obtain

$$\int_1^\infty t^{\frac{\alpha}{n}q}\frac{1}{B^{-1}(t)^q}dt = \int_{B^{-1}(1)}^\infty B(\ell)^{\frac{\alpha}{n}q}\frac{1}{\ell^q}B'(\ell)d\ell.$$

By (2.1),

$$\int_{B^{-1}(1)}^\infty B(\ell)^{\frac{\alpha}{n}q}\frac{1}{\ell^q}B'(\ell)d\ell \cong \int_{B^{-1}(1)}^\infty \frac{B(\ell)^{\frac{\alpha}{n}q+1}}{\ell^{q+1}}d\ell.$$

Hence 10 holds. □

Proof of Theorem 1.11. We plan to show Theorem 1.11 as follows:

- (1)$\Longrightarrow$ (9) $\Longrightarrow$ (8) $\Longrightarrow$ (1),
- (1)$\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1),
- (5)$\Longleftrightarrow$ (1)$\Longrightarrow$ (6)$\Longrightarrow$ (7)$\Longrightarrow$ (1).

(1) Condition (9) follows readily from (1) and the Fefferman–Stein inequality. In fact, by assumption 1, $M_B f(x) \leqslant CM^{(p)}f(x)$ for all $x \in \mathbb{R}^n$. Furthermore by the Fefferman–Stein inequality for $p = 1$,

$$\begin{aligned} w(\{x \in \mathbb{R}^n : M_B f(x) > \lambda\}) &\leqslant w(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda^p\}) \\ &\leqslant \frac{C}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p Mw(x)dx. \end{aligned}$$

Hence, we obtain (9).

(2) We verify (9) $\Longrightarrow$ (8). By (2.5),

$$w\left(\left\{x\in\mathbb{R}^n : \frac{M_\alpha f(x)}{M_{\overline{B},\alpha}\left(u^{\frac{1}{p}}\right)(x)} > \lambda\right\}\right) \leqslant w\left(\left\{x\in\mathbb{R}^n : 2M_B\left(fu^{-\frac{1}{p}}\right)(x) > \frac{\lambda}{2}\right\}\right).$$

By assumption (9),

$$w\left(\left\{x \in \mathbb{R}^n : M_B\left(fu^{-\frac{1}{p}}\right)(x) > \frac{\lambda}{2}\right\}\right) \leqslant \left\{\frac{C}{\lambda}\left(\int_{\mathbb{R}^n} |f(x)|^p \frac{Mw(x)}{u(x)}dx\right)^{\frac{1}{p}}\right\}^p.$$

Hence, we obtain (8).

(3) Thirdly, we verify (8) $\Longrightarrow$ (1). Fix $t>1$, $Q_0=Q(0,1)$, $|Q_1|=t$, $Q_0\subset Q_1$. Taking $w\equiv 1$, $f=u=\chi_{Q_0}$, for every $\lambda>0$, we calculate $\|f\|_{L^p(Mw)}=1$. Meanwhile,

$$w\left(\left\{x\in\mathbb{R}^n:\frac{M_\alpha f(x)}{M_{\overline{B},\alpha}\left(u^{\frac{1}{p}}\right)(x)}>\lambda\right\}\right)$$
$$\geqslant w\left(\left\{x\in 3Q_1,|x|>\ell(Q_1),\frac{M_\alpha\left(\chi_{Q_0}\right)(x)}{M_{\overline{B},\alpha}\left(\chi_{Q_0}\right)(x)}>\lambda\right\}\right).$$

If $x\in 3Q_1$, then

$$M_\alpha\left(\chi_{Q_0}\right)(x)\geqslant \ell(3Q_1)^\alpha \fint_{3Q_1}\chi_{Q_0}(y)dy\cong t^{\frac{\alpha}{n}-1}. \tag{3.4}$$

If $x\in 3Q_1$ and $|x|>\ell(Q_1)(>1)$, then $\ell(Q_1)<|x|<\ell(3Q_1)$. By property (2.7) in Lemma 2.2,

$$M_{\overline{B},\alpha}\left(\chi_{Q_0}\right)(x)\cong |x|^\alpha\overline{B}^{-1}\left(|x|^n\right)^{-1}\leqslant \ell(3Q_1)^\alpha\overline{B}^{-1}\left(\ell\left(Q_1\right)^n\right)^{-1}=3^\alpha t^{\frac{\alpha}{n}}\overline{B}^{-1}(t)^{-1}.$$

By (2.4),

$$M_{\overline{B},\alpha}\left(\chi_{Q_0}\right)(x)\cong t^{\frac{\alpha}{n}-1}B^{-1}(t). \tag{3.5}$$

these inequalities (3.4) and (3.5) give

$$\frac{M_\alpha\left(\chi_{Q_0}\right)(x)}{M_{\overline{B},\alpha}\left(\chi_{Q_0}\right)(x)}\gtrsim B^{-1}(t)^{-1}.$$

Taking $\lambda\cong B^{-1}(t)^{-1}$, we obtain

$$w\left(\left\{x\in 3Q_1,|x|>\ell(Q_1),\frac{M_\alpha\left(\chi_{Q_0}\right)(x)}{M_{\overline{B},\alpha}\left(\chi_{Q_0}\right)(x)}>\lambda\right\}\right)^{\frac{1}{p}}$$
$$\gtrsim w\left(3Q_1\setminus\{|x|\leqslant\ell(Q_1)\}\right)^{\frac{1}{p}}\cong |Q_1|^{\frac{1}{p}}=t^{\frac{1}{p}}.$$

This implies that

$$\frac{t^{\frac{1}{p}}}{B^{-1}(t)}\cong\lambda t^{\frac{1}{p}}\lesssim\lambda w\left(\left\{x\in 3Q_1,|x|>\ell(Q_1),\frac{M_\alpha\left(\chi_{Q_0}\right)(x)}{M_{\overline{B},\alpha}\left(\chi_{Q_0}\right)(x)}>\lambda\right\}\right)^{\frac{1}{p}}.$$

By assumption (8),

$$B^{-1}(t)^{-1}t^{\frac{1}{p}}\lesssim 1 \text{ and } B\left(t^{\frac{1}{p}}\right)\lesssim t.$$

Letting $s\cong t^{\frac{1}{p}}$, we obtain (1).

(4) We verify (1)$\Longrightarrow$ (3). By assumption (1), $M_{B,\alpha}f(x) \lesssim M_{\alpha p}\left(|f|^p\right)(x)^{\frac{1}{p}}$ holds. We check condition (1.26) for $\alpha \to \alpha p$, p and $q \to 1$:

$$\sup_Q |Q|^{\frac{\alpha p}{n}+1-1}\left(\fint_Q w(x)dx\right)\left(\operatorname*{ess\,inf}_{x\in Q} M_{\alpha p}w(x)\right)^{-1}$$
$$\leqslant \sup_Q |Q|^{\frac{\alpha p}{n}+1-1}\left(\fint_Q w(x)dx\right)\left(\ell(Q)^{\alpha p}\fint_Q w(y)dy\right)^{-1} = 1.$$

By Proposition 1.25,

$$w\left(\left\{x\in\mathbb{R}^n : M_{\alpha p}\left(|f|^p\right)(x) > \lambda^p\right\}\right) \lesssim \left(\frac{1}{\lambda}\left(\int_{\mathbb{R}^n}|f(x)|^p M_{\alpha p}w(x)dx\right)^{\frac{1}{p}}\right)^p.$$

Therefore we obtain (3).

(5) We verify (3)$\Longrightarrow$ (4). By (2.5),

$$w\left(\left\{x\in\mathbb{R}^n : \frac{M_\alpha f(x)}{M_{\overline{B}}\left(u^{\frac{1}{p}}\right)(x)} > \lambda\right\}\right) \leqslant w\left(\left\{x\in\mathbb{R}^n : M_{B,\alpha}\left(fu^{-\frac{1}{p}}\right)(x) > \frac{\lambda}{2}\right\}\right).$$

By assumption (3),

$$w\left(\left\{x\in\mathbb{R}^n : 2M_{B,\alpha}\left(fu^{-\frac{1}{p}}\right)(x) > \frac{\lambda}{2}\right\}\right) \lesssim \left\{\frac{1}{\lambda}\left(\int_{\mathbb{R}^n}|f(x)|^p\frac{M_{\alpha p}w(x)}{u(x)}dx\right)^{\frac{1}{p}}\right\}^p.$$

Hence, we obtain (4).

(6) We verify (4) $\Longrightarrow$ (1). Fix $t>1$. Letting $Q_0 = Q(0,1)$, $|Q_1| = t$, $Q_0 \subset Q_1$, $w(x) = \dfrac{1}{(1+|x|)^{\alpha p}}$ and $f = u = \chi_{Q_0}$, by Lemma 2.6,

$$\left(\int_{\mathbb{R}^n}|f(x)|^p\frac{M_{\alpha p}w(x)}{u(x)}dx\right)^{\frac{1}{p}} \lesssim 1. \tag{3.6}$$

Meanwhile,

$$w\left(\left\{x\in\mathbb{R}^n : \frac{M_\alpha f(x)}{M_{\overline{B}}\left(u^{\frac{1}{p}}\right)(x)} > \lambda\right\}\right)$$
$$\geqslant w\left(\left\{x\in 3Q_1, |x| > \ell(Q_1), \frac{M_\alpha\left(\chi_{Q_0}\right)(x)}{M_{\overline{B}}\left(\chi_{Q_0}\right)(x)} > \lambda\right\}\right).$$

If $x \in 3Q_1$, then (3.4) holds. If $x\in 3Q_1$ and $|x| > \ell(Q_1) (> 1)$, note that $\ell(Q_1) < |x| < \ell(3Q_1)$. by property (2.7) in Lemma 2.2.

$$M_{\overline{B}}\left(\chi_{Q_0}\right)(x) \cong \overline{B}^{-1}\left(|x|^n\right)^{-1} \leqslant \overline{B}^{-1}\left(\ell\left(Q_1\right)^n\right)^{-1} = \overline{B}^{-1}\left(\ell\left(Q_1\right)^n\right)^{-1} = \overline{B}^{-1}(t)^{-1}.$$

By (2.4),

$$M_{\overline{B}}\left(\chi_{Q_0}\right)(x) \lesssim \frac{B^{-1}(t)}{t}. \tag{3.7}$$

these inequalities (3.4) and (3.7) give

$$\frac{M_\alpha\left(\chi_{Q_0}\right)(x)}{M_{\overline{B},\alpha}\left(\chi_{Q_0}\right)(x)} \gtrsim \frac{t^{\frac{\alpha}{n}}}{B^{-1}(t)}.$$

Taking $\lambda \cong \frac{t^{\frac{\alpha}{n}}}{B^{-1}(t)}$, we obtain

$$w\left(\left\{x \in 3Q_1, |x| > \ell(Q_1), \frac{M_\alpha\left(\chi_{Q_0}\right)(x)}{M_{\overline{B}}\left(\chi_{Q_0}\right)(x)} > \lambda\right\}\right)^{\frac{1}{p}}$$

$$\gtrsim\left(\int_{3Q_1\setminus\{|x|\leqslant\ell(Q_1)\}} \frac{1}{(1+|x|)^{\alpha p}}dx\right)^{\frac{1}{p}} \gtrsim \left(\int_{3Q_1\setminus\{|x|\leqslant\ell(Q_1)\}} \frac{1}{\ell(Q_1)^{\alpha p}}dx\right)^{\frac{1}{p}}$$

$$=|Q_1|^{-\frac{\alpha}{n}}\left(\int_{3Q_1\setminus\{|x|\leqslant\ell(Q_1)\}} dx\right)^{\frac{1}{p}} \cong |Q_1|^{\frac{1}{p}-\frac{\alpha}{n}} = t^{\frac{1}{p}-\frac{\alpha}{n}}.$$

This implies that

$$\frac{t^{\frac{1}{p}}}{B^{-1}(t)} \cong \lambda t^{\frac{1}{p}-\frac{\alpha}{n}} \lesssim \lambda w\left(\left\{x \in 3Q_1, |x| > \ell(Q_1), \frac{M_\alpha\left(\chi_{Q_0}\right)(x)}{M_{\overline{B}}\left(\chi_{Q_0}\right)(x)} > \lambda\right\}\right)^{\frac{1}{p}}.$$

By assumption (4),

$$\frac{t^{\frac{1}{p}}}{B^{-1}(t)} \lesssim 1 \text{ and } B\left(t^{\frac{1}{p}}\right) \leqslant t.$$

Letting $s \cong t^{\frac{1}{p}}$, we obtain (1).

(7) We will prove that (5)$\Longrightarrow$ (1). For every $t>1$, let $|Q_1| = t$, $|Q_0| = 1$ such that $Q_1 \supset Q_0$ and $f = \chi_{Q_0}$. Note that $\{x \in \mathbb{R}^n : M_{B,\alpha}f(x) > \lambda\} \supset \{x \in Q_1 : M_{B,\alpha}\left(\chi_{Q_1}\right)(x) > \lambda\}$. If $x \in Q_1$, then

$$M_{B,\alpha}\left(\chi_{Q_0}\right)(x) \geqslant \ell(Q_1)^\alpha \left\|\chi_{Q_0}\right\|_{B,Q_1} = \ell(Q_1)^\alpha B^{-1}\left(|Q_1|\right)^{-1} = t^{\frac{\alpha}{n}} B^{-1}(t)^{-1}.$$

Letting $\lambda = t^{\frac{\alpha}{n}} B^{-1}(t)^{-1}$,

$$|\{x \in \mathbb{R}^n : M_{B,\alpha}f(x) > \lambda\}| \geqslant \left|\left\{x \in Q_1 : M_{B,\alpha}f(x) > t^{\frac{\alpha}{n}} B^{-1}(t)^{-1}\right\}\right| = |Q_1| = t.$$

Meanwhile, by (5),

$$\left|\left\{x \in \mathbb{R}^n : M_{B,\alpha}\left(\chi_{Q_0}\right)(x) > \lambda\right\}\right| \lesssim \left(\frac{1}{\lambda}\int_{\mathbb{R}^n} |f(x)|^p dx\right)^q t^{-\frac{\alpha}{n}q} B^{-1}(t)^q.$$

That is,

$$t^{\frac{1}{q}+\frac{\alpha}{n}} \lesssim B^{-1}(t) \text{ and } B\left(t^{\frac{1}{q}+\frac{\alpha}{n}}\right) \lesssim t.$$

Letting $s \cong t^{\frac{1}{q}+\frac{\alpha}{n}}$, we see

$$B(s) \lesssim s^{\left(\frac{1}{q}+\frac{\alpha}{n}\right)^{-1}}. \tag{3.8}$$

Since $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, we obtain

$$B(s) \lesssim s^p,$$

that is, 1 holds.

(8) We verify (1) $\Longrightarrow$ (5). By (1),

$$M_{B,\alpha}f(x) \lesssim M_{\alpha p}(|f|^p)(x)^{\frac{1}{p}}. \tag{3.9}$$

This implies that

$$|\{x\in\mathbb{R}^n : M_{B,\alpha}f(x)>\lambda\}| \lesssim \left|\left\{x\in\mathbb{R}^n : M_{\alpha p}(|f|^p)(x)>\lambda^p\right\}\right|. \tag{3.10}$$

Applying statement (B) of Proposition 1.12 for $\frac{p}{q}=1-\frac{\alpha p}{n}$,

$$\left|\left\{x\in\mathbb{R}^n : M_{\alpha p}(|f|^p)(x)>\lambda^p\right\}\right|^{\frac{p}{q}} \lesssim \frac{1}{\lambda^p}\||f|^p\|_{L^1(\mathbb{R}^n)} = \frac{1}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^p dx. \tag{3.11}$$

By (3.10) and (3.11), we obtain (5):

$$|\{x\in\mathbb{R}^n : M_{B,\alpha}f(x)>\lambda\}| \leqslant \left(\frac{1}{\lambda}\left(\int_{\mathbb{R}^n}|f(x)|^p dx\right)^{\frac{1}{p}}\right)^q.$$

(9) We verify that (1) $\Longrightarrow$ (6). By (1), inequality (3.9) holds. We check condition (1.26) with $\alpha=\frac{n}{p}-\frac{n}{q}$, $u(x)=w(x)^{\frac{p}{q}}$, and $v(x)=Mw(x)^{\frac{p}{q}}$. For every cube $Q\subset\mathbb{R}^n$,

$$\begin{aligned}&|Q|^{\frac{\alpha p}{n}+\frac{p}{q}-1}\left(⨍_Q u(x)^{\frac{q}{p}}dx\right)^{\frac{p}{q}}\left(\operatorname*{ess\,inf}_{x\in Q} v(x)\right)^{-1}\\ =&\left(⨍_Q w(x)dx\right)^{\frac{p}{q}}\left(\operatorname*{ess\,inf}_{x\in Q} Mw(x)^{\frac{p}{q}}\right)^{-1} \leqslant \left(⨍_Q w(x)dx\right)^{\frac{p}{q}}\left(\left(⨍_Q w(y)dy\right)^{\frac{p}{q}}\right)^{-1}\\ =&1<\infty.\end{aligned}$$

Applying Proposition 1.25, we learn

$$\begin{aligned}&w(\{x\in\mathbb{R}^n : M_{B,\alpha}f(x)>\lambda\})^{\frac{1}{q}} \leqslant \left\{u^{\frac{q}{p}}\left(\left\{x\in\mathbb{R}^n : M_{\alpha p}(|f|^p)(x)>\lambda^p\right\}\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}\\ \lesssim&\left\{\frac{1}{\lambda^p}\||f|^p\|_{L^1(v)}\right\}^{\frac{1}{p}} = \frac{1}{\lambda}\left(\int_{\mathbb{R}^n}|f(x)|Mw(x)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}.\end{aligned}$$

(10) We verify (6) $\Longrightarrow$ (7). By (2.5),

$$w\left(\left\{x\in\mathbb{R}^n : \frac{M_\alpha f(x)}{M_{\bar{B}}\left(u^{\frac{1}{q}}\right)(x)}>\lambda\right\}\right)\leqslant w\left(\left\{x\in\mathbb{R}^n : 2M_{B,\alpha}\left(fu^{-\frac{1}{q}}\right)(x)>\lambda\right\}\right).$$

By (6),

$$w\left(\left\{x\in\mathbb{R}^n : 2M_{B,\alpha}\left(fu^{-\frac{1}{q}}\right)(x)>\lambda\right\}\right)$$
$$\lesssim\left(\frac{1}{\lambda}\left(\int_{\mathbb{R}^n}|f(x)|^p\left(\frac{Mw(x)}{u(x)}\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}\right)^q.$$

(11) Finally, we verify that (7) $\Longrightarrow$ (1). For $t>1$, letting $Q_0=Q(0,1)$ and $|Q_1|=t$ such that $Q_0\subset Q_1$, $w=\chi_{3Q_1}$ and $f=u=\chi_{Q_0}$, for $\lambda>0$,

$$\lambda w\left(\left\{x\in\mathbb{R}^n : \frac{M_\alpha(f)(x)}{M_{\overline{B}}\left(u^{\frac{1}{q}}\right)(x)}>\lambda\right\}\right)^{\frac{1}{q}}$$

(3.12)

$$\geqslant\lambda\left|\left\{x\in 3Q_1 : |x|>\ell(Q_1),\frac{M_\alpha\left(\chi_{Q_0}\right)(x)}{M_{\overline{B}}\left(\chi_{Q_0}\right)(x)}>\lambda\right\}\right|^{\frac{1}{q}}$$

and

(3.13)

$$\left(\int_{\mathbb{R}^n}|f(x)|^p\left(\frac{Mw(x)}{u(x)}\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}=\left(\int_{Q_0}M\left(\chi_{3Q_1}\right)(x)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}\leqslant|Q_0|^{\frac{1}{p}}=1<\infty$$

hold. If $x\in 3Q_1$, then

$$M_\alpha\left(\chi_{Q_0}\right)(x)\geqslant\ell(3Q_1)^\alpha\left(\fint_{3Q_1}\chi_{Q_0(y)}dy\right)\cong|Q_1|^{\frac{\alpha}{n}-1}=t^{\frac{\alpha}{n}-1}. \tag{3.14}$$

If $x\in 3Q_1$ and $|x|>\ell(Q_1)(>1)$, by property (2.7) in Lemma 2.2,

$$M_{\overline{B}}\left(\chi_{Q_0}\right)(x)\cong\overline{B}^{-1}\left(|x|^n\right)^{-1}\leqslant\overline{B}^{-1}\left(|Q_1|\right)^{-1}=\overline{B}^{-1}(t)^{-1}. \tag{3.15}$$

(3.13) and (3.15) imply that

$$\frac{M_\alpha\left(\chi_{Q_0}\right)(x)}{M_{\overline{B}}\left(\chi_{Q_0}\right)(x)}\gtrsim t^{\frac{\alpha}{n}-1}\overline{B}^{-1}(t).$$

Taking $\lambda\cong t^{\frac{\alpha}{n}-1}\overline{B}^{-1}(t)$, we have

$$w\left(\left\{x\in\mathbb{R}^n : \frac{M_\alpha(f)(x)}{M_{\overline{B}}\left(u^{\frac{1}{q}}\right)(x)}>\lambda\right\}\right)\gtrsim|Q_1|=t.$$

Hence,

$$\lambda w\left(\left\{x\in\mathbb{R}^n : \frac{M_\alpha(f)(x)}{M_{\overline{B}}\left(u^{\frac{1}{q}}\right)(x)}>\lambda\right\}\right)^{\frac{1}{q}} \gtrsim \lambda|Q_1|^{\frac{1}{q}} \cong t^{\frac{\alpha}{n}-1+\frac{1}{q}}\overline{B}^{-1}(t). \tag{3.16}$$

By (2.4),

$$t^{\frac{\alpha}{n}-1+\frac{1}{q}}\overline{B}^{-1}(t) \geqslant t^{\frac{\alpha}{n}+\frac{1}{q}}\frac{1}{B^{-1}(t)} \tag{3.17}$$

holds. By assumption (7), inequalities (3.12), (3.13), (3.16) and (3.17) give

$$\frac{t^{\frac{\alpha}{n}+\frac{1}{q}}}{B^{-1}(t)} \lesssim 1 \text{ and } B\left(t^{\frac{\alpha}{n}+\frac{1}{q}}\right)\leqslant t.$$

Taking $s\cong t^{\frac{\alpha}{n}+\frac{1}{q}}$, we conclude

$$B(s)\lesssim s^{\left(\frac{\alpha}{n}+\frac{1}{q}\right)^{-1}}.$$

Since $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, we obtain (1). □

Proof of Theorem 1.19. By Lemma 2.3, we have only to analyze the mixed weighted norm of $M_{V,B,\alpha,3\mathscr{D}}f$. We may assume that $f\in L^\infty_{\rm c}$ as before. For $k\in\mathbb{Z}$, let $\Omega_k := \left\{x\in\mathbb{R}^n : M_{V,B,\alpha,3\mathscr{D}}f(x)>a^k\right\}$ in Lemma 2.5. Since $f\in L^\infty_{\rm c}$, we have

$$\mathbb{R}^n=\bigcup_{k\in\mathbb{Z}}\Omega_k\setminus\Omega_{k+1}.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^n} M_{V,B,\alpha,3\mathscr{D}}f(x)^p v(x)V(x)dx &= \sum_{k\in\mathbb{Z}}\int_{\Omega_k\setminus\Omega_{k+1}} M_{V,B,\alpha,3\mathscr{D}}f(x)^p v(x)V(x)dx\\ &\leqslant \sum_{k\in\mathbb{Z}} a^{(k+1)p}\int_{\Omega_k\setminus\Omega_{k+1}} v(x)V(x)dx.\end{aligned}$$

Since $\Omega_k\setminus\Omega_{k+1}\subset\bigcup_j Q_{k,j}$,

$$\sum_{k\in\mathbb{Z}} a^{(k+1)p}\int_{\Omega_k\setminus\Omega_{k+1}} v(x)V(x)dx \leqslant a^p\sum_{k,j\in\mathbb{Z}}\left(\int_{Q_{k,j}} v(x)V(x)dx\right)a^{kp}.$$

Since $a^{kp}<V(3Q_{k,j})^{\frac{\alpha p}{n}}\|f\|^p_{B,3Q_{k,j}}$,

$$\begin{aligned}\sum_{k,j\in\mathbb{Z}}\left(\int_{Q_{k,j}} v(x)V(x)dx\right)a^{kp} &\leqslant \sum_{k,j\in\mathbb{Z}}\left(\int_{Q_{k,j}} v(x)V(x)dx\right)V(3Q_{k,j})^{\frac{\alpha p}{n}}\|f\|^p_{B,3Q_{k,j}}\\ &= \sum_{k,j\in\mathbb{Z}}\left(\frac{1}{V(Q_{k,j})}\int_{Q_{k,j}} v(x)V(x)dx\right)V(3Q_{k,j})^{\frac{\alpha p}{n}}\|f\|^p_{B,3Q_{k,j}}V(Q_{k,j}).\end{aligned}$$

Since $V(Q_{k,j}) \leqslant CV(E_{k,j})$,

$$\begin{aligned}
&\sum_{k,j\in\mathbb{Z}} \left(\frac{1}{V(Q_{k,j})}\int_{Q_{k,j}} v(x)V(x)dx\right) V(3Q_{k,j})^{\frac{\alpha p}{n}} \|f\|_{B,3Q_{k,j}}^p V(Q_{k,j})\\
&\leqslant C\sum_{k,j\in\mathbb{Z}} \left(\frac{1}{V(Q_{k,j})}\int_{Q_{k,j}} v(x)V(x)dx\right) V(3Q_{k,j})^{\frac{\alpha p}{n}} \|f\|_{B,3Q_{k,j}}^p V(E_{k,j}).
\end{aligned}$$

By (2.6), $\|f\|_{B,3Q_{k,j}} \leqslant 2\left\|fw^{\frac{1}{p}}\right\|_{C,3Q_{k,j}}\left\|w^{-\frac{1}{p}}\right\|_{A,3Q_{k,j}}$. This implies that

$$\begin{aligned}
&\sum_{k,j\in\mathbb{Z}} \left(\frac{1}{V(Q_{k,j})}\int_{Q_{k,j}} v(x)V(x)dx\right) V(3Q_{k,j})^{\frac{\alpha p}{n}} \|f\|_{B,3Q_{k,j}}^p V(E_{k,j})\\
&\leqslant 2^p\sum_{k,j} V(3Q_{k,j})^{\frac{\alpha p}{n}} \left\|fw^{\frac{1}{p}}\right\|_{C,3Q_{k,j}}^p \left\|w^{-\frac{1}{p}}\right\|_{A,3Q_{k,j}}^p \left(\frac{1}{V(Q_{k,j})}\int_{Q_{k,j}} v(x)V(x)dx\right) V(E_{k,j}).
\end{aligned}$$

By (1.28) and (1.14)

$$\begin{aligned}
&\sum_{k,j} V(3Q_{k,j})^{\frac{\alpha p}{n}} \left\|fw^{\frac{1}{p}}\right\|_{C,3Q_{k,j}}^p \left\|w^{-\frac{1}{p}}\right\|_{A,3Q_{k,j}}^p \left(\frac{1}{V(Q_{k,j})}\int_{Q_{k,j}} v(x)V(x)dx\right) V(E_{k,j})\\
&\leqslant C^pK_1^p\sum_{k,j}\left\|fw^{\frac{1}{p}}\right\|_{C,3Q_{k,j}}^p V(E_{k,j}) = C^pK_1^p\sum_{k,j}\int_{E_{k,j}}\left\|fw^{\frac{1}{p}}\right\|_{C,3Q_{k,j}}^p V(x)dx\\
&\leqslant C^pK_1^p\sum_{k,j}\int_{E_{k,j}}\left\{M_C\left(fw^{\frac{1}{p}}\right)(x)\right\}^p V(x)dx\\
&= C^pK_1^p\sum_{k}\int_{\Omega_k\setminus\Omega_{k+1}}\left\{M_C\left(fw^{\frac{1}{p}}\right)(x)\right\}^p V(x)dx\\
&= C^pK_1^p\int_{\mathbb{R}^n}\left\{M_C\left(fw^{\frac{1}{p}}\right)(x)\right\}^p V(x)dx.
\end{aligned}$$

By assumption (1.15),

$$\begin{aligned}
\int_{\mathbb{R}^n}\left\{M_C\left(fw^{\frac{1}{p}}\right)(x)\right\}^p V(x)dx &\leqslant C\int_{\mathbb{R}^n}\left(|f(x)|w(x)^{\frac{1}{p}}\right)^p W(x)dx\\
&= C\int_{\mathbb{R}^n}|f(x)|^p w(x)W(x)dx.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.21. By Lemma 2.3, we have only to analyze the mixed weighted norm of $M_{B,\alpha,3\mathscr{D}}f$. We may assume that $f \in L^\infty_{\rm c}$. For $k \in \mathbb{Z}$, let $\Omega_k :=$ $\left\{x\in\mathbb{R}^n : M_{B,\alpha,3\mathscr{D}}f(x) > a^k\right\}$ in Lemma 2.5. Since $f \in L^\infty_{\rm c}$, we have

$$\mathbb{R}^n = \bigcup_{k\in\mathbb{Z}} \Omega_k\setminus\Omega_{k+1}.$$

Based on this partition, we decompose the integrals:

$$\int_{\mathbb{R}^n} M_{B,\alpha,3\mathscr{D}} f(x)^q v(x)V(x)dx = \sum_{k\in\mathbb{Z}} \int_{\Omega_k\setminus\Omega_{k+1}} M_{B,\alpha,3\mathscr{D}} f(x)^q v(x)V(x)dx.$$

By the definition of Ω_k,

$$\sum_{k\in\mathbb{Z}} \int_{\Omega_k\setminus\Omega_{k+1}} M_{B,\alpha,3\mathscr{D}} f(x)^q v(x)V(x)dx \leqslant a^q \sum_{k\in\mathbb{Z}} a^{kq} \int_{\Omega_k\setminus\Omega_{k+1}} v(x)V(x)dx.$$

Since $\Omega_k\setminus\Omega_{k+1} \subset \bigcup_j Q_{k,j}$,

$$\sum_{k\in\mathbb{Z}} a^{kq} \int_{\Omega_k\setminus\Omega_{k+1}} v(x)V(x)dx \leqslant \sum_{k,j\in\mathbb{Z}} a^{kq} \int_{Q_{k,j}} v(x)V(x)dx.$$

Since $a^{kq} < \ell(3Q_{k,j})^{\alpha q} \|f\|_{B,3Q_{k,j}}^q$,

$$\sum_{k,j\in\mathbb{Z}} a^{kq} \int_{Q_{k,j}} v(x)V(x)dx \leqslant \sum_{k,j\in\mathbb{Z}} \ell(3Q_{k,j})^{\alpha q} \|f\|_{B,3Q_{k,j}}^q \int_{Q_{k,j}} v(x)V(x)dx.$$

By (2.6), $\|f\|_{B,3Q_{k,j}} \leqslant 2 \left\| f w^{\frac{1}{p}} \right\|_{C,3Q_{k,j}} \left\| w^{-\frac{1}{p}} \right\|_{A,3Q_{k,j}}$. This implies that

$$\begin{aligned}
&\sum_{k,j\in\mathbb{Z}} \ell(3Q_{k,j})^{\alpha q} \|f\|_{B,3Q_{k,j}}^q \int_{Q_{k,j}} v(x)V(x)dx \\
\leqslant& 2^q \sum_{k,j\in\mathbb{Z}} \ell(3Q_{k,j})^{\alpha q} \left\| f w^{\frac{1}{p}} \right\|_{C,3Q_{k,j}}^q \left\| w^{-\frac{1}{p}} \right\|_{A,3Q_{k,j}}^q \int_{Q_{k,j}} v(x)V(x)dx.
\end{aligned}$$

Since $V(Q_{k,j}) \leqslant CV(E_{k,j})$,

$$\begin{aligned}
&\sum_{k,j\in\mathbb{Z}} \ell(3Q_{k,j})^{\alpha q} \left\| f w^{\frac{1}{p}} \right\|_{C,3Q_{k,j}}^q \left\| w^{-\frac{1}{p}} \right\|_{A,3Q_{k,j}}^q \int_{Q_{k,j}} v(x)V(x)dx \\
\leqslant& \sum_{k,j\in\mathbb{Z}} \ell(3Q_{k,j})^{\alpha q} \left\| f w^{\frac{1}{p}} \right\|_{C,3Q_{k,j}}^q \left\| w^{-\frac{1}{p}} \right\|_{A,3Q_{k,j}}^q \left(\frac{1}{V(Q_{k,j})} \int_{Q_{k,j}} v(x)V(x)dx \right) V(E_{k,j}).
\end{aligned}$$

By (1.28) and (1.19),

$$\begin{aligned}
&\sum_{k,j\in\mathbb{Z}} \ell(3Q_{k,j})^{\alpha q} \left\| f w^{\frac{1}{p}} \right\|_{C,3Q_{k,j}}^q \left\| w^{-\frac{1}{p}} \right\|_{A,3Q_{k,j}}^q \left(\frac{1}{V(Q_{k,j})} \int_{Q_{k,j}} v(x)V(x)dx \right) V(E_{k,j}) \\
\leqslant& C^q K_2^q \sum_{k,j\in\mathbb{Z}} \left(\ell(3Q_{k,j})^{\alpha} \left\| f w^{\frac{1}{p}} \right\|_{C,3Q_{k,j}} \right)^q V(E_{k,j}) \\
=& C^q K_2^q \sum_{k,j\in\mathbb{Z}} \int_{E_{k,j}} \left(\ell(3Q_{k,j})^{\alpha} \left\| f w^{\frac{1}{p}} \right\|_{C,3Q_{k,j}} \right)^q V(x)dx \\
\leqslant& C^q K_2^q \sum_{k,j\in\mathbb{Z}} \int_{E_{k,j}} M_{C,\alpha} \left(f w^{\frac{1}{p}} \right) (x)^q V(x)dx.
\end{aligned}$$

Therefore, we have

$$\left(\int_{\mathbb{R}^n} M_{B,\alpha}f(x)^q v(x)V(x)dx\right)^{\frac{1}{q}} \leqslant CK_2\left(\int_{\mathbb{R}^n} M_{C,\alpha}\left(fw^{\frac{1}{p}}\right)(x)^q V(x)dx\right)^{\frac{1}{q}}.$$

By assumption (1.20),

$$\left(\int_{\mathbb{R}^n} M_{C,\alpha}\left(fw^{\frac{1}{p}}\right)(x)^q V(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)W(x)dx\right)^{\frac{1}{p}}.$$

This completes the proof. □

4. Applications and related results

REMARK 5. We consider the duality for I_α: for non-negative measurable functions f and g,

$$\int_{\mathbb{R}^n} I_\alpha f(x)g(x)dx = \int_{\mathbb{R}^n} f(x)I_\alpha g(x)dx.$$

Note that

$$\left(\int_{\mathbb{R}^n} I_\alpha f(x)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n} f(x)^p v(x)dx\right)^{\frac{1}{p}} \tag{4.1}$$

holds for all nonnegative measurable functions f and g if and only if the "dual" inequality

$$\left(\int_{\mathbb{R}^n} I_\alpha f(x)^{p'} v(x)^{1-p'}dx\right)^{\frac{1}{p'}} \leqslant C\left(\int_{\mathbb{R}^n} f(x)^{q'} u(x)^{1-q'}dx\right)^{\frac{1}{q'}} \tag{4.2}$$

holds for all nonnegative measurable functions f and g.

In contrast, for non-negative measurable functions f and g,

$$\int_{\mathbb{R}^n} Mf(x)g(x)dx = \int_{\mathbb{R}^n} f(x)Mg(x)dx \tag{4.3}$$

fails as the example of $f = \chi_{(0,1)^n}$ and $g(x) \equiv 1$. In fact, $\int_{\mathbb{R}^n} Mf(x)g(x)dx = \infty > \int_{\mathbb{R}^n} f(x)Mg(x)dx = 1$. This implies that it is not trivial whether or not the boundedness of $M : L^p(u) \to L^p(v)$ yields to the dual inequality $M : L^{p'}(v^{1-p'}) \to L^{p'}(u^{1-p'})$ in general weights u and v. In the case of Proposition 1.18, the dual inequality holds. In fact, since $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$,

$$\|Mf\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)} \iff w \in A_p \Leftrightarrow w^{1-p'} \in A_{p'}$$
$$\Leftrightarrow \|Mf\|_{L^{p'}(w^{1-p'})} \leqslant C\|f\|_{L^{p'}(w^{1-p'})}.$$

In contrast, let us consider the classical weighted inequality of Fefferman and Stein:

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p Mw(x)dx. \tag{4.4}$$

Then the natural "dual" inequality

$$\int_{\mathbb{R}^n} Mf(x)^{p'} (Mw(x))^{1-p'} dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^{p'} w(x)^{1-p'} dx \tag{4.5}$$

fails even as the example of $f = w = \chi_{(0,1)^n}$ shows. Corollary 1.9 can be located as a modificated result in inequality (4.5).

From these observations we learn that it is natural and important to ask ourselves whether the "dual" of the weighted inequalities of the maximal operators M or M_α holds.

REMARK 6. According to [10, 11], we have the following inequality: If $k = 1,2,3,\ldots,$ then there exists a constant $C = C_{n,k}$ such that for all bounded functions f with support contained in Q

$$\|f\|_{L(\log L)^k,Q} \leqslant C ⨍_Q M^k f(y)dy, \tag{4.6}$$

where $M^k = M \circ M \circ \cdots \circ M$ denotes the k-fold composition of M. Estimate 4.6 gives the following inequality: Let $0 \leqslant \alpha < n$,

$$M_{L(\log L)^k,\alpha}(f)(x) \leqslant CM_\alpha \left(M^k f\right)(x). \tag{4.7}$$

Theorem 1.7 1 gives the dual type inequality. The proofs of Corollaries 4.1, 4.2 and 4.3 below are based on the proof of Corollary 1.8 in [11, p.151].

COROLLARY 4.1. *Let $0 \leqslant \alpha < n$, $1 < p < \infty$ satisfy $1 \leqslant p' < \frac{n}{\alpha}$. Suppose that u is a weight. Then*

$$\int_{\mathbb{R}^n} M_\alpha f(x)^p \cdot M_{\alpha p'} \left(M^{[p']}u\right)(x)^{1-p} dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p u(x)^{1-p} dx$$

for all measurable functions f and all weights u.

REMARK 7. In Corollary 4.1 the result is sharp in the sense that it does not hold for $M^{[p']}$ replaced by the pointwise smaller operator $M^{[p']-1}$ as the example of $f = u = \chi_{(0,1)^n}$ shows.

Theorem 1.7 (3) gives the dual type inequality.

COROLLARY 4.2. *Let $0 \leqslant \alpha < n$ and $1 < p < \frac{n}{\alpha}$. Let also u and w be weights. Then*

$$\int_{\mathbb{R}^n} M_\alpha f(y)^p M^{[p'+1]}u(y)^{1-p} w(y)dy \leqslant C \int_{\mathbb{R}^n} |f(y)|^p u(y)^{1-p} \cdot M_{\alpha p} w(y)dy. \tag{4.8}$$

As we will check after the proof of Theorem 4.10, Corollary 4.2 is sharp in the sense that it does not hold with $M^{[p'+1]}$ replaced by the pointwise smaller operator $M^{[p']}$.

Likewise Theorem 1.16 (9) gives the dual type inequality.

COROLLARY 4.3. *Let $0 \leqslant \alpha < n$, $1 < p < \frac{n}{\alpha}$, and $1 < q < \infty$. Suppose that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then,*

$$\left(\int_{\mathbb{R}^n} M_\alpha f(x)^q \frac{dx}{M^{[p'+1]}u(x)^{\frac{q}{p'}}}\right)^{\frac{1}{q}} \leqslant C \left(\int_{\mathbb{R}^n} |f(x)|^p u(x)^{1-p} dx\right)^{\frac{1}{p}} \tag{4.9}$$

for all measurable functions f and u.

We prove Corollary 4.3 only since Corollaries 4.1 and 4.2 can be proved similarly.

Proof of Corollary 4.3. Let $A(t) = t\log^{[p']}(1+t)$ and choose a Young function B so that $B(t) \simeq t^{p'} \log^{[p']}(1+t)$. Write $v = u^{\frac{p'}{q}}$. Then

$$M_{\overline{B}}\left(u^{\frac{1}{q}}\right)(x) = M_{\overline{B}}\left(v^{\frac{p-1}{p}}\right)(x) \leqslant CM^{[p'+1]}v(x)^{\frac{q}{p'}} \quad (x \in \mathbb{R}^n).$$

Hence

$$\left(\int_{\mathbb{R}^n} M_\alpha f(x)^q \frac{w(x)}{M_{\overline{B}}^{\left(\frac{1}{q}\right)} u(x)} dx\right)^{\frac{1}{q}} \geqslant C \left(\int_{\mathbb{R}^n} M_\alpha f(x)^q \frac{w(x)}{M_{\overline{B}}(v)(x)^{\frac{q}{p'}}} dx\right)^{\frac{1}{q}}.$$

By Theorem 1.16 9

$$\left(\int_{\mathbb{R}^n} M_\alpha f(x)^q \frac{w(x)}{M^{[p'+1]}v(x)^{\frac{q}{p'}}} dx\right)^{\frac{1}{q}} \leqslant C \left(\int_{\mathbb{R}^n} |f(x)|^p \left(\frac{Mw(x)}{u(x)}\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$
$$= \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)^{1-p} Mw(x)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}.$$

Taking $w = 1$, and replacing the weight v with the weight u, we obtain

$$\left(\int_{\mathbb{R}^n} M_\alpha f(x)^q \frac{dx}{M^{[p'+1]}u(x)^{\frac{q}{p'}}}\right)^{\frac{1}{q}} \leqslant C \left(\int_{\mathbb{R}^n} |f(x)|^p u(x)^{1-p} dx\right)^{\frac{1}{p}}. \quad \square$$

Given $0<\alpha<n$ and $b\in\mathrm{BMO}$, define the first order commutator $[b,I_\alpha]$ to be the operator

$$[b,I_\alpha]f(x)=\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}f(y)dy.$$

Theorem 1.7 has an application to the commutator $[b,I_\alpha]$ on weighted Lebesgue spaces.

THEOREM 4.4. *Let $0<\alpha<n$ and $1<p<\frac{n}{\alpha}$. If $b\in\mathrm{BMO}(\mathbb{R}^n)$ and $w\in A_\infty(\mathbb{R}^n)$. Then,*

$$\|[b,I_\alpha]f\|_{L^p(w)}\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^p(M_{\alpha p}(w))}\ (f\in L^\infty_c),$$

where

$$\|b\|_{\mathrm{BMO}}:=\sup_{Q\subset\mathbb{R}^n}\fint_Q|b(x)-m_Q(b)|dx.$$

To prove Theorem 4.4, we use Lemmas 4.5, 4.6, and 4.7. Denote by f^* the decreasing rearrangement of a measurable function f. The following inequality is a well-known result (see [1, p.127]):

LEMMA 4.5. *We assume that $f^*(t)\to 0$ $(t\to\infty)$. If $w\in A_\infty(\mathbb{R}^n)$, then for all $p>0$, there exists a constant C such that*

$$\int_{\mathbb{R}^n}M_{\mathscr{D}}f(x)^p w(x)dx\leqslant C\int_{\mathbb{R}^n}M^{\#}f(x)^p w(x)dx,$$

where $M^{\#}$ is the sharp maximal operator:

$$M^{\#}f(x):=\sup_{Q:\text{cube}}\chi_Q(x)\fint_Q|f(y)-m_Q(f)|\,dy,$$

and $M_{\mathscr{D}}$ is the dyadic maximal operator:

$$M_{\mathscr{D}}f(x):=\sup_{x\in Q\in\mathscr{D}(\mathbb{R}^n)}\fint_Q|f(y)|dy.$$

The following point-wise inequality is due to [1].

LEMMA 4.6. *Let $B(t)=t\log(e+t)$, Given α, $0<\alpha<n$, $b\in\mathrm{BMO}$ and a non-negative function $f\in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, there exists a constant C such that for all x,*

$$M^{\#}\left([b,I_\alpha]f\right)(x)\leqslant C\|b\|_{\mathrm{BMO}}\left(I_\alpha f(x)+M_{B,\alpha}f(x)\right).$$

The following inequality is a well-known result (see [7, p.143]) which we use for the proof of Theorem 4.4.

LEMMA 4.7. *If* $w \in A_\infty(\mathbb{R}^n)$*, then* $\|I_\alpha f\|_{L^p(w)} \leqslant C\|M_\alpha f\|_{L^p(w)}$ *for all* $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

Proof of Theorem 4.4. Since $|[b,I_\alpha]f(x)| \leqslant M_{\mathscr{D}}([b,I_\alpha]f)(x)$ by the Lebesgue differentiation theorem,

$$\|[b,I_\alpha]f\|_{L^p(w)} \leqslant \|M_{\mathscr{D}}([b,I_\alpha]f)\|_{L^p(w)}.$$

Since $[b,I_\alpha]f \in L^u(\mathbb{R}^n)$ as long as u satisfies

$$1 < u < \infty, \quad \frac{1}{u} + \frac{\alpha}{n} \in (0,1),$$

$([b,I_\alpha]f)^*(\lambda) \to 0$ as $\lambda \to \infty$. Since $w \in A_\infty(\mathbb{R}^n)$, we are in the position of using Lemma 4.5. The result is

$$\|M_{\mathscr{D}}([b,I_\alpha]f)\|_{L^p(w)} \leqslant C\left\|M^\#([b,I_\alpha]f)\right\|_{L^p(w)}.$$

By Lemma 4.6, if $B(t) = t\log(e+t)$, then

$$\left\|M^\#([b,I_\alpha]f)\right\|_{L^p(w)} \leqslant C\|b\|_{\mathrm{BMO}}\left(\|I_\alpha f\|_{L^p(w)} + \|M_{B,\alpha}f\|_{L^p(w)}\right).$$

By Lemma 4.7 and (1.4) in Theorem 1.7,

$$\|I_\alpha f\|_{L^p(w)} \leqslant C\|M_{B,\alpha}f\|_{L^p(w)}.$$

By (2),

$$\|M_{B,\alpha}f\|_{L^p(w)} \leqslant C\|f\|_{L^p(M_{\alpha p}(w))}.$$

This implies that we get the desired result. □

Theorem 1.16 has an application to the commutator $[b,I_\alpha]$ on weighted Lebesgue spaces.

THEOREM 4.8. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $w \in A_\infty(\mathbb{R}^n)$*, for all* $f \in L^\infty_{\mathrm{c}}$,

$$\|[b,I_\alpha]f\|_{L^q(w)} \leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^p\left((Mw)^{\frac{p}{q}}\right)}.$$

The proof of Theorem 4.8 is omitted since it is similar to the one of Theorem 4.4. Since

$$\int_{\mathbb{R}^n}[b,I_\alpha]f(x)\cdot g(x)dx = -\int_{\mathbb{R}^n}[b,I_\alpha]g(x)\cdot f(x)dx$$

for all $f,g \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, we can dualize Theorems 4.4 and 4.8.

THEOREM 4.9. *Let* $0 < \alpha < n$ *and* $1 < p < \frac{n}{\alpha}$. *Suppose that* $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(A) If $w \in A_\infty(\mathbb{R}^n)$, then for all $f \in L^\infty_c(\mathbb{R}^n)$ and weights w

$$\left(\int_{\mathbb{R}^n} |[b,I_\alpha]f(x)|^{p'} M_{\alpha p}w(x)^{-\frac{p'}{p}}dx\right)^{\frac{1}{p'}} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f(x)|^{p'} w(x)^{-\frac{p'}{p}}dx\right)^{\frac{1}{p'}}.$$

(B) Suppose that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $w \in A_\infty(\mathbb{R}^n)$. Then for all $f \in L^\infty_c(\mathbb{R}^n)$ and weights w

$$\left(\int_{\mathbb{R}^n} |[b,I_\alpha]f(x)|^{p'} Mw(x)^{-\frac{p'}{q}}dx\right)^{\frac{1}{p'}} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f(x)|^{q'} w(x)^{-\frac{q'}{q}}dx\right)^{\frac{1}{q'}}.$$

We now discuss the sharpness of Corollary 4.2. We have an explicit formula for the iterated operator of one-dimensional uncentered Hardy–Littlewood maximal operator M.

THEOREM 4.10. *Let $Q = (a,b)$ be an interval and $k \in \mathbb{N}$. Then*

$$M^k(\chi_Q)(x) = \chi_Q(x) + \chi_{Q^c}(x)\left(\frac{b-a}{\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}}\sum_{j=0}^{k-1}\frac{1}{j!}\left(\log\frac{\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}}{b-a}\right)^j\right) \quad (x \in \mathbb{R}),$$

in particular, if $x > b$, then,

$$M^k(\chi_Q)(x) = \frac{b-a}{x-a}\sum_{j=0}^{k-1}\frac{1}{j!}\left(\log\frac{x-a}{b-a}\right)^j$$

and if $x < a$ then,

$$M^k(\chi_Q)(x) = \frac{b-a}{b-x}\sum_{j=0}^{k-1}\frac{1}{j!}\left(\log\frac{b-x}{b-a}\right)^j.$$

In particular, if $Q = (0,1)$ and $x > 1$, then,

$$M^k\left(\chi_{(0,1)}\right)(x) = \frac{1}{x}\sum_{j=0}^{k-1}\frac{1}{j!}(\log x)^j.$$

It is interesting that the k-th approximation of e^t at $t = 0$ naturally appears in the right-hand side. We prove Theorem 4.10.

Proof of Theorem 4.10. We induct on $k \in \mathbb{N}$. Firstly, we prove the base case of $k = 1$. Clearly, if $x \in Q$, then, $M(\chi_Q)(x) = 1$. If $x > b$, then,

$$M(\chi_Q)(x) = \frac{1}{x-a}\int_a^x \chi_{(a,b)}(y)dy = \frac{1}{x-a}\int_a^b dy = \frac{b-a}{x-a}.$$

If $x<a$, then,

$$M(\chi_Q)(x)=\frac{1}{b-x}\int_x^b \chi_{(a,b)}(y)dy=\frac{1}{b-x}\int_a^b dy=\frac{b-a}{b-x}.$$

Hence, in the case of $k=1$, Theorem 4.10 holds.

We assume that Theorem 4.10 is true in the case of $k=N$. Obviously, if $x\in Q$, then $M^{N+1}(\chi_Q)(x)=1$. If $x>b$, then,

$$\begin{aligned}M^{N+1}(\chi_Q)(x)&=\frac{1}{x-a}\int_a^x M^N(\chi_Q)(y)dy\\&=\frac{1}{x-a}\left(\int_a^b M^N(\chi_Q)(y)dy+\int_b^x M^N(\chi_Q)(y)dy\right).\end{aligned}$$

By the assumption of the induction,

$$\begin{aligned}M^{N+1}(\chi_Q)(x)&=\frac{b-a}{x-a}\left(1+\sum_{j=0}^{N-1}\frac{1}{j!}\int_b^x\frac{1}{y-a}\left(\log\frac{y-a}{b-a}\right)^j dy\right)\\&=\frac{b-a}{x-a}\left(1+\sum_{j=0}^{N-1}\frac{1}{(j+1)!}\left(\log\frac{x-a}{b-a}\right)^{j+1}\right)\\&=\frac{b-a}{x-a}\sum_{j=0}^{N}\frac{1}{j!}\left(\log\frac{x-a}{b-a}\right)^j.\end{aligned}$$

If $x<a$, then,

$$\begin{aligned}M^{N+1}(\chi_Q)(x)&=\frac{1}{b-x}\int_x^b M^N(\chi_Q)(y)dy\\&=\frac{1}{b-x}\left(\int_x^a M^N(\chi_Q)(y)dy+\int_a^b M^N(\chi_Q)(y)dy\right).\end{aligned}$$

Similar argument of the case $x>b$ gives

$$M^{N+1}(\chi_Q)(x)=\frac{b-a}{b-x}\sum_{j=0}^{N}\frac{1}{j!}\left(\log\frac{b-x}{b-a}\right)^j.$$

Therefore, Theorem 4.10 holds in the case of $k=N+1$. This completes the proof. □

We end this section with the proof of the sharpness of Corollary 4.2.

REMARK 8. Corollary 4.2 is sharp in the sense that it does not hold with $M^{[p'+1]}$ replaced by the pointwise smaller operator $M^{[p']}$, as the example of $f=u=\chi_{(0,1)^n}$ and $w=\dfrac{1}{(1+|\cdot|)^{\alpha p}}$ shows. In fact, by Lemma 2.6,

$$\int_{\mathbb{R}^n}|f(y)|^p u(y)^{1-p}M_{\alpha p}w(y)dy\leqslant C.$$

Meanwhile,

$$\int_{\mathbb{R}^n} M_\alpha f(y)^p M^{[p']}u(y)^{1-p}w(y)dy \geqslant \int_e^\infty M_\alpha\left(\chi_{(0,1)}\right)(y)^p M^{[p']}\left(\chi_{(0,1)}\right)(y)^{1-p}\frac{dy}{(1+y)^{\alpha p}}.$$

By Theorem 4.10,

$$\begin{aligned}&\int_e^\infty M_\alpha\left(\chi_{(0,1)}\right)(y)^p M^{[p']}\left(\chi_{(0,1)}\right)(y)^{1-p}\frac{dy}{(1+y)^{\alpha p}}\\ \geqslant & C\int_e^\infty \left(y^{\alpha-1}\right)^p\left(\frac{(\log y)^{([p']-1)}}{y}\right)^{1-p}\frac{dy}{(1+y)^{\alpha p}} = \int_e^\infty y^{\alpha p-1}(\log y)^{([p']-1)(1-p)}\frac{dy}{(1+y)^{\alpha p}}.\end{aligned}$$

By the change of variables $s = \log y$,

$$\int_e^\infty y^{\alpha p-1}(\log y)^{([p']-1)(1-p)}\frac{dy}{(1+y)^{\alpha p}} = \int_1^\infty \left(\frac{e^s}{1+e^s}\right)^{\alpha p} s^{([p']-1)(1-p)}ds.$$

Since the function $F(s) = \left(\frac{e^s}{1+e^s}\right)^{\alpha p}$ is an increasing function,

$$\begin{aligned}\int_1^\infty \left(\frac{e^s}{1+e^s}\right)^{\alpha p} s^{([p']-1)(1-p)}ds &\geqslant F(1)\int_1^\infty s^{([p']-1)(1-p)}ds\\ &= \frac{e^{\alpha p}}{(1+e)^{\alpha p}}\int_1^\infty s^{([p']-1)(1-p)}ds.\end{aligned}$$

Since $([p']-1)(1-p) \geqslant (p'-1)(1-p) = -1$,

$$\frac{e^{\alpha p}}{(1+e)^{\alpha p}}\int_1^\infty s^{([p']-1)(1-p)}ds = \infty.$$

Therefore,

$$\int_{\mathbb{R}^n} M_\alpha f(x)^p M^{[p']}u(x)^{1-p}w(x)dx \leqslant C\int_{\mathbb{R}^n} |f(x)|^p u(x)^{1-p}M_{\alpha p}w(x)dx$$

fails.

5. Appendix–Proof of Proposition 1.23

We prove Proposition 1.23. For every cube Q, we assume $\mathrm{supp}(f) \subset Q$ and that $f(x) \geqslant 0$ for almost every $x \in \mathbb{R}^n$. Let $B_p(t) := B\left(t^{\frac{1}{p}}\right)$. We use the following scaling property of the maximal operators: $M_{B,\alpha}f(x) = M_{B_p,\alpha p}(f^p)(x)^{\frac{1}{p}}$. From (2.18) we have

$$\left|\left\{x \in Q : M_{B_p,\alpha p}(f^p\chi_Q)(x) > t\right\}\right|^{\frac{n-\alpha p}{n}} \leqslant C_1\int_{\{x\in Q: f(x)\geqslant t/C_2\}} B_p\left(\frac{f(x)^p}{t}\right)dx. \tag{5.1}$$

Since

$$\left(\int_Q M_{B,\alpha}(f\chi_Q)(y)^q dy\right)^{\frac{1}{q}} = \left(\int_Q M_{B_p,\alpha p}\left(f^p\chi_Q\right)(y)^{\frac{q}{p}} dy\right)^{\frac{1}{q}}$$
$$= \left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}}\left|\left\{x\in Q : M_{B_p,\alpha p}\left(f^p\chi_Q\right)(x) > t\right\}\right|\frac{dt}{t}\right)^{\frac{1}{q}},$$

we have

$$\left(\int_0^\infty t^{\frac{q}{p}}\left|\left\{x\in Q : M_{B_p,\alpha p}\left(f^p\chi_Q\right)(x) > t\right\}\right|\frac{dt}{t}\right)^{\frac{1}{q}}$$
$$\leqslant C\left(\int_0^\infty t^{\frac{q}{p}}\left(\int_{\{x\in Q: f(x)^p > t/c\}} B_p\left(\frac{f(x)^p}{t}\right)dx\right)^{\frac{n}{n-\alpha p}}\frac{dt}{t}\right)^{\frac{1}{q}}$$

by inequality (5.1). Since $\frac{n}{n-\alpha p} = \frac{q}{p}$, we have

$$\left(\int_0^\infty t^{\frac{q}{p}}\left(\int_{\{x\in Q: f(x)^p > t/C_2\}} B_p\left(\frac{f(x)^p}{t}\right)dx\right)^{\frac{n}{n-\alpha p}}\frac{dt}{t}\right)^{\frac{1}{q}}$$
$$=\left\{\left(\int_0^\infty t^{\frac{q}{p}}\left(\int_{\{x\in Q: f(x)^p > t/C_2\}} B_p\left(\frac{f(x)^p}{t}\right)dx\right)^{\frac{q}{p}}\frac{dt}{t}\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}$$
$$=\left\{\left(\int_0^\infty \left(\int_Q tB_p\left(\frac{f(x)^p}{t}\right)\chi_{\{x\in\mathbb{R}^n: f(x)^p > t/C_2\}}(x)dx\right)^{\frac{q}{p}}\frac{dt}{t}\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}.$$

We use the Minkowski inequality:

$$\left\{\left(\int_0^\infty \left(\int_Q tB_p\left(\frac{f(x)^p}{t}\right)\chi_{\{x\in\mathbb{R}^n: f(x)^p > t/C_2\}}(x)dx\right)^{\frac{q}{p}}\frac{dt}{t}\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}$$
$$\leqslant\left\{\int_Q\left(\int_0^\infty t^{\frac{q}{p}}B_p\left(\frac{f(x)^p}{t}\right)^{\frac{q}{p}}\chi_{\{t>0: t<C_2 f(x)^p\}}(t)\,\frac{dt}{t}\right)^{\frac{p}{q}}dx\right\}^{\frac{1}{p}}$$
$$=\left\{\int_Q\left(\int_0^{C_2 f(x)^p} t^{\frac{q}{p}}B\left(\frac{f(x)}{t^{1/p}}\right)^{\frac{q}{p}}\frac{dt}{t}\right)^{\frac{p}{q}}dx\right\}^{\frac{1}{p}}.$$

By the change of the variables $s = \frac{f(x)}{t^{1/p}}$,

$$\int_0^{C_2 f(x)^p} t^{\frac{q}{p}}B\left(\frac{f(x)}{t^{1/p}}\right)^{\frac{q}{p}}\frac{dt}{t} = p\int_{C_2^{-\frac{1}{p}}}^\infty\left(\frac{f(x)}{s}\right)^q B(s)^{\frac{q}{p}}\frac{ds}{s} = pf(x)^q\int_{C_2^{-\frac{1}{p}}}^\infty\frac{B(s)^{\frac{q}{p}}}{s^{q+1}}ds.$$

Hence,

$$\left\{\int_Q \left(\int_0^{C_2 f(x)^p} t^{\frac{q}{p}} B\left(\frac{f(x)}{t^{1/p}}\right)^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{p}{q}} dx\right\}^{\frac{1}{p}} = \left\{\int_Q \left(p f(x)^q \int_{C_2^{-\frac{1}{p}}}^{\infty} \frac{B(s)^{\frac{q}{p}}}{s^{q+1}} ds\right)^{\frac{p}{q}} dx\right\}^{\frac{1}{p}}$$

$$= \left\{\left(\int_Q p^{\frac{p}{q}} f(x)^p dx\right) \cdot \left(\int_{C_2^{-\frac{1}{p}}}^{\infty} \frac{B(s)^{\frac{q}{p}}}{s^{q+1}} ds\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}$$

$$= p^{\frac{1}{q}} \left(\int_Q f(x)^p dx\right)^{\frac{1}{p}} \cdot \left(\int_{C_2^{-\frac{1}{p}}}^{\infty} \frac{B(s)^{\frac{q}{p}}}{s^{q+1}} ds\right)^{\frac{1}{q}}.$$

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