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REFINEMENTS OF THE HADAMARD AND CAUCHY–SCHWARZ INEQUALITIES WITH TWO INEQUALITIES OF THE PRINCIPAL ANGLES

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Abstract. By discussing two volume formulae for the parallelotope, some refinements of the Hadamard and Cauchy-Schwarz inequalities are given and a class of principal inequalities related a parallelotope is established. This class of principal inequalities have a close relation to the Hadamard and Fischer determinant inequalities. By using the interlacing property, a principal inequality related to two subspaces is given which has a close relation to the Koteljanskii determinant inequality. Analysis indicates that these two principal inequalities can be extended to two class of principal inequalities easily.

1. Introduction

Polytope is an important topic in n-dimensional Euclidean space and some profound results have been established. Simplex and parallelotope are main objects and many properties and conclusions have been established. Some results of the volume formulae for the simplices and parallelotopes are established. For example, a new volume formula for the simplex was established [1], a result for the volume of the largest parallelotope contained in a given simplex was established [2]; some volume formulae of parallelotopes and zonotopes were discussed and some determinant inequalities involved positive definite matrices were proved [3]; a volume formula associated with $m \times n$ matrices was discussed in [4].

Another active topic is the vertex angle of a simplex. A weighted matrix inequality was established and some inequalities on vertex angles of a n-dimensional simplex were discussed [5]. By using the vertex angle, the generalized sine theorem and inequalities for the simplices were established [6]. Two property theorems for the inner and outer bisection planes of the dihedral angles of the simplex were suggested [7]. Angles between two subspaces of dimension p and q in the Euclidean space were discussed [8].

Matrix eigenvalue plays an important role in the matrix equation theory [9, 10, 11, 12] and system and control theory [13, 14]. Much work has been done on the

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eigenvalues of the matrix and some matrix eigenvalues inequalities were established [15, 16, 17, 18]. For example, a new proof method for the arithmetic and geometric mean inequality of singular values for any two matrices was given [19]. By using principal angles and singular values, some volume formulae of parallelotopes were discussed in [20].

In this paper, by introducing two volume formulae of the parallelotope, the relation of the well-known Hadamard and Cauchy-Schwarz inequalities is explored, based on it, some refinements of the Hadamard and Cauchy-Schwarz inequalities are suggested. Using the volume formulae, a class of principal inequalities related to a parallelotope determined by a matrix is established. This class of principal inequalities have a close relation to the determinant Hadamard and Fischer inequalities. By using the interlacing property and discussing the principal angles of two subspaces, a class of principal inequalities of two subspaces is given. Using this principal inequality, the Koteljanskii determinant inequality can be proved [3].

The rest of this paper is organized as follows. Section 2 gives some notation and four lemmas as well as the definition of the principal angle. Section 3 introduces two volume formulae for the parallelotope. Some refinements of the Hadamard and Cauchy-Schwarz inequalities are given in Section 4. Section 5 suggests an principal inequality related to a parallelotope. A principal inequality related to two subspaces is discussed in Section 6. Finally, we offer some concluding remarks in Section 7.

2. Basic preliminaries

Let us introduce some notation and lemmas first. For a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, symbol $\lambda_i[\mathbf{A}]$, $i = 1, 2, \cdots, n$, denotes the eigenvalues of matrix $\mathbf{A}$ with order $\lambda_1[\mathbf{A}] \geqslant \lambda_2[\mathbf{A}] \geqslant \cdots \geqslant \lambda_n[\mathbf{A}]$. Symbol $\det(\mathbf{A})$ represents the determinant of a square matrix $\mathbf{A}$. Set $\mathbf{A}_m := [\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m]$, symbol $\mathscr{A}_m$ denotes the subspace spanned by vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m$, namely, $\mathscr{A}_m := \mathrm{span}\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m\}$, especially, $\mathscr{A}_{[i_1:i_k]}$ represents the subspace spanned by vectors $\boldsymbol{\alpha}_{i_1}, \boldsymbol{\alpha}_{i_2}, \cdots, \boldsymbol{\alpha}_{i_k}$, where $i_1, i_2, \cdots, i_k$ are some consecutive integers not starting from 1. Symbol $\dim(\mathscr{A})$ represents the dimension of subspace $\mathscr{A} \subset \mathbb{R}^n$. $\|\boldsymbol{\alpha}\|$ denotes the vector norm and is defined by formula $\|\boldsymbol{\alpha}\| := \sqrt{\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}}$. $V_{\mathbf{A}}$ denotes the volume of the parallelotope formed by the column vectors of the matrix $\mathbf{A}$. $\mathbf{I}_n$ is an identity matrix with order $n \times n$. $\mathbf{O}$ is the zero matrix with proper order.

The eigenvalue properties of the real symmetric matrix are depicted by the following Courant-Fischer Minimax theorem and interlacing property [21].

LEMMA 1. (**Courant-Fisher Minimax Theorem**) *If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a real symmetric matrix, then*

$$\lambda_i[\mathbf{A}] = \max_{\dim(\mathbf{S})=i} \min_{\mathbf{0}\neq\mathbf{y}\in\mathbf{S}} \frac{\mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}, \ i = 1, 2, \cdots, n. \tag{2.1}$$

LEMMA 2. (**Interlacing Property**) *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, $\mathbf{A}_k := \mathbf{A}_{[i_1,i_2,\cdots,i_k]}$ is the principal submatrix of $\mathbf{A}$ with indices $i_1, i_2, \cdots, i_k$.*

$\lambda_i[\mathbf{A}_k]$ *denotes the eigenvalues of matrix* $\mathbf{A}_k$. *We have the following property:*

$$\lambda_i[\mathbf{A}] \geqslant \lambda_i[\mathbf{A}_k] \geqslant \lambda_{n-k+i}[\mathbf{A}],\ i = 1, 2, \cdots, k. \tag{2.2}$$

Let us recall the the Schmidt orthogonalization of a matrix.

LEMMA 3. (**Schmidt orthogonalization**) *Let* $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m \in \mathbb{R}^n$ *be the linear independent column vectors, there exists an upper triangular matrix* $\mathbf{R}$ *such that*

$$[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m] = [\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_m] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ 0 & r_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & r_{mm} \end{bmatrix}. \tag{2.3}$$

where $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_m$ *are the norm orthogonal vectors in* $\mathbb{R}^n$ *and* $\mathbf{R} := (r_{ij}) = [\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_m] \in \mathbb{R}^{m\times m}$ *is the upper triangular matrix with* $r_{ii} > 0$. *Here,* $m = 1, 2, \cdots, n$. *Equation (2.3) can be written as*

$$\mathbf{A}_m = \mathbf{B}_m \mathbf{R}_m,\ m = 1, 2, \cdots, n, \tag{2.4}$$

where $\mathbf{A}_m = [\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m]$, $\mathbf{B}_m = [\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_m]$ *and* $\mathbf{R}_m = [\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_m]$.

LEMMA 4. *For matrices* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *and* $\mathbf{B} \in \mathbb{R}^{n\times m}$, *the matrices* $\mathbf{AB}$ *and* $\mathbf{BA}$ *have the same nonzero eigenvalues.*

Proof. Suppose that $n \geqslant m$, using the block elementary row transformation, we have

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{O} \\ -\lambda^{-1}\mathbf{A} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \lambda\mathbf{I}_n & \mathbf{B} \\ \mathbf{A} & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{I}_n & \mathbf{B} \\ \mathbf{O} & \mathbf{I}_m - \lambda^{-1}\mathbf{AB} \end{bmatrix},$$

and

$$\begin{bmatrix} \mathbf{I}_n & -\mathbf{B} \\ \mathbf{O} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \lambda\mathbf{I}_n & \mathbf{B} \\ \mathbf{A} & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{BA} & \mathbf{O} \\ \mathbf{A} & \mathbf{I}_m \end{bmatrix}.$$

Taking the determinants of these two equations gives

$$\left| \begin{bmatrix} \lambda\mathbf{I}_n & \mathbf{B} \\ \mathbf{A} & \mathbf{I}_m \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda\mathbf{I}_n & \mathbf{B} \\ \mathbf{O} & \mathbf{I}_m - \lambda^{-1}\mathbf{AB} \end{bmatrix} \right| = |\lambda\mathbf{I}_n| \left|\mathbf{I}_m - \lambda^{-1}\mathbf{AB}\right| = \lambda^{n-m} |\lambda\mathbf{I}_m - \mathbf{AB}|,$$

and

$$\left| \begin{bmatrix} \lambda\mathbf{I}_n & \mathbf{B} \\ \mathbf{A} & \mathbf{I}_m \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{BA} & \mathbf{O} \\ \mathbf{A} & \mathbf{I}_m \end{bmatrix} \right| = |\mathbf{I}_m| \, |\lambda\mathbf{I}_n - \mathbf{BA}| = |\lambda\mathbf{I}_n - \mathbf{BA}|.$$

This shows that matrices $\mathbf{AB}$ and $\mathbf{BA}$ have the same nonzero eigenvalues. The proof is completed. □

The cosine of the principal angles of two subspaces is defined by the following formula [21, 22].

DEFINITION 1. (**Principal angles**) Let $\mathscr{F}$ and $\mathscr{G}$ be subspaces in $\mathbb{R}^n$ whose dimensions satisfy

$$p=\dim(\mathscr{F})\geqslant\dim(\mathscr{G})=q\geqslant 1.$$

If $\dim(\mathscr{F}\cap\mathscr{G})=0$ then the nonzero principal angles $\theta_1,\theta_2,\cdots,\theta_q\in[0,\pi/2]$ between $\mathscr{F}$ and $\mathscr{G}$ are defined recursively by

$$\cos\theta_k=\max_{\mathbf{u}\in\mathscr{F}}\max_{\mathbf{v}\in\mathscr{G}}\mathbf{u}^{\mathrm{T}}\mathbf{v}=\mathbf{u}_k^{\mathrm{T}}\mathbf{v}_k, \tag{2.5}$$

subject to:

$$\begin{aligned}&\|\mathbf{u}\|=\|\mathbf{v}\|=1,\\&\mathbf{u}^{\mathrm{T}}\mathbf{u}_i=0,\ i=1:k-1,\\&\mathbf{v}^{\mathrm{T}}\mathbf{v}_i=0,\ i=1:k-1.\end{aligned}$$

Here, the principal angles satisfy $0\leqslant\theta_1\leqslant\theta_2\cdots\leqslant\theta_q\leqslant\pi/2$. Referring to [20], $\sin\theta(\mathscr{F},\mathscr{G})$ can be defined as $\sin\theta(\mathscr{F},\mathscr{G}):=\sin\theta_1\sin\theta_2\cdots\sin\theta_q$ and $\cos\theta(\mathscr{F},\mathscr{G})$ can be defined as $\cos\theta(\mathscr{F},\mathscr{G}):=\cos\theta_1\cos\theta_2\cdots\cos\theta_q$. In particular, $\sin^2\theta(\mathscr{F},\mathscr{G})+\cos^2\theta(\mathscr{F},\mathscr{G})\leqslant 1$.

3. Two volume formulae for the parallelotope

Let $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m\in\mathbb{R}^n$ be the m linear independent vectors and set $\mathbf{A}_m:=[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m]\in\mathbb{R}^{n\times m}$. Symbol $V_{\mathbf{A}_m}$ denotes the volume of the parallelotope formed by the vectors $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m$. According to equation (2.4), we have the following volume formula:

$$\begin{aligned}V_{\mathbf{A}_m}&=\sqrt{\det(\mathbf{A}^{\mathrm{T}}\mathbf{A})}=\sqrt{\det((\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m)^{\mathrm{T}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m))}\\&=\sqrt{\det(\mathbf{R}_m^{\mathrm{T}}\mathbf{B}_m^{\mathrm{T}}\mathbf{B}_m\mathbf{R}_m)}=\sqrt{\det(\mathbf{R}_m^{\mathrm{T}}\mathbf{R}_m)}=\det(\mathbf{R}_m)=r_{11}r_{22}\cdots r_{mm}.\end{aligned} \tag{3.1}$$

Here, we can take $r_{ii}, i=1,2,\cdots,m$, as the distance from the end of the vector $\boldsymbol{\alpha}_i$ to the subspace $\mathscr{A}_{i-1}$ spanned by vectors $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_{i-1}$. Symbol $\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1})$ denotes the principal angle formed by vector $\boldsymbol{\alpha}_m$ and subspace $\mathscr{A}_{m-1}$, according to the definition of principal angle in Definition 1, we have

$$\begin{aligned}\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1})&=\min_{\mathbf{u}\in\operatorname{span}\mathbf{A}_{m-1}}\theta(\boldsymbol{\alpha}_m,\mathbf{u})=\arccos\left(\frac{\boldsymbol{\alpha}_m^{\mathrm{T}}\mathbf{P}_{\mathbf{A}_{m-1}}\boldsymbol{\alpha}_m}{\|\boldsymbol{\alpha}_m\|\|\mathbf{P}_{\mathbf{A}_{m-1}}\boldsymbol{\alpha}_m\|}\right)\\&=\theta(\boldsymbol{\alpha}_m,\mathbf{P}_{\mathbf{A}_{m-1}}\boldsymbol{\alpha}_m),\end{aligned} \tag{3.2}$$

where $\mathbf{P}_{\mathbf{A}_{m-1}}=\mathbf{A}_{m-1}\left(\mathbf{A}_{m-1}^{\mathrm{T}}\mathbf{A}_{m-1}\right)^{-1}\mathbf{A}_{m-1}^{\mathrm{T}}$ is the orthogonal projection matrix of $\mathbf{A}_{m-1}$. According to equation (2.4), we have

$$\mathbf{A}_{m-1}=[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_{m-1}]=[\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_{m-1}]\mathbf{R}_{m-1}.$$

Set $\mathbf{B}_{m-1}=[\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_{m-1}]$, we have

$$\begin{aligned}\mathbf{P}_{\mathbf{A}_{m-1}}&=\mathbf{A}_{m-1}\left(\mathbf{A}_{m-1}^{\mathrm{T}}\mathbf{A}_{m-1}\right)^{-1}\mathbf{A}_{m-1}^{\mathrm{T}}=\mathbf{B}_{m-1}\mathbf{R}_{m-1}\left(\mathbf{R}_{m-1}^{\mathrm{T}}\mathbf{B}_{m-1}^{\mathrm{T}}\mathbf{B}_{m-1}\mathbf{R}_{m-1}\right)^{-1}\mathbf{R}_{m-1}^{\mathrm{T}}\mathbf{B}_{m-1}^{\mathrm{T}}\\&=\mathbf{B}_{m-1}\mathbf{R}_{m-1}\left(\mathbf{R}_{m-1}^{-1}\mathbf{R}_{m-1}^{-\mathrm{T}}\right)\mathbf{R}_{m-1}^{\mathrm{T}}\mathbf{B}_{m-1}^{\mathrm{T}}=\mathbf{B}_{m-1}\mathbf{B}_{m-1}^{\mathrm{T}}.\end{aligned} \tag{3.3}$$

Using $\boldsymbol{\alpha}_m = r_{1m}\boldsymbol{\beta}_1 + r_{2m}\boldsymbol{\beta}_2 + \cdots + r_{m-1,m}\boldsymbol{\beta}_{m-1} + r_{mm}\boldsymbol{\beta}_m$ gives

$$\begin{aligned}\mathbf{P}_{\mathbf{A}_{m-1}}\boldsymbol{\alpha}_m &= \mathbf{B}_{m-1}\mathbf{B}_{m-1}^{\mathrm{T}}[\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_m]\boldsymbol{\gamma}_m = \mathbf{B}_{m-1}[\mathbf{I}_{m-1},\mathbf{O}]\boldsymbol{\alpha}_m \\ &= r_{1m}\boldsymbol{\beta}_1 + r_{2m}\boldsymbol{\beta}_2 + \cdots + r_{m-1,m}\boldsymbol{\beta}_{m-1} \\ &= \mathbf{B}_m\begin{bmatrix} r_{1m} \\ \vdots \\ r_{m-1,m} \\ 0 \end{bmatrix}. \end{aligned} \tag{3.4}$$

Using $\boldsymbol{\alpha}_m = r_{1m}\boldsymbol{\beta}_1 + r_{2m}\boldsymbol{\beta}_2 + \cdots + r_{m-1,m}\boldsymbol{\beta}_{m-1} + r_{mm}\boldsymbol{\beta}_m$ and equation (3.4) give

$$\begin{aligned}\cos\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1}) &= \frac{\boldsymbol{\alpha}_m^{\mathrm{T}}\mathbf{P}_{\mathbf{A}_{m-1}}\boldsymbol{\alpha}_m}{\|\boldsymbol{\alpha}_m\|\|\mathbf{P}_{\mathbf{A}_{m-1}}\boldsymbol{\alpha}_m\|} \\ &= \frac{r_{1m}^2 + r_{2m}^2 + \cdots + r_{m-1,m}^2}{\sqrt{r_{1m}^2 + r_{2m}^2 + \cdots + r_{mm}^2}\sqrt{r_{1m}^2 + r_{2m}^2 + \cdots + r_{m-1,m}^2}} \\ &= \frac{\sqrt{r_{1m}^2 + r_{2m}^2 + \cdots + r_{m-1,m}^2}}{\sqrt{r_{1m}^2 + r_{2m}^2 + \cdots + r_{mm}^2}}. \end{aligned} \tag{3.5}$$

According the well known formula $\sin^2\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1}) + \cos^2\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1}) = 1$, we have

$$\sin\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1}) = \frac{r_{m,m}}{\sqrt{r_{1m}^2 + r_{2m}^2 + \cdots + r_{mm}^2}} = \frac{r_{mm}}{\|\boldsymbol{\gamma}_m\|} \tag{3.6}$$

$$= \frac{r_{11}r_{22}\cdots r_{mm}}{\|\boldsymbol{\gamma}_m\| r_{11}r_{22}\cdots r_{m-1,m-1}} = \frac{V_{\mathbf{A}_m}}{\|\boldsymbol{\gamma}_m\| V_{\mathbf{A}_{m-1}}}. \tag{3.7}$$

Using equation (3.6), equation (3.1) can be rewritten as

$$\begin{aligned}\mathbf{V}_{\mathbf{A}_m} &= r_{11}r_{22}\cdots r_{mm} = r_{11}\|\boldsymbol{\gamma}_2\|\frac{r_{22}}{\|\boldsymbol{\gamma}_2\|}\cdots\|\boldsymbol{\gamma}_m\|\frac{r_{mm}}{\|\boldsymbol{\gamma}_m\|} \\ &= \|\boldsymbol{\gamma}_1\|\|\boldsymbol{\gamma}_2\|\sin\theta(\boldsymbol{\alpha}_2,\mathscr{A}_1)\cdots\|\boldsymbol{\gamma}_m\|\sin\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1}) \\ &= \|\boldsymbol{\gamma}_1\|\|\boldsymbol{\gamma}_2\|\cdots\|\boldsymbol{\gamma}_m\|\sin\theta(\boldsymbol{\alpha}_2,\mathscr{A}_1)\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\cdots\sin\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1}). \end{aligned} \tag{3.8}$$

Note that $\|\boldsymbol{\alpha}_i\| = \|\boldsymbol{\gamma}_i\|$, $i = 1,2,\cdots,m$ and using equation (3.8) give

$$\mathbf{V}_{\mathbf{A}_m} = \|\boldsymbol{\alpha}_1\|\|\boldsymbol{\alpha}_2\|\cdots\|\boldsymbol{\alpha}_m\|\sin\theta(\boldsymbol{\alpha}_2,\mathscr{A}_1)\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\cdots\sin\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1}). \tag{3.9}$$

So we obtain a volume formula for the parallelotope. For $\sin\theta(\boldsymbol{\alpha}_i,\mathscr{A}_{i-1}) \leqslant 1$, $i = 2,3,\cdots,m-1$, from equation (3.9) we obtain the following Hadamard inequality,

$$\mathbf{V}_{\mathbf{A}_m} \leqslant \|\boldsymbol{\alpha}_1\|\|\boldsymbol{\alpha}_2\|\cdots\|\boldsymbol{\alpha}_m\|. \tag{3.10}$$

Next, we give another volume formula for the parallelotope. Suppose that matrix $\mathbf{A}_m$ can be divided into two block matrices denoted by $\mathbf{C} := [\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_p]$ and

$\mathbf{D} := [\boldsymbol{\alpha}_{p+1}, \boldsymbol{\alpha}_{p+2}, \cdots, \boldsymbol{\alpha}_m]$, that is, $\mathbf{A}_m = [\mathbf{C}, \mathbf{D}]$. Then we have the following volume formula for the parallelotope,

$$\begin{aligned} \mathbf{V}^2_{\mathbf{A}_m} &= \det(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \det\left(\begin{bmatrix} \mathbf{C}^{\mathrm{T}} \\ \mathbf{D}^{\mathrm{T}} \end{bmatrix} [\mathbf{C}, \mathbf{D}]\right) = \det\left(\begin{bmatrix} \mathbf{C}^{\mathrm{T}}\mathbf{C} & \mathbf{C}^{\mathrm{T}}\mathbf{D} \\ \mathbf{D}^{\mathrm{T}}\mathbf{C} & \mathbf{D}^{\mathrm{T}}\mathbf{D} \end{bmatrix}\right) \\ &= \det(\mathbf{C}^{\mathrm{T}}\mathbf{C}) \det\left(\mathbf{D}^{\mathrm{T}}\mathbf{D} - \mathbf{D}^{\mathrm{T}}\mathbf{C}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{D}\right) \\ &= \det(\mathbf{C}^{\mathrm{T}}\mathbf{C}) \det(\mathbf{D}^{\mathrm{T}}\mathbf{D}) \det\left(\mathbf{I}_{m-p} - (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{D}\right), \end{aligned} \tag{3.11}$$

where $m - p = \dim(\mathbf{D})$. Introducing symbols $\mathscr{C} := \operatorname{span}\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_p\}$ and $\mathscr{D} := \operatorname{span}\{\boldsymbol{\alpha}_{p+1}, \boldsymbol{\alpha}_{p+2}, \cdots, \boldsymbol{\alpha}_m\}$, if we define

$$\sin^2\theta(\mathscr{C}, \mathscr{D}) := \det\left(\mathbf{I}_{m-p} - (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{D}\right) \tag{3.12}$$

and use equation (3.1) then it gives

$$V_{\mathbf{A}_m} = V_{\mathbf{C}} V_{\mathbf{D}} \sin\theta(\mathscr{C}, \mathscr{D}). \tag{3.13}$$

4. Refinements of the Hadamard and Cauchy-Schwarz inequalities

The Hadamard inequality is a basic and important inequality and many proof methods has been established [3, 23]. The Cauchy-Schwarz inequality is well-known and an important inequality. Many extensions and refinements have been done [24, 25, 26, 27, 28, 29], but the relation of these two important inequalities seldom mentioned in the literatures. To establish the refinements of the Hadamard and Cauchy-Schwarz inequalities, we make more discussion on $\sin\theta(\mathscr{C}, \mathscr{D})$.

Let $\mathbf{C} = \mathbf{B}_{\mathbf{C}}\mathbf{R}_{\mathbf{C}}$ and $\mathbf{D} = \mathbf{B}_{\mathbf{D}}\mathbf{R}_{\mathbf{D}}$ are the Schmidt orthogonal decomposition forms of the matrices $\mathbf{C}$ and $\mathbf{D}$, respectively. We have

$$\begin{aligned} \sin^2\theta(\mathscr{C}, \mathscr{D}) &= \det(\mathbf{I}_{m-p} - (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{D}) \\ &= \det\Big(\mathbf{I}_{m-p} - ((\mathbf{B}_{\mathbf{D}}\mathbf{R}_{\mathbf{D}})^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}\mathbf{R}_{\mathbf{D}})^{-1}(\mathbf{B}_{\mathbf{D}}\mathbf{R}_{\mathbf{D}})^{\mathrm{T}}\mathbf{B}_{\mathbf{C}}\mathbf{R}_{\mathbf{C}} \times \\ &\qquad \times((\mathbf{B}_{\mathbf{C}}\mathbf{R}_{\mathbf{C}})^{\mathrm{T}}\mathbf{B}_{\mathbf{C}}\mathbf{R}_{\mathbf{C}})^{-1}(\mathbf{B}_{\mathbf{C}}\mathbf{R}_{\mathbf{C}})^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}\mathbf{R}_{\mathbf{D}}\Big) \\ &= \det\left(\mathbf{I}_{m-p} - (\mathbf{R}_{\mathbf{D}}^{\mathrm{T}}\mathbf{R}_{\mathbf{D}})^{-1}(\mathbf{B}_{\mathbf{D}}\mathbf{R}_{\mathbf{D}})^{\mathrm{T}}\mathbf{B}_{\mathbf{C}}\mathbf{R}_{\mathbf{C}}(\mathbf{R}_{\mathbf{C}}^{\mathrm{T}}\mathbf{R}_{\mathbf{C}})^{-1}(\mathbf{B}_{\mathbf{C}}\mathbf{R}_{\mathbf{C}})^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}\mathbf{R}_{\mathbf{D}}\right) \\ &= \det\left(\mathbf{I}_{m-p} - \mathbf{R}_{\mathbf{D}}^{-1}\mathbf{B}_{\mathbf{D}}^{\mathrm{T}}\mathbf{B}_{\mathbf{C}}\mathbf{B}_{\mathbf{C}}^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}\mathbf{R}_{\mathbf{D}}\right) = \det\left(\mathbf{I}_{m-p} - \mathbf{B}_{\mathbf{D}}^{\mathrm{T}}\mathbf{B}_{\mathbf{C}}\mathbf{B}_{\mathbf{C}}^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}\right). \end{aligned} \tag{4.1}$$

According to equation (8) in [30], we have $0 \leqslant \lambda[\mathbf{B}_{\mathbf{D}}^{\mathrm{T}}\mathbf{B}_{\mathbf{C}}\mathbf{B}_{\mathbf{C}}^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}] \leqslant 1$. Since $\mathbf{B}_{\mathbf{D}}^{\mathrm{T}}\mathbf{B}_{\mathbf{C}}\mathbf{B}_{\mathbf{C}}^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}$ is a real symmetric matrix, there exists an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}^{\mathrm{T}}\mathbf{B}_{\mathbf{C}}\mathbf{B}_{\mathbf{C}}^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}\mathbf{Q} =: \boldsymbol{\Lambda} \in \mathbb{R}^{(m-p)\times(m-p)}$ is diagonal. Set $\boldsymbol{\Lambda} := \operatorname{diag}(\delta_1^2, \delta_2^2, \cdots, \delta_{m-p}^2)$, $\delta_i > 0$, then we have

$$\begin{aligned} \sin^2\theta(\mathscr{C}, \mathscr{D}) &= \det(\mathbf{I}_p)\det\left(\mathbf{I}_{m-p} - \mathbf{B}_{\mathbf{D}}^{\mathrm{T}}\mathbf{B}_{\mathbf{C}}\mathbf{B}_{\mathbf{C}}^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}\right) = \det\left(\mathbf{I}_{m-p} - \mathbf{Q}^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}^{\mathrm{T}}\mathbf{B}_{\mathbf{C}}\mathbf{B}_{\mathbf{C}}^{\mathrm{T}}\mathbf{B}_{\mathbf{D}}\mathbf{Q}\right) \\ &= \det(\mathbf{I}_{m-p} - \boldsymbol{\Lambda}) = (1 - \delta_1^2)(1 - \delta_2^2)\cdots(1 - \delta_{m-p}^2). \end{aligned} \tag{4.2}$$

Next, we prove that $\cos\theta_i = \delta_i$, $i = 1, 2, \cdots, m-p$. According to Definition 1 and using $\mathscr{B}_{\mathbf{C}}$ and $\mathscr{B}_{\mathbf{D}}$ represent the subspaces spanned by the vectors of the matrices $\mathbf{B}_{\mathbf{C}}$

and $\mathbf{B_D}$, respectively, since $\mathbf{B_C}$ and $\mathbf{B_D}$ are normal orthogonal matrices, we have

$$\cos\theta_i = \max_{\mathbf{u}\in\mathscr{B}_\mathbf{C}}\max_{\mathbf{v}\in\mathscr{B}_\mathbf{D}}\frac{\mathbf{u}^\mathrm{T}\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \max_{\substack{\mathbf{y}\in\mathbb{R}^p\\ \|\mathbf{y}\|=1}}\max_{\substack{\mathbf{z}\in\mathbb{R}^{m-p}\\ \|\mathbf{z}\|=1}}\mathbf{y}^\mathrm{T}(\mathbf{B_C^T B_D})\mathbf{z} = \sqrt{\max_{\substack{\mathbf{x}\in\mathbb{R}^{m-p}\\ \|\mathbf{x}\|=1}}\mathbf{x}^\mathrm{T}(\mathbf{B_D^T B_C B_C^T B_D})\mathbf{x}}$$
$$= \delta_i. \tag{4.3}$$

So equation (4.2) can be rewritten as

$$\begin{aligned}\sin^2\theta(\mathscr{C},\mathscr{D}) &= (1-\delta_1^2)(1-\delta_2^2)\cdots(1-\delta_{m-p}^2)\\ &= (1-\cos^2\theta_1)(1-\cos^2\theta_2)\cdots(1-\cos^2\theta_{m-p})\\ &= \sin^2\theta_1\sin^2\theta_2\cdots\sin^2\theta_{m-p}.\end{aligned}\tag{4.4}$$

Combining equations (3.13) and (4.4) gives

$$V_{\mathbf{A}_m} = V_\mathbf{C}V_\mathbf{D}\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-p}. \tag{4.5}$$

With these preparations, next we establish the refinements of the Hadamard and Cauchy-Schwarz inequalities. When $m=2$, equation (3.9) or (4.5) is

$$\begin{aligned}\mathbf{V}_{\mathbf{A}_2}^2 &= \begin{vmatrix}\boldsymbol{\alpha}_1^\mathrm{T}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_1^\mathrm{T}\boldsymbol{\alpha}_2\\ \boldsymbol{\alpha}_2^\mathrm{T}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2^\mathrm{T}\boldsymbol{\alpha}_2\end{vmatrix} = \|\boldsymbol{\alpha}_1\|^2\|\boldsymbol{\alpha}_2\|^2\sin^2\theta(\boldsymbol{\alpha}_2,\mathscr{A}_1)\\ &= \|\boldsymbol{\alpha}_1\|^2\|\boldsymbol{\alpha}_2\|^2\sin^2\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1).\end{aligned}\tag{4.6}$$

Manipulating equation (4.6) gives

$$\begin{aligned}\frac{\begin{vmatrix}\boldsymbol{\alpha}_1^\mathrm{T}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_1^\mathrm{T}\boldsymbol{\alpha}_2\\ \boldsymbol{\alpha}_2^\mathrm{T}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2^\mathrm{T}\boldsymbol{\alpha}_2\end{vmatrix}}{\|\boldsymbol{\alpha}_1\|^2\|\boldsymbol{\alpha}_2\|^2} &= \frac{\|\boldsymbol{\alpha}_1\|^2\|\boldsymbol{\alpha}_2\|^2-(\boldsymbol{\alpha}_1^\mathrm{T}\boldsymbol{\alpha}_2)^2}{\|\boldsymbol{\alpha}_1\|^2\|\boldsymbol{\alpha}_2\|^2} = 1-\frac{(\boldsymbol{\alpha}_1^\mathrm{T}\boldsymbol{\alpha}_2)^2}{\|\boldsymbol{\alpha}_1\|^2\|\boldsymbol{\alpha}_2\|^2}\\ &= 1-\cos^2\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1) = \sin^2\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1).\end{aligned}\tag{4.7}$$

Equation (4.7) signifies that the Cauchy-Schwarz inequality and the Hadamard inequality are the two sides of a coin, that is, formula

$$\frac{\begin{vmatrix}\boldsymbol{\alpha}_1^\mathrm{T}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_1^\mathrm{T}\boldsymbol{\alpha}_2\\ \boldsymbol{\alpha}_2^\mathrm{T}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2^\mathrm{T}\boldsymbol{\alpha}_2\end{vmatrix}}{\|\boldsymbol{\alpha}_1\|^2\|\boldsymbol{\alpha}_2\|^2} = \sin^2\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1) \leqslant 1$$

contains the Hadamard inequality and formula

$$\frac{(\boldsymbol{\alpha}_1^\mathrm{T}\boldsymbol{\alpha}_2)^2}{\|\boldsymbol{\alpha}_1\|^2\|\boldsymbol{\alpha}_2\|^2} = \cos^2\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1) \leqslant 1$$

gives the Cauchy-Schwarz inequality. Using equation (3.9) gives

$$\begin{aligned}\sqrt{\det(\mathbf{A}^{\mathrm{T}}\mathbf{A})} &= \sqrt{\det((\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m)^{\mathrm{T}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m))}\\ &= \|\boldsymbol{\alpha}_1\|\|\boldsymbol{\alpha}_2\|\cdots\|\boldsymbol{\alpha}_m\|\sin\theta(\boldsymbol{\alpha}_2,\mathscr{A}_1)\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\cdots\sin\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1})\\ &\leqslant \|\boldsymbol{\alpha}_1\|\|\boldsymbol{\alpha}_2\|\cdots\|\boldsymbol{\alpha}_m\|\sin\theta(\boldsymbol{\alpha}_2,\mathscr{A}_1)\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\\ &\quad\cdots\sin\theta(\boldsymbol{\alpha}_{m-1},\mathscr{A}_{m-2}) \qquad (4.8)\\ &\vdots\\ &\leqslant \|\boldsymbol{\alpha}_1\|\|\boldsymbol{\alpha}_2\|\cdots\|\boldsymbol{\alpha}_m\|\sin\theta(\boldsymbol{\alpha}_2,\mathscr{A}_1) \qquad (4.9)\\ &\leqslant \|\boldsymbol{\alpha}_1\|\|\boldsymbol{\alpha}_2\|\cdots\|\boldsymbol{\alpha}_m\|. \qquad (4.10)\end{aligned}$$

Clearly, equation (4.10) is the Hadamard inequality and equations (4.8) and (4.9) can be seen some refinements of the Hadamard inequality.

Since the Hadamard inequality and the Cauchy-Schwarz inequality are the two sides of a coin and the Hadamard inequality has the above perfect refinement versions, what is the refinement version of the Cauchy-Schwarz inequality? According to equations (3.12) and (4.4), we have

$$0\leqslant\lambda[(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{D}]\leqslant 1. \qquad (4.11)$$

This denotes

$$\begin{aligned}\det((\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{D}) &= \det((\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1})\det(\mathbf{D}^{\mathrm{T}}\mathbf{C}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{D})\\ &= \det(\boldsymbol{\Lambda}) = \delta_1^2\delta_2^2\cdots\delta_{m-p}^2\\ &= \cos^2\theta_1\cos^2\theta_2\cdots\cos^2\theta_{m-p}\leqslant 1. \qquad (4.12)\end{aligned}$$

For $\mathbf{C}=[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_p]$ and $\mathbf{D}=[\boldsymbol{\alpha}_{p+1},\boldsymbol{\alpha}_{p+2},\cdots,\boldsymbol{\alpha}_m]$, if $p=m-p$ then $\mathbf{C}$, $\mathbf{D}\in\mathbb{R}^{n\times m}$ and we have

$$\begin{aligned}&\det(\mathbf{D}^{\mathrm{T}}\mathbf{C}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{D}) = \cos^2\theta_1\cos^2\theta_2\cdots\cos^2\theta_{m-p}\det(\mathbf{D}^{\mathrm{T}}\mathbf{D}) \qquad (4.13)\\ &\det(\mathbf{D}^{\mathrm{T}}\mathbf{C})\det((\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1})\det(\mathbf{C}^{\mathrm{T}}\mathbf{D}) = \cos^2\theta_1\cos^2\theta_2\cdots\cos^2\theta_{m-p}\det(\mathbf{D}^{\mathrm{T}}\mathbf{D})\\ &\big(\det(\mathbf{D}^{\mathrm{T}}\mathbf{C})\big)^2 = \cos^2\theta_1\cos^2\theta_2\cdots\cos^2\theta_{m-p}\det(\mathbf{D}^{\mathrm{T}}\mathbf{D})\det(\mathbf{C}^{\mathrm{T}}\mathbf{C}). \qquad (4.14)\end{aligned}$$

Using equation (4.14) gives

$$\begin{aligned}\big(\det(\mathbf{D}^{\mathrm{T}}\mathbf{C})\big)^2 &\leqslant \cos^2\theta_1\cos^2\theta_2\cdots\cos^2\theta_{m-p+1}\det(\mathbf{D}^{\mathrm{T}}\mathbf{D})\det(\mathbf{C}^{\mathrm{T}}\mathbf{C}) \qquad (4.15)\\ &\vdots\\ &\leqslant \cos^2\theta_1\det(\mathbf{D}^{\mathrm{T}}\mathbf{D})\det(\mathbf{C}^{\mathrm{T}}\mathbf{C}) \qquad (4.16)\\ &\leqslant \det(\mathbf{D}^{\mathrm{T}}\mathbf{D})\det(\mathbf{C}^{\mathrm{T}}\mathbf{C}). \qquad (4.17)\end{aligned}$$

Clearly, equation (4.17) is an extension of the Cauchy-Schwarz inequality [22] and equations (4.15) and (4.16) are the refinement versions of the Cauchy-Schwarz inequality.

5. An inequality of the principal angles related to parallelotope

Using the results established in Section 4, the following conclusion describes the relation of the principal angles related to a parallelotope determined by a matrix.

THEOREM 2. *Let* $\boldsymbol{\alpha}_1,\cdots,\boldsymbol{\alpha}_p,\boldsymbol{\alpha}_{p+1},\cdots,\boldsymbol{\alpha}_m$ *be linear independent vectors and* $\mathbf{C}:=[\boldsymbol{\alpha}_1,\cdots,\boldsymbol{\alpha}_p]$ *and* $\mathbf{D}:=[\boldsymbol{\alpha}_{p+1},\cdots,\boldsymbol{\alpha}_m]$*. If* $p>m-p$ *then*

$$\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-p}\geqslant\sin\theta(\boldsymbol{\alpha}_2,\mathscr{A}_1)\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\cdots\sin\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1}),\quad(5.1)$$

where θ_i*,* $i=1,2,\cdots,m-p$*, are the principal angles determined by subspaces* $\mathscr{C}$ *and* $\mathscr{D}$ *and* $\theta(\boldsymbol{\alpha}_i,\mathscr{A}_{i-1})$*,* $i=2,3,\cdots,m$*, are the principal angles between* $\boldsymbol{\alpha}_i$ *and subspace* $\mathscr{A}_{i-1}$*.*

Proof. According to equations (3.9) and (4.5), we have

$$\begin{aligned}&\prod_{i=1}^{n}\|\boldsymbol{\alpha}_i\|\sin\theta(\boldsymbol{\alpha}_2,\mathscr{A}_1)\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\cdots\sin\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1})\\=\;&V_{\mathbf{C}}V_{\mathbf{D}}\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-p}.\end{aligned}\quad(5.2)$$

According equation (3.10), we have

$$V_{\mathbf{C}}\leqslant\|\boldsymbol{\alpha}_1\|\|\boldsymbol{\alpha}_2\|\cdots\|\boldsymbol{\alpha}_p\|,\quad(5.3)$$

$$V_{\mathbf{D}}\leqslant\|\boldsymbol{\alpha}_{p+1}\|\|\boldsymbol{\alpha}_{p+2}\|\cdots\|\boldsymbol{\alpha}_m\|.\quad(5.4)$$

Namely,

$$V_{\mathbf{C}}V_{\mathbf{D}}\leqslant\|\boldsymbol{\alpha}_1\|\|\boldsymbol{\alpha}_2\|\cdots\|\boldsymbol{\alpha}_p\|\|\boldsymbol{\alpha}_{p+1}\|\|\boldsymbol{\alpha}_{p+2}\|\cdots\|\boldsymbol{\alpha}_m\|=\left(\prod_{i=1}^{n}\|\boldsymbol{\alpha}_i\|\right).\quad(5.5)$$

Combining equations (5.2) and (5.5) gives equation (5.1). The proof of theorem 2 is completed. □

REMARK 1. Theorem 2 is a special case of the following class of principal inequalities. According to equation (4.4), equation (5.1) can be rewritten as

$$\sin\theta(\mathscr{C},\mathscr{D})\geqslant\sin\theta(\boldsymbol{\alpha}_2,\mathscr{A}_1)\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\cdots\sin\theta(\boldsymbol{\alpha}_m,\mathscr{A}_{m-1}).\quad(5.6)$$

Let $\mathscr{A}_{[i_1:i_k]}$ represents the subspace spanned by vectors $\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\cdots,\boldsymbol{\alpha}_{i_k}$. Here, $i_1,i_2,\cdots,$ i_k are some consecutive integers not starting from 1. For convenience, set $m=5$, $\mathscr{C}:=\mathscr{A}_3$ and $\mathscr{D}:=\mathscr{A}_{[4,5]}$, equation (5.6) can be rewritten as

$$\sin\theta(\mathscr{A}_3,\mathscr{A}_{[4,5]})\geqslant\sin\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1)\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\sin\theta(\boldsymbol{\alpha}_4,\mathscr{A}_3)\sin\theta(\boldsymbol{\alpha}_5,\mathscr{A}_4).\quad(5.7)$$

Using equation (3.13) recursively, we have

$$\begin{aligned}V_{\mathbf{A}_5}&=V_{\mathbf{A}_3}V_{\mathbf{A}_{[4,5]}}\sin\theta(\mathscr{A}_3,\mathscr{A}_{[4,5]})=\|\boldsymbol{\alpha}_1\|V_{\mathbf{A}_{[2,3]}}\sin\theta(\boldsymbol{\alpha}_1,\mathscr{A}_{[2,3]})V_{\mathbf{A}_{[4,5]}}\sin\theta(\mathscr{A}_3,\mathscr{A}_{[4,5]})\\&=\|\boldsymbol{\alpha}_1\|\|\boldsymbol{\alpha}_2\|\|\boldsymbol{\alpha}_3\|\sin\theta(\boldsymbol{\alpha}_1,\mathscr{A}_{[2,3]})\sin\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)V_{\mathbf{A}_{[4,5]}}\sin\theta(\mathscr{A}_3,\mathscr{A}_{[4,5]})\\&=\prod_{i=1}^{5}\|\boldsymbol{\alpha}_i\|\sin\theta(\boldsymbol{\alpha}_1,\mathscr{A}_{[2,3]})\sin\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)\sin\theta(\mathscr{A}_3,\mathscr{A}_{[4,5]})\sin\theta(\boldsymbol{\alpha}_4,\boldsymbol{\alpha}_5).\end{aligned}\quad(5.8)$$

Combining equations (3.9) and (5.8) gives

$$\begin{aligned} &\sin\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1)\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\sin\theta(\boldsymbol{\alpha}_4,\mathscr{A}_3)\sin\theta(\boldsymbol{\alpha}_5,\mathscr{A}_4) \\ =\ &\sin\theta(\boldsymbol{\alpha}_1,\mathscr{A}_{[2,3]})\sin\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)\sin\theta(\mathscr{A}_3,\mathscr{A}_{[4,5]})\sin\theta(\boldsymbol{\alpha}_4,\boldsymbol{\alpha}_5). \end{aligned} \tag{5.9}$$

From equation (5.9), we can obtain a class of principal inequalities. For example, since $\sin\theta(\boldsymbol{\alpha}_5,\mathscr{A}_4)\leqslant\sin\theta(\boldsymbol{\alpha}_4,\boldsymbol{\alpha}_5)$, we have

$$\begin{aligned} &\sin\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1)\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\sin\theta(\boldsymbol{\alpha}_4,\mathscr{A}_3) \\ \geqslant\ &\sin\theta(\boldsymbol{\alpha}_1,\mathscr{A}_{[2,3]})\sin\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)\sin\theta(\mathscr{A}_3,\mathscr{A}_{[4,5]}). \end{aligned}$$

And using $\sin\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1)\leqslant 1$ gives

$$\begin{aligned} &\sin\theta(\boldsymbol{\alpha}_3,\mathscr{A}_2)\sin\theta(\boldsymbol{\alpha}_4,\mathscr{A}_3)\sin\theta(\boldsymbol{\alpha}_5,\mathscr{A}_4) \\ \geqslant\ &\sin\theta(\boldsymbol{\alpha}_1,\mathscr{A}_{[2,3]})\sin\theta(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)\sin\theta(\mathscr{A}_3,\mathscr{A}_{[4,5]})\sin\theta(\boldsymbol{\alpha}_4,\boldsymbol{\alpha}_5). \end{aligned}$$

Of course, equation (5.7) can be got easily from equation (5.9). Referring to equation (3.7), a class of determinant inequalities such as the Hadamard or Fischer inequalities can be got easily.

6. An inequality of the principal angles related to two subspaces

In this section, by using the interlacing property, we will establish a principal angles inequality related to two subspaces.

THEOREM 3. *Let* $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_p$, $\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_m$ *be linear independent vectors in* $\mathbb{R}^n$ *and set* $\mathbf{U}:=[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2\cdots,\boldsymbol{\alpha}_p]$ *and* $\mathbf{V}:=[\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_m]$, *respectively. Setting* $\mathscr{U}:=\mathrm{span}\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_p\}$ *and* $\mathscr{V}:=\mathrm{span}\{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_m\}$. *Suppose that* $p\leqslant m$ *and the principal angles between* $\mathscr{U}$ *and* $\mathscr{V}$ *are* $\phi_1,\phi_2,\cdots,\phi_p$. *Let* $\mathbf{W}:=[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2\cdots,\boldsymbol{\alpha}_q]$ *and* $\mathscr{W}:=\mathrm{span}\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2\cdots,\boldsymbol{\alpha}_q\}$, $q<p$. *The principal angles between* $\mathscr{W}$ *and* $\mathscr{V}$ *are* $\psi_1,\psi_2,\cdots,\psi_q$. *Then we have the following inequality.*

$$\phi_j\leqslant\psi_j\leqslant\phi_{p-q+j},\ j=1,2,\cdots,q. \tag{6.1}$$

Proof. For convenience, suppose that vector groups $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_p$ and $\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_m$ are normal orthogonal. Using Definition 1 gives

$$\begin{aligned} \cos\phi_j &= \max_{\mathbf{u}\in\mathscr{U}}\max_{\mathbf{v}\in\mathscr{V}}\mathbf{u}^{\mathrm{T}}\mathbf{v} = \max_{\substack{\mathbf{y}\in\mathbb{R}^p\\ \|\mathbf{y}\|=1}}\max_{\substack{\mathbf{z}\in\mathbb{R}^m\\ \|\mathbf{z}\|=1}}\mathbf{y}^{\mathrm{T}}(\mathbf{U}^{\mathrm{T}}\mathbf{V})\mathbf{z} = \sqrt{\max_{\substack{\mathbf{x}\in\mathbb{R}^m\\ \|\mathbf{x}\|=1}}\mathbf{x}^{\mathrm{T}}(\mathbf{V}^{\mathrm{T}}\mathbf{U}\mathbf{U}^{\mathrm{T}}\mathbf{V})\mathbf{x}} \\ &= \sqrt{\lambda_j[\mathbf{V}^{\mathrm{T}}\mathbf{U}\mathbf{U}^{\mathrm{T}}\mathbf{V}]}. \end{aligned} \tag{6.2}$$

According to equation (6.2), we have

$$\cos^2\phi_j=\lambda_j[\mathbf{V}^{\mathrm{T}}\mathbf{U}\mathbf{U}^{\mathrm{T}}\mathbf{V}],\ j=1,2,\cdots,p. \tag{6.3}$$

Similarly, we have

$$\cos^2\psi_k = \lambda_k[\mathbf{V}^{\mathrm{T}}\mathbf{W}\mathbf{W}^{\mathrm{T}}\mathbf{V}],\ k = 1,2,\cdots,q. \tag{6.4}$$

According to Lemma 4, we have

$$\cos^2\phi_j = \lambda_j[\mathbf{V}^{\mathrm{T}}\mathbf{U}\mathbf{U}^{\mathrm{T}}\mathbf{V}] = \lambda_j[\mathbf{U}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{U}],\ j = 1,2,\cdots,p, \tag{6.5}$$

$$\cos^2\psi_k = \lambda_k[\mathbf{V}^{\mathrm{T}}\mathbf{W}\mathbf{W}^{\mathrm{T}}\mathbf{V}] = \lambda_k[\mathbf{W}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{W}],\ k = 1,2,\cdots,q. \tag{6.6}$$

Since $\mathscr{W} \subset \mathscr{U}$, for any unit vector $\mathbf{z} \in \mathbb{R}^q$ there exists a unit vector $\mathbf{y} \in \mathbb{R}^p$ such that

$$\frac{\mathbf{y}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{U}\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}} \geqslant \frac{\mathbf{z}^{\mathrm{T}}\mathbf{W}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{W}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{z}}. \tag{6.7}$$

According to Lemma 1 and equation (6.7), we obtain

$$\lambda_j[\mathbf{U}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{U}] \geqslant \lambda_j[\mathbf{W}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{W}],\ j = 1,2,\cdots,q. \tag{6.8}$$

Similarly, we have

$$\lambda_j[\mathbf{W}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{W}] \geqslant \lambda_{p-q+j}[\mathbf{U}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{U}],\ j = 1,2,\cdots,q. \tag{6.9}$$

Combining equations (6.8) and (6.9) gives

$$\lambda_j[\mathbf{U}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{U}] \geqslant \lambda_j[\mathbf{W}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{W}] \geqslant \lambda_{p-q+j}[\mathbf{U}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{U}],\ j = 1,2,\cdots,q. \tag{6.10}$$

Referring to equations (6.5) and (6.6) gives

$$\cos\phi_j \geqslant \cos\psi_j \geqslant \cos\phi_{p-q+j},\ j = 1,2,\cdots,q. \tag{6.11}$$

Note that the monotonically decreasing property of cosine in $[0,\pi/2]$. The proof of theorem 3 is completed. □

REMARK 2. From the proof of Theorem 3, we find that $\mathbf{W}$ can be formed by arbitrarily taking some column vectors from matrix $\mathbf{U}$ and equation (6.1) still holds. Further more, the matrix $\mathbf{W}$ can be formed by taking some column vectors from matrix $\mathbf{V}$, in this case, ψ_i denotes the principal angles between by subspaces $\mathscr{U}$ and $\mathscr{W}$ and the inequality in equation (6.1) still holds.

REMARK 3. According to equation (6.1) or equation (6.11), we have

$$\sin\phi_j \leqslant \sin\psi_j \leqslant \sin\phi_{p-q+j},\ j = 1,2,\cdots,q. \tag{6.12}$$

Using equation (6.12) gives

$$\sin\phi_1\sin\phi_2\cdots\sin\phi_q \leqslant \sin\psi_1\sin\psi_2\cdots\sin\psi_q. \tag{6.13}$$

Further more, we have

$$\sin\phi_1\sin\phi_2\cdots\sin\phi_q\sin\phi_{q+1}\cdots\sin\phi_p \leqslant \sin\psi_1\sin\psi_2\cdots\sin\psi_q. \tag{6.14}$$

From equation (6.14), the Koteljanskii determinant inequality can be proved [3].

7. Conclusions

By analyzing two volume formulae for the parallelotope, some refinements of the Hadamard and Cauchy-Schwarz inequalities are presented and a principal inequality related a paralleotope is established. By using the interlacing property, a principal inequality related to two subspaces is given. It is proved that these two principal inequalities can be extended to two class of principal inequalities easily. Analysis indicates that these two class of principal inequalities have a close relation to the several classic determinant inequalities.

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