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# SHARP BOUNDS ON THE SINC FUNCTION VIA THE FOURIER SERIES METHOD

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*Abstract.* In this paper we provide sharp bounds on the sinc function using the Fourier series method. Refinements of some remarkable trigonometric inequalities involving sinc function are given as well.

## 1. Introduction

The sinc function $sinc(x)$, also called the *sampling function* is a function that arises in many areas of mathematics and its applications (in communications engineering, for example) [11].

The sinc function is defined to be

$$sinc(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

In spectral domain, sinc function is the best representative for finite data length as the convolution kernel. In discrete Fourier transform (DFT), rectangular signal is transformed to the sinc function in frequency domain. It is related to signal uncertainty principle. Also sinc is used in wavelet basis functions.

The linear approximation $sin(x)=x$ or, equivalently, $sinc(x)=1$ for small values of $x$ is very important in applications.There are some remarkable approximations for the $sinc$ function through inequalities.

The following result is well known as Jordan inequality [16]:

$$\frac{2}{\pi} \leqslant sinc(x) \leqslant 1,\ x \in \left[0, \frac{\pi}{2}\right]. \tag{1.1}$$

Another famous inequality related to sinc function is the Cusa - Huygens inequality:

$$\frac{1+\cos x}{2} \leqslant sinc(x) \leqslant \frac{2+\cos x}{3},\ x \in \left[0, \frac{\pi}{2}\right]. \tag{1.2}$$

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These inequalities have been further refined by many authors in the past few years [4] - [7]. In [13] Redheffer proposed the inequality

$$sinc(x) \geqslant \frac{\pi^2 - x^2}{\pi^2 + x^2},\ x \neq 0. \tag{1.3}$$

The aim of the present paper is to refine these classical inequalities. The main idea is that the function sinc is even, therefore it may be expanded in a Fourier series, e.g.,

$$\frac{\sin x}{x} - 1 = a + b\cos x + c\cos 2x + \dots .$$

We define the function $F(x)$ by

$$F(x) = a + b\cos x + c\cos 2x.$$

The power series expansion of $\frac{\sin x}{x} - 1 - F(x)$ near $0$ is

$$(-a-b-c) + \left(\frac{b}{2} + 2c - \frac{1}{6}\right)x^2 + \frac{1}{120}(-5b - 80c + 1)x^4 + \frac{1}{5040}(7b + 448c - 1)x^6 + O(x^8).$$

In order to increase the speed of the function $F(x)$ approximating $\frac{\sin x}{x} - 1$ we vanish the first coefficients as follows:

$$\begin{cases} -a-b-c = 0 \\ \frac{b}{2} + 2c - \frac{1}{6} = 0 \\ -5b - 80c + 1 = 0. \end{cases}$$

We find $a = \frac{-11}{30}$, $b = \frac{17}{45}$, $c = \frac{-1}{90}$ and therefore we have

$$\frac{\sin x}{x} - 1 + \frac{11}{30} - \frac{17}{45}\cos x + \frac{1}{90}\cos 2x = \frac{-x^6}{1512} + \frac{29x^8}{453600} + O(x^{10}).$$

It follows that

$$\lim_{x \to 0} \frac{\frac{\sin x}{x} - 1 - F(x)}{x^6} = \frac{-1}{1512}.$$

In particular the speed of the function $F(x)$ approximating $\frac{\sin x}{x} - 1$ is given by the order estimate $O(x^6)$ as $x = 0$.

It is the first aim of our work to establish the following sharpened bounds for the sinc function.

THEOREM 1.1. *For any* $x \in \left(0, \frac{\pi}{2}\right)$, *one has*

$$1 - \frac{x^6}{1512} - \frac{(1-\cos x)(16-\cos x)}{45} < \frac{\sin x}{x} < 1 - \frac{x^6}{1512} + \frac{29x^8}{453600} - \frac{(1-\cos x)(16-\cos x)}{45} \tag{1.4}$$

As applications of theorem 1.1, it is the second aim of our paper to improve the inequalities (1.1), (1.2) and (1.3).

## 2. The proof of theorem 1.1

The left hand side of inequality (1.4) is equivalent to

$$7560\sin x - 4872x + 168x\cos^2 x - 2856x\cos x + 5x^7 \geqslant 0, \text{ for all } x \in \left[0, \frac{\pi}{2}\right].$$

We introduce the function

$$f(x) = 7560\sin x - 4872x + 168x\cos^2 x - 2856x\cos x + 5x^7.$$

Easy computation yields

$$\begin{aligned}
f'(x) &= 7\left(5x^6 + 408x\sin x + 24\cos^2 x - 48x\sin x\cos x + 672\cos x - 696\right),\\
f^{(2)}(x) &= 42\left(5x^5 - 44\sin x - 8\sin 2x + 68x\cos x - 8x\cos 2x\right),\\
f^{(3)}(x) &= 42\left(25x^4 - 68x\sin x + 16x\sin 2x + 24\cos x - 24\cos 2x\right),\\
f^{(4)}(x) &= 168\left(25x^3 - 23\sin x + 16\sin 2x - 17x\cos x + 8x\cos 2x\right),\\
f^{(5)}(x) &= 168\left(75x^2 + 17x\sin x - 16x\sin 2x - 40\cos x + 40\cos 2x\right)\\
&= 168\left[\left(75x^2 - 17x\sin x - 60x^2\right) + \left(60x^2 - 40\cos x + 40\cos 2x\right)\right].
\end{aligned}$$

The function

$$g(x) = 15x^2 - 17x\sin x$$

has the derivative

$$g'(x) = 30x + 17(\sin x + x\cos x) > 0$$

for all $x \in \left(0, \frac{\pi}{2}\right)$. Then $g$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. As $g(0) = 0$, we get $g(x) > 0$ on $\left(0, \frac{\pi}{2}\right)$. The function

$$h(x) = 60x^2 - 40\cos x + 40\cos 2x$$

has the derivatives

$$h'(x) = 20(6x - 4\sin 2x + 2\sin x)$$

and

$$h^{(2)} = 20(-16\cos^2 x + 2\cos x + 14) = 20\left[14\sin^2 x + 2\cos x(1-\cos x)\right].$$

Evidently $h^{(2)} > 0$ on $\left(0, \frac{\pi}{2}\right)$. Using a similar algorithm we obtain $h(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$. Therefore

$$f^{(5)}(x) = 168(g(x) + h(x)) > 0 \text{ on } \left(0, \frac{\pi}{2}\right).$$

Then $f^{(4)}$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. As $f^{(4)}(0) = 0$, we have $f^{(4)} > 0$ on $\left(0, \frac{\pi}{2}\right)$. Continuing the algorithm, finally we obtain $f(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$.

The right hand side of inequality (1.4) is equivalent to

$$453600\sin x - 292320x + 10080x\cos^2 x - 171360x\cos x - 29x^9 + 300x^7 \leqslant 0$$

for all $x \in \left[0, \frac{\pi}{2}\right]$. Let

$$s(x) = 453600\sin x - 292320x + 10080x\cos^2 x - 171360x\cos x - 29x^9 + 300x^7.$$

Then

$$\begin{aligned}
s'(x) &= -3\Big(87x^8 - 700x^6 - 57120x\sin x + 3360x\sin 2x - 94080\cos x - 1680\cos 2x \\
&\quad +95760\big),\\
s^{(2)}(x) &= -72\left(29x^7 - 175x^5 + 1540\sin x + 280\sin 2x - 2380x\cos x + 280x\cos 2x\right),\\
s^{(3)}(x) &= 504\left(-29x^6 + 125x^4 - 340x\sin x + 80x\sin 2x + 120\cos x - 120\cos 2x\right),\\
s^{(4)}(x) &= 1008\left(-87x^5 + 250x^3 - 230\sin x + 160\sin 2x - 170x\cos x + 80\cos 2x\right),\\
s^{(5)}(x) &= 1008\left(-435x^4 + 750x^2 + 170x\sin x - 160x\sin 2x - 400\cos x + 400\cos 2x\right),\\
s^{(6)}(x) &= 1008\left(-1740x^3 + 1500x + 570\sin x - 960\sin 2x + 170x\cos x - 320x\cos 2x\right),\\
s^{(7)}(x) &= 1008\left(-5220x^2 - 170x\sin x + 640x\sin 2x + 740\cos x - 2240\cos 2x + 1500\right)\\
&= 1008\left[x(-1110x - 170x\sin x + 640\sin 2x) + \left(-4110x^2 + 740\cos x\right.\right.\\
&\quad \left.\left.-2240\cos 2x + 1500\right)\right].
\end{aligned}$$

The function

$$r(x) = -1110x - 170x\sin x + 640\sin 2x$$

has the derivative

$$r'(x) = -20\sin^2\frac{x}{2}(256\cos x + 239).$$

Evidently $r'(x) < 0$ on $\left(0, \frac{\pi}{2}\right)$. Then $r$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$. As $r(0) = 0$, we get $r < 0$ on $\left(0, \frac{\pi}{2}\right)$. The function

$$p(x) = -4110x^2 + 740\cos x - 2240\cos 2x + 1500$$

has the derivatives

$$p'(x) = -8220x - 740\sin x + 4480\sin 2x$$

and

$$p''(x) = -40\sin^2\frac{x}{2}(896\cos x + 859) < 0 \text{ on } \left(0, \frac{\pi}{2}\right).$$

Using a similar algorithm we have $p(x) < 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$. Therefore

$$s^{(7)}(x) = 1008\,(xr(x) + p(x)) < 0 \text{ on } \left(0, \frac{\pi}{2}\right).$$

Then $s^{(6)}$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$. As $s^{(6)}(0) = 0$, we obtain $s^{(6)} < 0$ on $\left(0, \frac{\pi}{2}\right)$. Continuing the algorithm, finally we get $s(x) < 0$ for every $x \in \left(0, \frac{\pi}{2}\right)$. This completes the proof of theorem 1.1.

## 3. Applications of theorem 1.1

In this section we will prove that the bounds for the function $sinc(x)$ obtained in theorem 1.1 provide better approximations that the classical inequalities (1.1), (1.2), (1.3).

Firstly we will prove that our sharp bounds improve Jordan's inequality.

THEOREM 3.1. *(i) For every* $0 < x < \frac{\pi}{2}$, *one has*

$$\frac{-x^6}{1512} + \frac{29x^8}{453600} - \frac{(1-\cos x)(16-\cos x)}{45} < 0. \tag{3.1}$$

*(ii) For every* $0 < x < 1.5657$, *one has*

$$\frac{-x^6}{1512} - \frac{(1-\cos x)(16-\cos x)}{45} > \frac{2}{\pi} - 1. \tag{3.2}$$

*Proof.* (i) The inequality 3.1 is obvious, since

$$\frac{-x^6}{1512} + \frac{29x^8}{453600} = \frac{x^6\left(29x^2 - 300\right)}{453600} < 0$$

on $\left(0, \frac{\pi}{2}\right)$ and

$$\frac{-(1-\cos x)(16-\cos x)}{45} < 0$$

on $\left(0, \frac{\pi}{2}\right)$.

(ii) The difference

$$E(x) = \frac{-x^6}{1512} - \frac{1}{45}(1-\cos x)(16-\cos x) - \frac{2-\pi}{\pi}$$

has the numerical roots $x_1 \approx -1.5657$, $x_2 \approx 1.5657$. Since

$$E(0) = \frac{\pi-2}{\pi} > 0,$$

it follows that $E(x) > 0$ on $(0, 1.5657)$. $\square$

In order to improve Cusa - Huygens inequality, we rewrite the double inequality (1.4) as

$$1 - \frac{x^6}{1512} - \frac{(1-\cos x)(16-\cos x)}{45} - \frac{2+\cos x}{3} < \frac{\sin x}{x} - \frac{2+\cos x}{3}$$
$$< 1 - \frac{x^6}{1512} + \frac{29x^8}{453600} - \frac{(1-\cos x)(16-\cos x)}{45} + 1 - \frac{2+\cos x}{3}.$$

On the other hand,

$$-\frac{(1-\cos x)(16-\cos x)}{45} + 1 - \frac{2+\cos x}{3} = \frac{-1}{45}(\cos x - 1)^2.$$

Using again the double inequality (1.4), we find

$$\frac{-x^6}{1512} - \frac{(1-\cos x)(16-\cos x)}{45} + \frac{1-\cos x}{2} < \frac{\sin x}{x} - \frac{1+\cos x}{2}$$
$$< \frac{-x^6}{1512} + \frac{29x^8}{453600} - \frac{(1-\cos x)(16-\cos x)}{45} + \frac{1-\cos x}{2}$$

or equivalently

$$\frac{-x^6}{1512} + \frac{(1-\cos x)(2\cos x + 13)}{90} < \frac{\sin x}{x} - \frac{1+\cos x}{2}$$
$$< \frac{-x^6}{1512} + \frac{29x^8}{453600} + \frac{(1-\cos x)(2\cos x + 13)}{90}.$$

Therefore we can state the following theorem related to Cusa - Huygens inequality.

THEOREM 3.2. *(i) For all* $x \in \left(0, \frac{\pi}{2}\right)$, *one has*

$$\frac{-x^6}{1512} - \frac{(\cos x - 1)^2}{45} < \frac{\sin x}{x} - \frac{2+\cos x}{3} < \frac{-x^6}{1512} + \frac{29x^8}{453600} - \frac{(\cos x - 1)^2}{45}.$$

*(ii) For all* $x \in \left(0, \frac{\pi}{2}\right)$, *one has*

$$\frac{-x^6}{1512} + \frac{(1-\cos x)(2\cos x + 13)}{90} < \frac{\sin x}{x} - \frac{1+\cos x}{2}$$
$$< \frac{-x^6}{1512} + \frac{29x^8}{453600} + \frac{(1-\cos x)(2\cos x + 13)}{90}.$$

REMARK 3.1. 1. Since

$$\frac{-x^6}{1512}+\frac{29x^8}{453600}=\frac{x^6\left(29x^2-300\right)}{453600}<0$$

on $\left(0,\frac{\pi}{2}\right)$ and

$$\frac{-(\cos x-1)^2}{45}<0,$$

it follows that

$$\frac{\sin x}{x}-\frac{2+\cos x}{3}<\frac{-x^6}{1512}+\frac{29x^8}{453600}-\frac{(\cos x-1)^2}{45}<0,$$

hence we improve the second inequality of (1.2).

2. The results from (ii) improve the inequality of (1.2), because

$$\frac{-x^6}{1512}+\frac{(1-\cos x)(2\cos x+13)}{90}>0 \text{ for every } x\in\left(0,\frac{\pi}{2}\right).$$

Indeed, the above inequality takes the equivalent form $H(x)>0$ for all $x\in\left(0,\frac{\pi}{2}\right)$, where

$$H(x)=-5x^6-168\cos^2 x-924\cos x+1092.$$

We introduce the auxiliary function $a:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$, $a(x)=\frac{189x^4}{2}-630x^2-168\cos^2 x-924\cos x+1092$. Its derivatives are

$$\begin{aligned}
a'(x)&=42\left(9x^3-30x+22\sin x+4\sin 2x\right),\\
a^{(2)}(x)&=42\left(27x^2+22\cos x+8\cos 2x-30\right),\\
a^{(3)}(x)&=-84\left(-27x+11\sin x+8\sin 2x\right),\\
a^{(4)}(x)&=168\sin^2\frac{x}{2}\left(32\cos x+43\right).
\end{aligned}$$

It is easy to see that $a^{(4)}(x)>0$ on $\left(0,\frac{\pi}{2}\right)$. Then $a^{(3)}$ is strictly increasing on $\left(0,\frac{\pi}{2}\right)$. As $a^{(3)}(0)=0$, it follows that $a^{(3)}>0$ for all $x\in\left(0,\frac{\pi}{2}\right)$. Continuing the algorithm, finally we find that $a(x)>0$ for all $x\in\left(0,\frac{\pi}{2}\right)$. Therefore the function $H(x)=-5x^6-168\cos^2 x-924\cos x+1092$ can be rewritten as

$$H(x)=a(x)+x^2b(x),$$

where $b(x)=-5x^4-\frac{189x^2}{2}+630$. The polynomial function $b(x)$ has the real roots $x_{1,2}=\pm\frac{1}{2}\sqrt{\frac{3\sqrt{9569}-189}{5}}\approx\pm 2.2854$. Since $b(0)=630>0$, it follows that $b(x)>0$ on $\left(0,\frac{\pi}{2}\right)$. Hence we proved that $H(x)=a(x)+x^2b(x)>0$ for all $x\in\left(0,\frac{\pi}{2}\right)$.

Finally we will prove that lower bound of sinc function from inequality (1.4) provides a refinement of Redheffer inequality (1.3).

THEOREM 3.3. *For all* $x \in \left(0, \frac{\pi}{2}\right)$, *one has*

$$1 - \frac{x^6}{1512} - \frac{(1-\cos x)(16-\cos x)}{45} > \frac{\pi^2 - x^2}{\pi^2 + x^2}. \tag{3.3}$$

*Proof.* The inequality (3.3) takes the equivalent form

$$T(x) > 0,$$

where

$$T(x) = \frac{2x^2}{\pi^2 + x^2} - \frac{x^6}{1512} - \frac{(1-\cos x)(16-\cos x)}{45}.$$

Our proof has three steps.

*The first step.* We will establish the following inequality:

$$-\frac{(1-\cos x)(16-\cos x)}{45} > -\frac{x^2}{6} + \frac{x^4}{120} \text{ for all } x \in \left(0, \frac{\pi}{2}\right).$$

Indeed, the above inequality can be rewritten as

$$-3x^4 + 60x^2 - 128 + 136\cos x - 8\cos^2 x > 0$$

for all $x \in \left(0, \frac{\pi}{2}\right)$. The function $c(x) = -3x^4 + 60x^2 - 128 + 136\cos x - 8\cos^2 x$ has the derivatives

$$\begin{aligned} c'(x) &= 4\left(-3x^3 + 30x - 34\sin x + 2\sin 2x\right), \\ c^{(2)}(x) &= 4\left(-9x^2 - 34\cos x + 4\cos 2x + 30\right), \\ c^{(3)}(x) &= -8\left(9x - 17\sin x + 4\sin 2x\right), \\ c^{(4)}(x) &= -8\left(\cos x - 1\right)\left(16\cos x - 1\right). \end{aligned}$$

To find critical points of $c^{(4)}$, we solve the equation $c^{(4)}(x) = 0$ on $\left(0, \frac{\pi}{2}\right)$. The solution is $x = \arccos\frac{1}{16} \approx 86°25'0.042''$. Partition the domain $\left(0, \frac{\pi}{2}\right)$ into intervals with endpoints at the critical points: $\left(0, \arccos\frac{1}{16}\right)$ and $\left(\arccos\frac{1}{16}, \frac{\pi}{2}\right)$. The sign of $c^{(4)}$ is: $c^{(4)}(x) > 0$ on $\left(0, \arccos\frac{1}{16}\right)$ and $c^{(4)}(x) < 0$ on $\left(\arccos\frac{1}{16}, \frac{\pi}{2}\right)$. Then $c^{(3)}$ is strictly increasing on $\left(0, \arccos\frac{1}{16}\right)$ and respectively strictly decreasing on $\left(\arccos\frac{1}{16}, \frac{\pi}{2}\right)$. Since $c^{(3)}(0) = 0$ and $c^{(3)}\left(\frac{\pi}{2}\right) = \frac{-8(9\pi - 34)}{2} > 0$, it follows that $c^{(3)}(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$, hence $c^{(2)}$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. As $c^{(2)}(0) = 0$, we obtain that $c^{(2)}(x) > 0$ on $\left(0, \frac{\pi}{2}\right)$. Continuing the algorithm, finally we have $c(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$.

*The second step.* We will show the following inequality

$$\frac{2x^2}{\pi^2 + x^2} > \frac{2x^2}{\pi^2} - \frac{2x^4}{\pi^4}.$$

Indeed, the above inequality has the equivalent true form: $x^4 > 0$.

*The third step.* Using the above inequalities, we may write

$$T(x) > x^2 \cdot d(x),$$

where

$$d(x) = \left(\frac{2}{\pi^2} - \frac{1}{6}\right) + \left(\frac{1}{120} - \frac{2}{\pi^4}\right)x^2 - \frac{x^4}{1512}.$$

The polynomial function $d$ has the real roots $x_{1,2} \approx \pm 1.6082$. Since $d(0) = \frac{2}{\pi^2} - \frac{1}{6} > 0$, it follows that $d(x) > 0$ on $\left(0, \frac{\pi}{2}\right)$, hence $T(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$. □

## 4. Final remarks

We are convinced that the use of the Fourier series method is suitable for proving and refining many other analytical inequalities which appear in the fields of engineering and applied mathematics.

**Competing interests.** The author declares that he has no competing interests.

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