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SOME GENERALIZED NONLINEAR GAMIDOV TYPE INTEGRAL INEQUALITIES WITH MAXIMA IN TWO VARIABLES AND THEIR WEAKLY SINGULAR ANALOGUES

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Abstract. In this paper, some new nonlinear Gronwall-Bellman-Gamidov type integral inequalities with maxima in two variables and their weakly singular analogues are discussed. By using analysis techniques, such as change of variable, amplification method, differential and integration, inverse function, we estimated the upper bounds of the unknown functions. For illustrating the validity of the inequalities established, some examples are given to study the boundedness and uniqueness of solutions of a certain Gamidov type weakly singular integral equations.

1. Introduction

With the development of the theory of differential equations, The Gronwall-Bellman inequality [1,2] and Bihari inequality [3] are widely used in the qualitative and quantitative analysis of differential equations, as it can provide explicit bound for an unknown function lying in the inequality. The study of inequality has always been a hot topic, see the literature [4-39]. During the past few years, many researchers have established various generalizations of the Gronwall-Bellman inequality. For example, in [7-8], the authors discussed some Gronwall-Like type inequalities; in [9-14], some inequalities with weakly singular kernel were studied; in [4-6, 15-17], the authors researched some Gamidov type inequalities; in [18-20], generalized Volterra-Fredholm type inequalities were investigated.

In 1992, Banov and Simeonov [5] established the following useful integral inequality:

$$u(t) \leqslant c + \int_{\alpha}^{t} f(s)u(s)ds + \int_{\alpha}^{\beta} g(s)u(s)ds, \quad t \in [\alpha, \beta]. \tag{1.1}$$

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In 2007, Wu-Sheng Wang [8] discussed a generalized retard Gronwall-Like inequality in two variables:

$$u^p(x,y) \leqslant a(x,y) + \sum_{i=1}^{n} \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} f_i(x,y,s,t)\varphi_i(u(s,t))dtds. \tag{1.2}$$

In 2014, Kelong Cheng et al. [15] researched a generalized nonlinear Gronwall-Bellman-Gamidov type integral inequality:

$$u^m(t) \leqslant a(t) + b(t)\int_0^t f(s)u^n(s)ds + c(t)\int_0^T g(s)u^r(s)ds, \quad t \in [0,T], \tag{1.3}$$

and its weakly singular analogue:

$$\begin{aligned} u^m(t) \leqslant & a(t) + b(t)\int_0^t (t^{\alpha_1} - s^{\alpha_1})^{\beta_1 - 1} s^{\gamma_1 - 1} f(s)u^n(s)ds \\ & + c(t)\int_0^T (T^{\alpha_2} - s^{\alpha_2})^{\beta_2 - 1} s^{\gamma_2 - 1} g(s)u^r(s)ds, \quad t \in [0,T], \end{aligned} \tag{1.4}$$

where $m \geqslant n \geqslant 0$, $m \geqslant r \geqslant 0$.

Along with the development of automatic control theory and its applications to computational mathematics and modeling, many Gronwall-Bellman integral inequalities with the maxima of the unknown function are established, see [21-24].

In 2013, Yong Yan [23] studied a generalized nonlinear Gronwall-Bellman inequalities with maxima in two variables:

$$\begin{aligned} u(x,y) \leqslant & a(x,y) + \sum_{i=1}^{m} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)h_i(u(s,t))dsdt \\ & + \sum_{j=m+1}^{m+n} \int_{\alpha_j(x_0)}^{\alpha_j(x)} \int_{\beta_j(y_0)}^{\beta_j(y)} b_j(s,t)h_j\left(\max_{\xi \in [s-h,s]} u(\xi,t)\right)dsdt, \quad (x,y) \in \Delta, \end{aligned} \tag{1.5}$$

$$u(x,y) \leqslant \psi(x,y), \quad (x,y) \in \Psi. \tag{1.6}$$

In 2015, Yong Yan [24] investigated a generalized nonlinear weakly singular Volterra integral inequalities with maxima:

$$\begin{aligned} \varphi(u(t)) \leqslant & a(t) + \sum_{i=1}^{m} \int_{b_i(t_0)}^{b_i(t)} (t^{\alpha_i} - s^{\alpha_i})^{k_i(\beta_i - 1)} s^{q_i(\gamma_i - 1)} g_i(t,s)\omega_i(u(s))ds \\ & + \sum_{j=m+1}^{m+n} \int_{b_j(t_0)}^{b_j(t)} (t^{\alpha_j} - s^{\alpha_j})^{k_j(\beta_j - 1)} s^{q_j(\gamma_j - 1)} g_j(t,s)\omega_j \\ & \left(\max_{\xi \in [c_j(s)-h, c_j(s)]} f(u(\xi))\right)ds, \quad t \in [t_0,t_1), \end{aligned} \tag{1.7}$$

$$u(t) \leqslant \psi(t), \quad t \in [b^*(t_0) - h, t_0], \tag{1.8}$$

where $b^*(t_0) = \min\left\{\min_{1\leqslant i\leqslant m} b_i(t_0), \min_{m+1\leqslant j\leqslant m+n} c_j(b_j(t_0))\right\}$.

However, we notice that there are few literature to investigate the Gamidov type integral inequalities with the unknown function that composed with the given function in the integrals. In this paper, based on the work of Wang [8], Cheng [15] and Yan [24], we deal with some classes of nonlinear Gamidov type integral inequalities with maxima in two variables and their weakly singular analogues. The other importance of the paper is the applications that show the boundedness and uniqueness of solutions for weakly singular Gamidov type integral equations with maxima.

2. Preliminary knowledge

In what follows, R denotes the set of real numbers, $R_+ = [0,+\infty)$, $R_0 = (0,+\infty)$. $C^1(U,V)$ denotes the class of all continuously differentiable functions defined on set U with range in the set V, $C(U,V)$ denotes the class of all continuous functions defined on set U with range in the set V.

Consider the sets Δ, Ψ, Λ defined by:

$$\Delta = \left\{(x,y)\in R^2 : x\in[x_0,M), y\in[y_0,N)\right\};$$

$$\Psi = \left\{(x,y)\in R^2 : x\in[\beta b_*(x_0),x_0], y\in[y_0,N)\right\};$$

$$\Lambda = \left\{(x,y)\in R^2 : x\in[\beta b_*(x_0),M), y\in[y_0,N)\right\} = \Delta\bigcup\Psi;$$

where $b_*(x_0) = \min_{1\leqslant i\leqslant 4} b_i(x_0)$, $0<\beta<1$.

For convenience, we cite some useful lemmas in the discussion of our proof as follows.

LEMMA 2.1. (See [25]). *Assume that $a\geqslant 0$, $m\geqslant n\geqslant 0$, and $m\neq 0$. then for any $K>0$,*

$$a^{\frac{n}{m}} \leqslant \frac{n}{m}K^{\frac{n-m}{m}}a + \frac{m-n}{m}K^{\frac{n}{m}}. \tag{2.1}$$

LEMMA 2.2. (See [9]). *Let α, β, γ and p be positive constants. Then*

$$\int_0^t (t^\alpha - s^\alpha)^{p(\beta-1)} s^{p(\gamma-1)} ds = \frac{t^\theta}{\alpha} B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t\in R_+, \tag{2.2}$$

where $B[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}ds$ ($Re\ \xi>0, Re\ \eta>0$) is the well-known B-function and $\theta = p[\alpha(\beta-1)+\gamma-1]+1\geqslant 0$.

LEMMA 2.3. *(Discrete Jensen Inequality). Let $A_1, A_2, \cdots, A_n$ be nonnegative real numbers and $r>1$ is a real number. Then*

$$(A_1 + A_2 + \cdots + A_n)^r \leqslant n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r). \tag{2.3}$$

3. Main results

THEOREM 3.1. *Let the following conditions be fulfilled:*
(i) The functions $b_i \in C^1([x_0,M),R_+)$ *and* $c_i \in C^1([y_0,N),[y_0,N))$ $(i=1,2,3,4)$ *are nondecreasing with* $b_i(x) \leqslant x$ *on* $[x_0,M)$, $c_i(y) \leqslant y$ *on* $[y_0,N)$ *and* $c_i(y_0)=y_0$;
(ii) The functions $a(x,y) \in C(\Delta,[1,\infty))$, $f_i(x,y) \in C(\Delta,R_+)$ $(i=1,2,3,4)$ *are nondecreasing in each of the variables,* $h_i(x,y) \in C(\Lambda,R_+)$ $(i=1,2,3,4)$;
(iii) The function $\phi(x,y) \in C(\Psi,R_+)$ *satisfies* $\max_{(\iota,\tau)\in\Psi}\phi(\iota,\tau) \leqslant a(x_0,y_0)$;
(iv) The functions $\omega_i \in C(R_+,C_+)$ *are nondecreasing with* $\omega_i(u)>0$ *for* $u>0$ $(i=1,2)$, *such that* $\omega_1 \propto \omega_2$, *and* ω_i *are submultiplicative, that is* $\omega_i(tx) \geqslant t\omega_i(x)$ *for* $0 \leqslant t \leqslant 1$;
(v) The function $u \in C(\Lambda,R_+)$ *satisfies:*

$$\begin{aligned} u(x,y) \leqslant & na(x,y)+f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} h_1(s,t)\omega_1(u(s,t))dsdt \\ & + f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\omega_2\left(\max_{\xi\in[\beta s,s]} u(\xi,t)\right)dsdt \\ & + f_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} h_3(s,t)u(s,t)dsdt \\ & + f_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\max_{\xi\in[\beta s,s]} u(\xi,t)dsdt, \quad (x,y)\in\Delta, \end{aligned} \tag{3.1}$$

$$u(x,y) \leqslant \phi(x,y), \quad (x,y) \in \Psi. \tag{3.2}$$

Then, we have the following explicit estimation

$$u(x,y) \leqslant a(x,y)W_1^{-1}\left\{W_2^{-1}\left\{W_2\left[W_1\left(1+G_1^{-1}(B_1(M,N))\right)+A_1(x,y)\right]+B_1(x,y)\right\}\right\}, \tag{3.3}$$

for all $(x,y)\in\Delta$, *where* W_i^{-1} *is the inverse function of* W_i:

$$W_1(z)=\int_c^z \frac{ds}{\omega_1(s)}, \quad W_2(z)=\int_c^z \frac{\omega_1(W_1^{-1}(s))}{\omega_2(W_1^{-1}(s))}ds, \tag{3.4}$$

$c>0$ *is a given constant.*

$$A_1(x,y)=f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} h_1(s,t)dsdt, \tag{3.5}$$

$$B_1(x,y)=f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)dsdt, \tag{3.6}$$

$$\begin{aligned} D_1(M,N)=&f_3(M,N)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} h_3(s,t)\frac{\widetilde{a}(s,t)}{a(x_0,y_0)}dsdt \\ & + f_4(M,N)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\frac{\widetilde{a}(s,t)}{a(x_0,y_0)}dsdt>0, \end{aligned} \tag{3.7}$$

$$G_1(u) = W_2\left(W_1\left(\frac{u}{D_1(M,N)}\right)\right) - W_2\left(W_1(1+u) + A_1(M,N)\right), \tag{3.8}$$

where $G_1(u)$ *is increasing in* R_+.

Proof. Define the nondecreasing function $\widetilde{a}(x,y) \in C(\Lambda,[1,+\infty))$ by

$$\widetilde{a}(x,y) = \begin{cases} a(x,y), & (x,y) \in \Delta, \\ a(x_0,y_0), & (x,y) \in \Psi. \end{cases} \tag{3.9}$$

From inequalities (3.1), (3.2), conditions (iii), (iv) and $\frac{1}{\widetilde{a}(x,y)} \leqslant 1$, we obtain

$$\begin{aligned} \frac{u(x,y)}{a(x,y)} \leqslant & 1 + f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} h_1(s,t)\omega_1\left(\frac{u(s,t)}{a(x,y)}\right)dsdt \\ & + f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\omega_2\left(\frac{\max_{\xi\in[\beta s,s]} u(\xi,t)}{a(x,y)}\right)dsdt \\ & + f_3(M,N)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} h_3(s,t)\frac{u(s,t)}{a(x,y)}dsdt \\ & + f_4(M,N)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\frac{\max_{\xi\in[\beta s,s]} u(\xi,t)}{a(x,y)}dsdt, \quad (x,y) \in \Delta, \end{aligned}$$

that is

$$\begin{aligned} \frac{u(x,y)}{\widetilde{a}(x,y)} \leqslant & 1 + f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} h_1(s,t)\omega_1\left(\frac{u(s,t)}{\widetilde{a}(s,t)}\right)dsdt \\ & + f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\omega_2\left(\frac{\max_{\xi\in[\beta s,s]} u(\xi,t)}{\widetilde{a}(s,t)}\right)dsdt \\ & + f_3(M,N)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} h_3(s,t)\frac{\widetilde{a}(s,t)}{a(x_0,y_0)}\cdot\frac{u(s,t)}{\widetilde{a}(s,t)}dsdt \\ & + f_4(M,N)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\frac{\widetilde{a}(s,t)}{a(x_0,y_0)}\cdot\frac{\max_{\xi\in[\beta s,s]} u(\xi,t)}{\widetilde{a}(s,t)}dsdt, \\ & (x,y) \in \Delta, \end{aligned} \tag{3.10}$$

$$\frac{u(x,y)}{\widetilde{a}(x,y)} \leqslant \frac{\phi(x,y)}{a(x_0,y_0)} \leqslant 1, \quad (x,y) \in \Psi. \tag{3.11}$$

For $s \in [b_*(x_0), b_*(M)]$, $t \in [y_0, N)$, we have

$$\frac{\max_{\xi\in[\beta s,s]} u(\xi,t)}{\widetilde{a}(s,t)} = \frac{u(\xi_1,t)}{\widetilde{a}(s,t)} \leqslant \frac{u(\xi_1,t)}{\widetilde{a}(\xi_1,t)} \leqslant \max_{\xi\in[\beta s,s]}\frac{u(\xi,t)}{\widetilde{a}(\xi,t)}. \tag{3.12}$$

Let

$$z(x,y) = \frac{u(x,y)}{\widetilde{a}(x,y)}. \tag{3.13}$$

From (3.12), it follows that the inequalities (3.10), (3.11) may be written in the form

$$\begin{aligned} z(x,y) \leqslant & 1 + f_1(x,y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t)\omega_1(z(s,t))dsdt \\ & + f_2(x,y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\omega_2\left(\max_{\xi\in[\beta s,s]} z(\xi,t)\right)dsdt + C(M,N), \\ & (x,y)\in\Delta, \end{aligned} \tag{3.14}$$

$$z(x,y) \leqslant 1, \quad (x,y)\in\Psi, \tag{3.15}$$

where

$$\begin{aligned} C(M,N) = & f_3(M,N) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} h_3(s,t)\frac{\widetilde{a}(s,t)}{a(x_0,y_0)} z(s,t)dsdt \\ & + f_4(M,N) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\frac{\widetilde{a}(s,t)}{a(x_0,y_0)} \max_{\xi\in[\beta s,s]} z(\xi,t)dsdt. \end{aligned} \tag{3.16}$$

$\forall X \in [x_0,M),\ Y \in [y_0,N)$, for all $(x,y) \in [x_0,X]\times[y_0,Y] \triangleq \Delta_1$, we have

$$\begin{aligned} z(x,y) \leqslant & 1 + f_1(X,Y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t)\omega_1(z(s,t))dsdt \\ & + f_2(X,Y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\omega_2\left(\max_{\xi\in[\beta s,s]} z(\xi,t)\right)dsdt + C(M,N), \end{aligned} \tag{3.17}$$

for $(x,y) \in [\beta b_*(x_0),x_0]\times[y_0,Y] \triangleq \Psi_1$, we have

$$z(x,y) \leqslant 1. \tag{3.18}$$

Define the function $z_1(x,y) \in C(\Lambda_1,R_+)$ $(\Lambda_1 = \Delta_1\cup\Psi_1)$ by:

$$z_1(x,y) = \begin{cases} 1 + f_1(X,Y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} h_1(s,t)\omega_1(z(s,t))dsdt \\ +f_2(X,Y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\omega_2\left(\max_{\xi\in[\beta s,s]} z(\xi,t)\right)dsdt \\ +C(M,N), \quad (x,y)\in\Delta_1, \\ \\ 1+C(M,N), \quad (x,y)\in\Psi_1. \end{cases}$$

Which is positive and nondecreasing in each of the variables, and

$$z_1(x_0,y) = 1 + C(M,N). \tag{3.19}$$

From (3.17), (3.18) and the definition of the $z_1(x,y)$, we have

$$z(x,y) \leqslant z_1(x,y), \quad (x,y)\in\Lambda_1, \tag{3.20}$$

$$\begin{aligned}\max_{\xi\in[\beta x,x]} z(\xi,y) &\leqslant \max_{\xi\in[\beta x,x]} z_1(\xi,y)\\ &\leqslant z_1(x,y), \quad (x,y)\in[b_*(x_0),b_*(X)]\times[y_0,Y).\end{aligned} \tag{3.21}$$

Differentiating $z_1(x,y)$ on Δ_1 with respect to x, and from (3.20), (3.21), we have

$$\begin{aligned}\frac{\partial z_1(x,y)}{\partial x} =& f_1(X,Y)b_1'(x)\int_{c_1(y_0)}^{c_1(y)} h_1(b_1(x),t)\omega_1(z(b_1(x),t))dt\\ &+f_2(X,Y)b_2'(x)\int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x),t)\omega_2\left(\max_{\xi\in[\beta b_2(x),b_2(x)]} z(\xi,t)\right)dt\\ \leqslant& f_1(X,Y)b_1'(x)\int_{c_1(y_0)}^{c_1(y)} h_1(b_1(x),t)\omega_1(z_1(b_1(x),t))dt\\ &+f_2(X,Y)b_2'(x)\int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x),t)\omega_2(z_1(b_2(x),t))dt,\end{aligned} \tag{3.22}$$

by the monotonicity of $\omega_i,\ z_1$ and the property of $b_i,\ c_i\ (i=1,2)$, we get

$$\begin{aligned}\frac{(\partial/\partial x)z_1(x,y)}{\omega_1(z_1(x,y))} \leqslant& f_1(X,Y)b_1'(x)\int_{c_1(y_0)}^{c_1(y)} h_1(b_1(x),t)dt\\ &+f_2(X,Y)b_2'(x)\int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x),t)\frac{\omega_2(z_1(b_2(x),t))}{\omega_1(z_1(b_2(x),t))}dt.\end{aligned} \tag{3.23}$$

Replace x to s, and integrating it from x_0 to x, we obtain

$$\begin{aligned}W_1(z_1(x,y)) \leqslant& W_1(z_1(x_0,y))+f_1(X,Y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} h_1(s,t)dsdt\\ &+f_2(X,Y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\frac{\omega_2(z_1(s,t))}{\omega_1(z_1(s,t))}dsdt\\ \leqslant& W_1(z_1(x_0,y))+A_1(X,Y)\\ &+f_2(X,Y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\frac{\omega_2(z_1(s,t))}{\omega_1(z_1(s,t))}dsdt,\end{aligned} \tag{3.24}$$

where $A_1(X,Y)$ is defined in (3.5). Let $z_2(x,y)$ denote the function of the right-hand side of (3.24), which is positive and nondecreasing in each of the variables, and

$$z_2(x_0,y) = W_1(z_1(x_0,y))+A_1(X,Y), \tag{3.25}$$

$$z_1(x,y) \leqslant W_1^{-1}(z_2(x,y)). \tag{3.26}$$

Differentiating $z_2(x,y)$ on Δ_1 with respect to x, and from (3.26), we have

$$\begin{aligned}\frac{\partial z_2(x,y)}{\partial x} =& f_2(X,Y)b_2'(x)\int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x),t)\frac{\omega_2(z_1(b_2(x),t))}{\omega_1(z_1(b_2(x),t))}dt\\ \leqslant& f_2(X,Y)b_2'(x)\int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x),t)\frac{\omega_2(W_1^{-1}(z_2(b_2(x),t)))}{\omega_1(W_1^{-1}(z_2(b_2(x),t)))}dt.\end{aligned} \tag{3.27}$$

From the condition $\omega_1 \propto \omega_2$, we obtain that $\frac{\omega_2}{\omega_1}$ is nondecreasing, by the monotonicity of z_2 and the property of $b_2,\ c_2$, we get

$$\frac{\omega_1(W_1^{-1}(z_2(x,y)))(\partial/\partial x)z_2(x,y)}{\omega_2(W_1^{-1}(z_2(x,y)))} \leqslant f_2(X,Y)b_2'(x)\int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x),t)dt. \tag{3.28}$$

Replace x to s, and integrating it from x_0 to x, we obtain

$$W_2(z_2(x,y)) \leqslant W_2(z_2(x_0,y)) + B_1(x,y,X,Y), \tag{3.29}$$

where

$$B_1(x,y,X,Y) = f_2(X,Y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)dsdt, \tag{3.30}$$

obviously, $B_1(x,y,x,y) = B_1(x,y)$, which is defined in (3.6). From (3.19), (3.20), (3.25), (3.26) and (3.29), we get

$$\begin{aligned} z(x,y) &\leqslant z_1(x,y) \leqslant W_1^{-1}(z_2(x,y)) \\ &\leqslant W_1^{-1}\left\{W_2^{-1}\left[W_2\left(W_1(1+C(M,N)) + A_1(X,Y)\right) + B_1(x,y,X,Y)\right]\right\}. \end{aligned} \tag{3.31}$$

Since $X,\ Y$ are chosen arbitrarily, we have

$$\begin{aligned} z(x,y) &\leqslant z_1(x,y) \\ &\leqslant W_1^{-1}\left\{W_2^{-1}\left[W_2\left(W_1(1+C(M,N)) + A_1(x,y)\right) + B_1(x,y)\right]\right\}, \quad (x,y)\in\Delta. \end{aligned} \tag{3.32}$$

By (3.20), (3.21) and the definition of $C(M,N)$, we have

$$\begin{aligned} C(M,N) &\leqslant z_1(M,N)D_1(M,N) \\ &\leqslant W_1^{-1}\left\{W_2^{-1}\left[W_2\left(W_1(1+C(M,N)) + A_1(M,N)\right) + B_1(M,N)\right]\right\}D_1(M,N), \end{aligned}$$

i.e.

$$W_2\left[W_1\left(\frac{C(M,N)}{D_1(M,N)}\right)\right] - W_2\left[W_1\left(1+C(M,N)\right) + A_1(M,N)\right] \leqslant B_1(M,N), \tag{3.33}$$

where $D_1(M,N)$ is defined in (3.7). By (3.8), we have

$$C(M,N) \leqslant G_1^{-1}(B_1(M,N)). \tag{3.34}$$

Combining (3.13), (3.32) and (3.34), we get the desired result (3.3). □

If $\omega_1(u) = \omega_2(u) = u$ in Theorem 3.1, we get an interesting result as follows.

COROLLARY 3.2. *Assume that the conditions (i)-(iii) of Theorem 3.1 are satisfied,* $u(x,y) \in C(\Lambda, R_+)$ *satisfies*

$$\begin{aligned} u(x,y) \leqslant & a(x,y) + f_1(x,y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t) u(s,t) ds dt \\ & + f_2(x,y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t) \max_{\xi \in [\beta s, s]} u(\xi,t) ds dt \\ & + f_3(x,y) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} h_3(s,t) u(s,t) ds dt \\ & + f_4(x,y) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} h_4(s,t) \max_{\xi \in [\beta s, s]} u(\xi,t) ds dt, \quad (x,y) \in \Delta, \end{aligned} \tag{3.35}$$

$$u(x,y) \leqslant \phi(x,y), \quad (x,y) \in \Psi. \tag{3.36}$$

Then

$$u(x,y) \leqslant \frac{a(x,y)}{1 - D_2(M,N)} \exp\left(A_1(x,y) + B_1(x,y)\right), \quad (x,y) \in \Delta, \tag{3.37}$$

where

$$\begin{aligned} D_2(x,y) = & f_3(M,N) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} \frac{\widetilde{a}(s,t)}{a(x_0,y_0)} h_3(s,t) \exp\left(A_1(s,t) + B_1(s,t)\right) ds dt \\ & + f_4(M,N) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} \frac{\widetilde{a}(s,t)}{a(x_0,y_0)} h_4(s,t) \exp\left(A_1(s,t) + B_1(s,t)\right) ds dt < 1. \end{aligned} \tag{3.38}$$

Proof. Applying Theorem 3.1 to (3.35), (3.36), then

$$W_1(z) = W_2(z) = \int_c^z \frac{ds}{s} = \ln z - \ln c, \quad G_1(u) = \frac{u}{D_2(M,N)} - u,$$

obviously, $G_1(u)$ is a strictly increasing function, we get the desired result. □

THEOREM 3.3. *Let the following conditions be fulfilled:*
(i) The conditions (i), (ii) of Theorem 3.1 are satisfied;
(ii) The functions $\omega_i \in C(R_+, R_+)$ *are nondecreasing with* $\omega_i(u) > 0$ *for* $u > 0$ $(i = 1,2)$, *such that* $\omega_1 \propto \omega_2$, *and* ω_i *are subadditive and submultiplicative, that is* $\omega_i(x+y) \leqslant \omega_i(x) + \omega_i(y)$, $\omega_i(tx) \geqslant t\omega_i(x)$ *for* $0 \leqslant t \leqslant 1$*;*
(iii) The function $\phi \in C(\Psi, R_+)$ *satisfies* $\max_{(\iota,\tau)\in\Psi} \phi(\iota,\tau) \leqslant a^{\frac{1}{l}}(x_0,y_0)$*;*
(iv) The function $u \in C(\Lambda, R_+)$ *satisfies:*

$$\begin{aligned} u^l(x,y) \leqslant & a(x,y) + f_1(x,y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t) \omega_1(u(s,t)) ds dt \\ & + f_2(x,y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t) \omega_2\left(\max_{\xi \in [\beta s, s]} u(\xi,t)\right) ds dt \end{aligned}$$

$$+ f_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} h_3(s,t)u^m(s,t)dsdt$$

$$+ f_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\max_{\xi\in[\beta s,s]} u^r(\xi,t)dsdt, \quad (x,y)\in\Delta, \quad (3.39)$$

$$u(x,y)\leqslant\phi(x,y), \quad (x,y)\in\Psi, \quad (3.40)$$

where l, m, r are constants and satisfy $l\geqslant m\geqslant 0$, $l\geqslant r\geqslant 0$, $l\geqslant 1$. Then, we have the following explicit estimation

$$u(x,y)\leqslant\Big\{a(x,y)+E_1(x,y)W_1^{-1}\Big\{W_2^{-1}\Big\{W_2\Big[W_1\Big(1+G_2^{-1}(B_1(M,N))\Big) + A_1(x,y)\Big]+B_1(x,y)\Big\}\Big\}\Big\}^{\frac{1}{l}}, \quad (x,y)\in\Delta, \quad (3.41)$$

where

$$E_1(x,y) =1+f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} h_1(s,t)\omega_1\left(\frac{1}{l}K^{\frac{1-l}{l}}\widetilde{a}(s,t)+\frac{l-1}{l}K^{\frac{1}{l}}\right)dsdt$$
$$+ f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\omega_2\left(\frac{1}{l}K^{\frac{1-l}{l}}\widetilde{a}(s,t)+\frac{l-1}{l}K^{\frac{1}{l}}\right)dsdt$$
$$+ f_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} h_3(s,t)\left(\frac{m}{l}K^{\frac{m-l}{l}}\widetilde{a}(s,t)+\frac{l-m}{l}K^{\frac{m}{l}}\right)dsdt$$
$$+ f_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\left(\frac{r}{l}K^{\frac{r-l}{l}}\widetilde{a}(s,t)+\frac{l-r}{l}K^{\frac{r}{l}}\right)dsdt, \quad (3.42)$$

for any constant $K\geqslant 1$.

$$D_3(M,N) =f_3(M,N)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} \frac{m\widetilde{E}_1(s,t)}{lE_1(x_0,y_0)}h_3(s,t)K^{\frac{m-l}{l}}dsdt$$
$$+ f_4(M,N)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} \frac{r\widetilde{E}_1(s,t)}{lE_1(x_0,y_0)}h_4(s,t)K^{\frac{r-l}{l}}dsdt, \quad (3.43)$$

$$\widetilde{E}_1(x,y) = \begin{cases} E_1(x,y), & (x,y)\in\Delta, \\ E_1(x_0,y_0), & (x,y)\in\Psi. \end{cases} \quad (3.44)$$

$$G_2(u) =W_2\left(W_1\left(\frac{u}{D_3(M,N)}\right)\right)-W_2\Big(W_1(1+u)+A_1(M,N)\Big) \quad (3.45)$$

where $G_2(u)$ is increasing in R_+. W_1, W_2 are defined as in (3.4).

Proof. Define the functions $z(x,y) \in C(\Lambda, R_+)$ by

$$z(x,y) = \begin{cases} f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} h_1(s,t)\omega_1(u(s,t))dsdt \\ +f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\omega_2\Big(\max_{\xi\in[\beta s,s]} u(\xi,t)\Big)dsdt \\ +f_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} h_3(s,t)u^m(s,t)dsdt \\ +f_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\max_{\xi\in[\beta s,s]} u^r(\xi,t)dsdt, \quad (x,y)\in\Delta, \\ \\ f_3(x_0,y_0)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} h_3(s,t)u^m(s,t)dsdt \\ +f_4(x_0,y_0)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\max_{\xi\in[\beta s,s]} u^r(\xi,t)dsdt, \quad (x,y)\in\Psi. \end{cases}$$

Obviously, $z(x,y)$ is nonnegative and nondecreasing in x and y. From inequalities (3.39), (3.40), and by the Lemma 2.1, we have

$$\begin{aligned} u(x,y) &\leqslant \Big(a(x,y)+z(x,y)\Big)^{\frac{1}{l}} \leqslant \Big(\widetilde{a}(x,y)+z(x,y)\Big)^{\frac{1}{l}} \\ &\leqslant \frac{1}{l}K^{\frac{1-l}{l}}\Big(\widetilde{a}(x,y)+z(x,y)\Big)+\frac{l-1}{l}K^{\frac{1}{l}}, \quad (x,y)\in\Delta, \end{aligned} \tag{3.46}$$

$$\begin{aligned} u(x,y) &\leqslant \phi(x,y) \leqslant a^{\frac{1}{l}}(x_0,y_0) \leqslant \Big(\widetilde{a}(x,y)+z(x,y)\Big)^{\frac{1}{l}} \\ &\leqslant \frac{1}{l}K^{\frac{1-l}{l}}\Big(\widetilde{a}(x,y)+z(x,y)\Big)+\frac{l-1}{l}K^{\frac{1}{l}}, \quad (x,y)\in\Psi, \end{aligned} \tag{3.47}$$

where $\widetilde{a}(x,y)$ is defined in (3.9). Moreover, from above inequalities, for $(x,y) \in [b_*(x_0), b_*(M)] \times [y_0, N)$, we have

$$\begin{aligned} \max_{\xi\in[\beta x,x]} u(\xi,y) &\leqslant \Big(\widetilde{a}(x,y)+z(x,y)\Big)^{\frac{1}{l}} \leqslant \frac{1}{l}K^{\frac{1-l}{l}}\Big(\widetilde{a}(x,y)+z(x,y)\Big)+\frac{l-1}{l}K^{\frac{1}{l}}, \\ u^m(x,y) &\leqslant \Big(\widetilde{a}(x,y)+z(x,y)\Big)^{\frac{m}{l}} \leqslant \frac{m}{l}K^{\frac{m-l}{l}}\Big(\widetilde{a}(x,y)+z(x,y)\Big)+\frac{l-m}{l}K^{\frac{m}{l}}, \\ \max_{\xi\in[\beta x,x]} u^r(\xi,y) &\leqslant \Big(\widetilde{a}(x,y)+z(x,y)\Big)^{\frac{r}{l}} \leqslant \frac{r}{l}K^{\frac{r-l}{l}}\Big(\widetilde{a}(x,y)+z(x,y)\Big)+\frac{l-r}{l}K^{\frac{r}{l}}, \end{aligned} \tag{3.48}$$

for any constant $K \geqslant 1$. From (3.48) and property of ω_i $(i=1,2)$, the function $z(x,y)$ may be written in the form:

$$\begin{aligned} z(x,y) \leqslant & f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} h_1(s,t)\omega_1\left[\frac{1}{l}K^{\frac{1-l}{l}}\Big(\widetilde{a}(s,t)+z(s,t)\Big)+\frac{l-1}{l}K^{\frac{1}{l}}\right]dsdt \\ & + f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\omega_2\left[\frac{1}{l}K^{\frac{1-l}{l}}\Big(\widetilde{a}(s,t)+z(s,t)\Big)+\frac{l-1}{l}K^{\frac{1}{l}}\right]dsdt \end{aligned}$$

$$
\begin{aligned}
&+f_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}h_3(s,t)\left[\frac{m}{l}K^{\frac{m-l}{l}}\left(\widetilde{a}(s,t)+z(s,t)\right)+\frac{l-m}{l}K^{\frac{m}{l}}\right]dsdt\\
&+f_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}h_4(s,t)\left[\frac{r}{l}K^{\frac{r-l}{l}}\left(\widetilde{a}(s,t)+z(s,t)\right)+\frac{l-r}{l}K^{\frac{r}{l}}\right]dsdt\\
\leqslant&E_1(x,y)+f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)}h_1(s,t)\omega_1(z(s,t))dsdt\\
&+f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)}h_2(s,t)\omega_2(z(s,t))dsdt\\
&+f_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}h_3(s,t)\frac{m}{l}K^{\frac{m-l}{l}}z(s,t)dsdt\\
&+f_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}h_4(s,t)\frac{r}{l}K^{\frac{r-l}{l}}z(s,t)dsdt,\quad (x,y)\in\Delta, \qquad (3.49)
\end{aligned}
$$

$$
\begin{aligned}
z(x,y)\leqslant&f_3(x_0,y_0)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}h_3(s,t)\left[\frac{m}{l}K^{\frac{m-l}{l}}\left(\widetilde{a}(s,t)+z(s,t)\right)+\frac{l-m}{l}K^{\frac{m}{l}}\right]dsdt\\
&+f_4(x_0,y_0)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}h_4(s,t)\left[\frac{r}{l}K^{\frac{r-l}{l}}\left(\widetilde{a}(s,t)+z(s,t)\right)+\frac{l-r}{l}K^{\frac{r}{l}}\right]dsdt\\
\leqslant&E_1(x_0,y_0)+f_3(x_0,y_0)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}h_3(s,t)\frac{m}{l}K^{\frac{m-l}{l}}z(s,t)dsdt\\
&+f_4(x_0,y_0)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}h_4(s,t)\frac{r}{l}K^{\frac{r-l}{l}}z(s,t)dsdt,\quad (x,y)\in\Psi, \qquad (3.50)
\end{aligned}
$$

where $E_1(x,y)$ is defined in (3.42), which is nonnegative and nondecreasing in each of the variables. Inequalities (3.49), (3.50) satisfies the conditions of Theorem 3.1, applying the result of Theorem 3.1, we get

$$
z(x,y)\leqslant E_1(x,y)W_1^{-1}\left\{W_2^{-1}\left\{W_2\left[W_1\left(1+G_2^{-1}(B_1(M,N))\right)+A_1(x,y)\right]+B_1(x,y)\right\}\right\}, \qquad (3.51)
$$

where $A_1(x,y)$, $B_1(x,y)$ and $G_2(u)$ are defined in (3.5), (3.6) and (3.45), respectively. Combining (3.46) and (3.51), we get the desired result. □

Take $l=2$, $m=r=1$, $\omega_1(u)=\omega_2(u)=u$ in Theorem 3.3, a new Gamidov-Ou-Iang type inequalities is obtained as follows.

COROLLARY 3.4. *Suppose that the conditions (i) of Theorem 3.3 are satisfied, and the function* $\phi\in C(\Psi,R_+)$ *satisfies* $\max_{(\iota,\tau)\in\Psi}\phi(\iota,\tau)\leqslant a^{\frac{1}{2}}(x_0,y_0)$. *If* $u\in C(\Lambda,R_+)$ *satisfies*

$$
\begin{aligned}
u^2(x,y)\leqslant&a(x,y)+f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)}h_1(s,t)u(s,t)dsdt\\
&+f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)}h_2(s,t)\max_{\xi\in[\beta s,s]}u(\xi,t)dsdt
\end{aligned}
$$

$$+ f_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} h_3(s,t)u(s,t)dsdt$$
$$+ f_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\max_{\xi\in[\beta s,s]} u(\xi,t)dsdt, \quad (x,y)\in\Delta, \tag{3.52}$$

$$u(x,y) \leqslant \phi(x,y), \quad (x,y)\in\Psi. \tag{3.53}$$

Then, we have the following explicit estimation

$$u(x,y) \leqslant \left[a(x,y) + \frac{E_2(x,y)}{1-D_4(M,N)}\exp\left(A_1(x,y)+B_1(x,y)\right)\right]^{\frac{1}{2}}, \quad (x,y)\in\Delta, \tag{3.54}$$

where

$$E_2(x,y) = 1 + f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} h_1(s,t)\left(\frac{1}{2}K^{-\frac{1}{2}}\widetilde{a}(s,t) + \frac{1}{2}K^{\frac{1}{2}}\right)dsdt$$
$$+ f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} h_2(s,t)\left(\frac{1}{2}K^{-\frac{1}{2}}\widetilde{a}(s,t) + \frac{1}{2}K^{\frac{1}{2}}\right)dsdt$$
$$+ f_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} h_3(s,t)\left(\frac{1}{2}K^{-\frac{1}{2}}\widetilde{a}(s,t) + \frac{1}{2}K^{\frac{1}{2}}\right)dsdt$$
$$+ f_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} h_4(s,t)\left(\frac{1}{2}K^{-\frac{1}{2}}\widetilde{a}(s,t) + \frac{1}{2}K^{\frac{1}{2}}\right)dsdt, \tag{3.55}$$

$$D_4(M,N) = f_3(M,N)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} \frac{\widetilde{E}_2(s,t)}{2E_2(x_0,y_0)} h_3(s,t)K^{-\frac{1}{2}}\exp\left(A_1(s,t)+B_1(s,t)\right)dsdt$$
$$+ f_4(M,N)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} \frac{\widetilde{E}_2(s,t)}{2E_2(x_0,y_0)} h_4(s,t)K^{-\frac{1}{2}}$$
$$\exp\left(A_1(s,t)+B_1(s,t)\right)dsdt < 1. \tag{3.56}$$

$$\widetilde{E}_2(x,y) = \begin{cases} E_2(x,y), & (x,y)\in\Delta, \\ E_2(x_0,y_0), & (x,y)\in\Psi. \end{cases} \tag{3.57}$$

Proof. Applying Theorem 3.3 to (3.52), (3.53), then

$$W_1(z) = W_2(z) = \int_c^z \frac{ds}{s} = \ln z - \ln c, \quad G_2(u) = \frac{u}{D_4(M,N)} - u,$$

obviously, $G_2(u)$ is a strictly increasing function, we get the desired result. □

THEOREM 3.5. *Let the following conditions be fulfilled:*
(i) The conditions (i)-(ii) of Theorem 3.3 are satisfied;
(ii) The function $\phi \in C(\Psi, R_+)$ *satisfies* $0 < \max_{(\iota,\tau)\in\Psi}\phi(\iota,\tau) \leqslant 5^{\frac{q-1}{ql}} a^{\frac{1}{l}}(x_0,y_0)$;
(iii) $\alpha_{ki}\in(0,1]$, $\beta_i\in(0,1)$ *and* $p(\gamma_{ki}-1)+1>0$, $p(\beta_i-1)+1>0$ *such that* $\frac{1}{p}+$

$\alpha_{ki}(\beta_i-1)+\gamma_{ki}-1\geqslant 0$ $(p>1;\ k=1,2;\ i=1,2,3,4)$;
(iv) The function $u\in C(\Lambda,R_+)$ *satisfies:*

$$\begin{aligned}
u^l(x,y)\leqslant &a(x,y)\\
&+f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)}(x^{\alpha_{11}}-s^{\alpha_{11}})^{\beta_1-1}s^{\gamma_{11}-1}(y^{\alpha_{21}}-t^{\alpha_{21}})^{\beta_1-1}t^{\gamma_{21}-1}h_1(s,t)\\
&\times\omega_1(u(s,t))dsdt\\
&+f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)}(x^{\alpha_{12}}-s^{\alpha_{12}})^{\beta_2-1}s^{\gamma_{12}-1}(y^{\alpha_{22}}-t^{\alpha_{22}})^{\beta_2-1}t^{\gamma_{22}-1}h_2(s,t)\\
&\times\omega_2\left(\max_{\xi\in[\beta s,s]}u(\xi,t)\right)dsdt\\
&+f_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}(M^{\alpha_{13}}-s^{\alpha_{13}})^{\beta_3-1}s^{\gamma_{13}-1}(N^{\alpha_{23}}-t^{\alpha_{23}})^{\beta_3-1}t^{\gamma_{23}-1}\\
&\times h_3(s,t)u^m(s,t)dsdt\\
&+f_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}(M^{\alpha_{14}}-s^{\alpha_{14}})^{\beta_4-1}s^{\gamma_{14}-1}(N^{\alpha_{24}}-t^{\alpha_{24}})^{\beta_4-1}t^{\gamma_{24}-1}\\
&\times h_4(s,t)\max_{\xi\in[\beta s,s]}u^r(\xi,t)dsdt,\quad (x,y)\in\Delta,
\end{aligned}\tag{3.58}$$

$$u(x,y)\leqslant\phi(x,y),\quad (x,y)\in\Psi.\tag{3.59}$$

Then, we have the following explicit estimation

$$\begin{aligned}
u(x,y)\leqslant&\Bigg\{5^{q-1}a^q(x,y)+E_3(x,y)\overline{W}_1^{-1}\Bigg\{\overline{W}_2^{-1}\Bigg\{\overline{W}_2\Bigg[\overline{W}_1\left(1+G_3^{-1}(B_2(M,N))\right)\\
&+A_2(x,y)\Bigg]+B_2(x,y)\Bigg\}\Bigg\}\Bigg\}^{\frac{1}{ql}},\quad (x,y)\in\Delta,
\end{aligned}\tag{3.60}$$

where

$$\begin{aligned}
E_3(x,y)=&1\\
&+F_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)}H_1(s,t)2^{q-1}\omega_1^q\left(\frac{1}{ql}K^{\frac{1-ql}{ql}}\widetilde{a}_1(s,t)+\frac{ql-1}{ql}K^{\frac{1}{ql}}\right)dsdt\\
&+F_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)}H_2(s,t)2^{q-1}\omega_2^q\left(\frac{1}{ql}K^{\frac{1-ql}{ql}}\widetilde{a}_1(s,t)+\frac{ql-1}{ql}K^{\frac{1}{ql}}\right)dsdt\\
&+F_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}H_3(s,t)\left(\frac{m}{l}K^{\frac{m-l}{l}}\widetilde{a}_1(s,t)+\frac{l-m}{l}K^{\frac{m}{l}}\right)dsdt\\
&+F_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}H_4(s,t)\left(\frac{r}{l}K^{\frac{r-l}{l}}\widetilde{a}_1(s,t)+\frac{l-r}{l}K^{\frac{r}{l}}\right)dsdt,
\end{aligned}\tag{3.61}$$

$$\begin{aligned}
F_i(x,y)=&5^{q-1}f_i^q(x,y)e_i^q(x,y),\ (i=1,2),\\
F_j(x,y)=&5^{q-1}f_j^q(x,y)e_j^q(M,N),\ (j=3,4),
\end{aligned}$$

$$H_i(x,y) = h_i^q(x,y),\ (i=1,2,3,4), \tag{3.62}$$

$$\begin{aligned} e_i(x,y) =& \left(\frac{x^{\theta_{1i}} y^{\theta_{2i}}}{\alpha_{1i}\alpha_{2i}}\right)^{\frac{1}{p}} \\ & \times \left[B\left(\frac{p(\gamma_{1i}-1)+1}{\alpha_{1i}}, p(\beta_i-1)+1\right) B\left(\frac{p(\gamma_{2i}-1)+1}{\alpha_{2i}}, p(\beta_i-1)+1\right)\right]^{\frac{1}{p}}, \\ \theta_{ki} =& p[\alpha_{ki}(\beta_i-1)+\gamma_{ki}-1]+1 \geqslant 0,\ (k=1,2;\ i=1,2,3,4), \end{aligned} \tag{3.63}$$

$$\overline{W}_1(z) = \int_c^z \frac{ds}{\omega_1^q(s)}, \quad \overline{W}_2(z) = \int_c^z \frac{\omega_1^q(\overline{W}_1^{-1}(s))}{\omega_2^q(\overline{W}_1^{-1}(s))} ds, \tag{3.64}$$

$$A_2(x,y) = F_1(x,y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} H_1(s,t) 2^{q-1} dsdt, \tag{3.65}$$

$$B_2(x,y) = F_2(x,y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} H_2(s,t) 2^{q-1} dsdt, \tag{3.66}$$

$$\begin{aligned} D_5(M,N) =& F_3(M,N) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} H_3(s,t) \frac{m\widetilde{E}_3(s,t)}{lE_3(x_0,y_0)} K^{\frac{m-l}{l}} dsdt \\ & + F_4(M,N) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} H_4(s,t) \frac{r\widetilde{E}_3(s,t)}{lE_3(x_0,y_0)} K^{\frac{r-l}{l}} dsdt, \end{aligned} \tag{3.67}$$

$$G_3(u) = \overline{W}_2\left(\overline{W}_1\left(\frac{u}{D_5(M,N)}\right)\right) - \overline{W}_2\left(\overline{W}_1(1+u) + A_2(M,N)\right), \tag{3.68}$$

where $G_3(u)$ is increasing on R_+.

Proof. Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, then $q>0$. Since $p(\beta_i-1)+1>0$, $p(\gamma_{ki}-1)+1>0$, $\frac{1}{p}+\alpha_{ki}(\beta_i-1)+\gamma_{ki}-1 \geqslant 0$ for $k=1,2;\ i=1,2,3,4$. Using the Hölder inequality to (3.58), and applying Lemma 2.2, we get

$$\begin{aligned} & u^l(x,y) \\ \leqslant & a(x,y) \\ & + f_1(x,y) \\ & \times \left[\int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} (x^{\alpha_{11}} - s^{\alpha_{11}})^{p(\beta_1-1)} s^{p(\gamma_{11}-1)} (y^{\alpha_{21}} - t^{\alpha_{21}})^{p(\beta_1-1)} t^{p(\gamma_{21}-1)} dsdt\right]^{\frac{1}{p}} \\ & \times \left[\int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1^q(s,t) \omega_1^q(u(s,t)) dsdt\right]^{\frac{1}{q}} \\ & + f_2(x,y) \\ & \times \left[\int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} (x^{\alpha_{12}} - s^{\alpha_{12}})^{p(\beta_2-1)} s^{p(\gamma_{12}-1)} (y^{\alpha_{22}} - t^{\alpha_{22}})^{p(\beta_2-1)} t^{p(\gamma_{22}-1)} dsdt\right]^{\frac{1}{p}} \\ & \times \left[\int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2^q(s,t) \omega_2^q\left(\max_{\xi\in[\beta s,s]} u(\xi,t)\right) dsdt\right]^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&+f_3(x,y)\\
&\times\left[\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}(M^{\alpha_{13}}-s^{\alpha_{13}})^{p(\beta_3-1)}s^{p(\gamma_{13}-1)}(N^{\alpha_{23}}-t^{\alpha_{23}})^{p(\beta_3-1)}t^{p(\gamma_{23}-1)}dsdt\right]^{\frac{1}{p}}\\
&\times\left[\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}h_3^q(s,t)u^{qm}(s,t)dsdt\right]^{\frac{1}{q}}\\
&+f_4(x,y)\\
&\times\left[\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}(M^{\alpha_{14}}-s^{\alpha_{14}})^{p(\beta_4-1)}s^{p(\gamma_{14}-1)}(N^{\alpha_{24}}-t^{\alpha_{24}})^{p(\beta_4-1)}t^{p(\gamma_{24}-1)}dsdt\right]^{\frac{1}{p}}\\
&\times\left[\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}h_4^q(s,t)\max_{\xi\in[\beta s,s]}u^{qr}(\xi,t)dsdt\right]^{\frac{1}{q}}\\
\leqslant& a(x,y)+f_1(x,y)e_1(x,y)\left[\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)}h_1^q(s,t)\omega_1^q(u(s,t))dsdt\right]^{\frac{1}{q}}\\
&+f_2(x,y)e_2(x,y)\left[\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)}h_2^q(s,t)\omega_2^q\left(\max_{\xi\in[\beta s,s]}u(\xi,t)\right)dsdt\right]^{\frac{1}{q}}\\
&+f_3(x,y)e_3(M,N)\left[\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}h_3^q(s,t)u^{qm}(s,t)dsdt\right]^{\frac{1}{q}}\\
&+f_4(x,y)e_4(M,N)\left[\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}h_4^q(s,t)\max_{\xi\in[\beta s,s]}u^{qr}(\xi,t)dsdt\right]^{\frac{1}{q}},\ (x,y)\in\Delta,
\end{aligned}
\tag{3.69}
$$

$$
u(x,y)\leqslant\phi(x,y)\leqslant 5^{\frac{q-1}{ql}}a^{\frac{1}{l}}(x_0,y_0),\quad (x,y)\in\Psi \tag{3.70}
$$

where $e_i(x,y)$ are defined in (3.63). It is easy to see that $e_i(x,y)$ is nondecreasing in each of the variables. Applying Lemma 2.3 to (3.69), we get

$$
\begin{aligned}
u^{ql}(x,y)\leqslant& a_1(x,y)+F_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)}H_1(s,t)\omega_1^q(u(s,t))dsdt\\
&+F_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)}H_2(s,t)\omega_2^q\left(\max_{\xi\in[\beta s,s]}u(\xi,t)\right)dsdt\\
&+F_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}H_3(s,t)u^{qm}(s,t)dsdt\\
&+F_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}H_4(s,t)\max_{\xi\in[\beta s,s]}u^{qr}(\xi,t)dsdt,\quad (x,y)\in\Delta,
\end{aligned}
\tag{3.71}
$$

where $F_i(x,y)$, $H_i(x,y)$ $(i=1,2,3,4)$ are defined in (3.62), $a_1(x,y)=5^{q-1}a^q(x,y)$, and $F_i(x,y)$, $a_1(x,y)$ are nondecreasing in each of the variables. Let

$$
z(x,y)=u^q(x,y). \tag{3.72}
$$

From (3.70) and (3.71), we have

$$\begin{aligned} z^l(x,y) \leqslant & a_1(x,y) + F_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} H_1(s,t)\omega_1^q(z^{\frac{1}{q}}(s,t))dsdt \\ & + F_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} H_2(s,t)\omega_2^q\left(\max_{\xi\in[\beta s,s]} z^{\frac{1}{q}}(\xi,t)\right)dsdt \\ & + F_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} H_3(s,t)z^m(s,t)dsdt \\ & + F_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} H_4(s,t)\max_{\xi\in[\beta s,s]} z^r(\xi,t)dsdt, \quad (x,y)\in\Delta, \end{aligned} \tag{3.73}$$

$$z(x,y) \leqslant 5^{\frac{q-1}{l}} a^{\frac{q}{l}}(x_0,y_0) = a_1^{\frac{1}{l}}(x_0,y_0), \quad (x,y)\in\Psi. \tag{3.74}$$

Inequalities (3.73), (3.74) are similar to (3.39), (3.40), separately. Let

$$v(x,y) = \begin{cases} F_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} H_1(s,t)\omega_1^q(z^{\frac{1}{q}}(s,t))dsdt \\ +F_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} H_2(s,t)\omega_2^q\left(\max_{\xi\in[\beta s,s]} z^{\frac{1}{q}}(\xi,t)\right)dsdt \\ +F_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} H_3(s,t)z^m(s,t)dsdt \\ +F_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} H_4(s,t)\max_{\xi\in[\beta s,s]} z^r(\xi,t)dsdt, \quad (x,y)\in\Delta, \\ \\ F_3(x_0,y_0)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)} H_3(s,t)z^m(s,t)dsdt \\ +F_4(x_0,y_0)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)} H_4(s,t)\max_{\xi\in[\beta s,s]} z^r(\xi,t)dsdt, \quad (x,y)\in\Psi. \end{cases}$$

Obviously, $v(x,y)$ is a positive and nondecreasing function. And from (3.73), (3.74), we have

$$z(x,y) \leqslant (a_1(x,y)+v(x,y))^{\frac{1}{l}} \leqslant (\widetilde{a}_1(x,y)+v(x,y))^{\frac{1}{l}}, \quad (x,y)\in\Delta, \tag{3.75}$$

$$z(x,y) \leqslant a_1^{\frac{1}{l}}(x_0,y_0) \leqslant (\widetilde{a}_1(x,y)+v(x,y))^{\frac{1}{l}}, \quad (x,y)\in\Psi, \tag{3.76}$$

where

$$\widetilde{a}_1(x,y) = \begin{cases} a_1(x,y), & (x,y)\in\Delta, \\ a_1(x_0,y_0), & (x,y)\in\Psi, \end{cases} \tag{3.77}$$

which is nondecreasing in each of the variables. Moreover, from (3.75) and (3.76), for $(x,y)\in\Lambda$, we have

$$z^{\frac{1}{q}}(x,y) \leqslant (\widetilde{a}_1(x,y)+v(x,y))^{\frac{1}{ql}} \leqslant \frac{1}{ql}K^{\frac{1-ql}{ql}}\left(\widetilde{a}_1(x,y)+v(x,y)\right) + \frac{ql-1}{ql}K^{\frac{1}{ql}},$$

$$\max_{\xi\in[\beta x,x]} z^{\frac{1}{q}}(\xi,y) \leqslant \max_{\xi\in[\beta x,x]}\left(\widetilde{a}_1(\xi,y)+v(\xi,y)\right)^{\frac{1}{ql}} \leqslant \left(\max_{\xi\in[\beta x,x]}\widetilde{a}_1(\xi,y)+\max_{\xi\in[\beta x,x]} v(\xi,y)\right)^{\frac{1}{ql}}$$

$$\leqslant \frac{1}{ql}K^{\frac{1-ql}{ql}}\left(\widetilde{a}_1(x,y)+v(x,y)\right)+\frac{ql-1}{ql}K^{\frac{1}{ql}},$$

$$z^m(x,y) \leqslant (\widetilde{a}_1(x,y)+v(x,y))^{\frac{m}{l}} \leqslant \frac{m}{l}K^{\frac{m-l}{l}}\left(\widetilde{a}_1(x,y)+v(x,y)\right)+\frac{l-m}{l}K^{\frac{m}{l}},$$

$$\max_{\xi\in[\beta x,x]} z^r(\xi,y) \leqslant \max_{\xi\in[\beta x,x]}\left(\widetilde{a}_1(\xi,y)+v(\xi,y)\right)^{\frac{r}{l}} \leqslant \frac{r}{l}K^{\frac{r-l}{l}}\left(\widetilde{a}_1(x,y)+v(x,y)\right) + \frac{l-r}{l}K^{\frac{r}{l}}, \tag{3.78}$$

for any constant $K \geqslant 1$. By the subadditive of ω_i $(i=1,2)$ and the Lemma 2.3, we have

$$\begin{aligned}\omega_1^q(z^{\frac{1}{q}}(x,y)) &\leqslant \omega_1^q\left[\frac{1}{ql}K^{\frac{1-ql}{ql}}\left(\widetilde{a}_1(x,y)+v(x,y)\right)+\frac{ql-1}{ql}K^{\frac{1}{ql}}\right]\\ &\leqslant \left[\omega_1\left(\frac{1}{ql}K^{\frac{1-ql}{ql}}\widetilde{a}_1(x,y)+\frac{ql-1}{ql}K^{\frac{1}{ql}}\right)+\omega_1(v(x,y))\right]^q\\ &\leqslant 2^{q-1}\omega_1^q\left(\frac{1}{ql}K^{\frac{1-ql}{ql}}\widetilde{a}_1(x,y)+\frac{ql-1}{ql}K^{\frac{1}{ql}}\right)+2^{q-1}\omega_1^q(v(x,y)),\\ \omega_2^q\left(\max_{\xi\in[\beta x,x]} z^{\frac{1}{q}}(\xi,y)\right) &\leqslant 2^{q-1}\omega_2^q\left(\frac{1}{ql}K^{\frac{1-ql}{ql}}\widetilde{a}_1(x,y)+\frac{ql-1}{ql}K^{\frac{1}{ql}}\right)+2^{q-1}\omega_2^q(v(x,y)),\end{aligned} \tag{3.79}$$

Substituting (3.78) and (3.79) into the definition of $v(x,y)$, and applying the method of proof of Theorem 3.3, we get

$$z(x,y) \leqslant \left\{a_1(x,y)+E_3(x,y)\overline{W}_1^{-1}\left\{\overline{W}_2^{-1}\left\{\overline{W}_2\left[\overline{W}_1\left(1+G_3^{-1}(B_2(M,N))\right) + A_2(x,y)\right]+B_2(x,y)\right\}\right\}\right\}^{\frac{1}{l}}, \quad (x,y)\in\Delta, \tag{3.80}$$

where $E_3(x,y)$, $\overline{W}_i(z)$, $A_2(x,y)$, $B_2(x,y)$ and $G_3(u)$ are defined in (3.61), (3.64), (3.65), (3.66) and (3.68), separately. Combining (3.72) and (3.80), we can easily get (3.60). □

If $\alpha_{1i}=\alpha_{2i}=1$, $\gamma_{1i}=\gamma_{2i}=1$, $0<\beta_i<1$ $(i=1,2,3,4)$ in (3.58), we get the following result.

COROLLARY 3.6. *Let the conditions (i)-(ii) of Theorem 3.5 be satisfied, suppose that $u\in C(\Lambda,R_+)$ satisfies:*

$$\begin{aligned}u^l(x,y) \leqslant& a(x,y)+f_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)}(x-s)^{\beta_1-1}(y-t)^{\beta_1-1}h_1(s,t)\omega_1(u(s,t))dsdt\\ &+f_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)}(x-s)^{\beta_2-1}(y-t)^{\beta_2-1}h_2(s,t)\omega_2\left(\max_{\xi\in[\beta s,s]}u(\xi,t)\right)dsdt\end{aligned}$$

$$+ f_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}(M-s)^{\beta_3-1}(N-t)^{\beta_3-1}h_3(s,t)u^m(s,t)dsdt$$
$$+ f_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}(M-s)^{\beta_4-1}(N-t)^{\beta_4-1}h_4(s,t)\max_{\xi\in[\beta s,s]}u^r(\xi,t)dsdt,$$
$$(x,y)\in\Delta, \tag{3.81}$$
$$u(x,y)\leqslant\phi(x,y), \quad (x,y)\in\Psi. \tag{3.82}$$

Then, we have the following explicit estimation

$$u(x,y)\leqslant\Bigg\{5^{q-1}a^q(x,y)+E_4(x,y)\overline{W}_1^{-1}\Bigg\{\overline{W}_2^{-1}\Bigg\{\overline{W}_2\Bigg[\overline{W}_1\Bigg(1+G_4^{-1}(B_3(M,N))\Bigg) + A_3(x,y)\Bigg]+B_3(x,y)\Bigg\}\Bigg\}\Bigg\}^{\frac{1}{ql}}, \quad (x,y)\in\Delta, \tag{3.83}$$

where

$$\begin{aligned}E_4(x,y)=&1\\ &+\widetilde{F}_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)}H_1(s,t)2^{q-1}\\ &\times\omega_1^q\left(\frac{1}{ql}K^{\frac{1-ql}{ql}}\widetilde{a}_1(s,t)+\frac{ql-1}{ql}K^{\frac{1}{ql}}\right)dsdt\\ &+\widetilde{F}_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)}H_2(s,t)2^{q-1}\\ &\times\omega_2^q\left(\frac{1}{ql}K^{\frac{1-ql}{ql}}\widetilde{a}_1(s,t)+\frac{ql-1}{ql}K^{\frac{1}{ql}}\right)dsdt\\ &+\widetilde{F}_3(x,y)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}H_3(s,t)\left(\frac{m}{l}K^{\frac{m-l}{l}}\widetilde{a}_1(s,t)+\frac{l-m}{l}K^{\frac{m}{l}}\right)dsdt\\ &+\widetilde{F}_4(x,y)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}H_4(s,t)\left(\frac{r}{l}K^{\frac{r-l}{l}}\widetilde{a}_1(s,t)+\frac{l-r}{l}K^{\frac{r}{l}}\right)dsdt,\end{aligned} \tag{3.84}$$

$$\begin{aligned}\widetilde{F}_i(x,y)=&5^{q-1}f_i^q(x,y)\widetilde{e}_i^q(x,y),\ (i=1,2),\\ \widetilde{F}_j(x,y)=&5^{q-1}f_j^q(x,y)\widetilde{e}_j^q(M,N),\ (j=3,4),\end{aligned} \tag{3.85}$$

$$\widetilde{e}_i(x,y)=(xy)^{\frac{\theta_i}{p}}[B(1,p(\beta_i-1)+1)]^{\frac{2}{p}}, \quad \theta_i=p(\beta_i-1)+1\geqslant 0, \quad i=1,2,3,4, \tag{3.86}$$

$$A_3(x,y)=\widetilde{F}_1(x,y)\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)}H_1(s,t)2^{q-1}dsdt, \tag{3.87}$$

$$B_3(x,y)=\widetilde{F}_2(x,y)\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)}H_2(s,t)2^{q-1}dsdt, \tag{3.88}$$

$$D_6(M,N)=\widetilde{F}_3(M,N)\int_{b_3(x_0)}^{b_3(M)}\int_{c_3(y_0)}^{c_3(N)}\frac{m\widetilde{E}_4(s,t)}{lE_4(x_0,y_0)}H_3(s,t)K^{\frac{m-l}{l}}dsdt$$

$$+\widetilde{F}_4(M,N)\int_{b_4(x_0)}^{b_4(M)}\int_{c_4(y_0)}^{c_4(N)}\frac{r\widetilde{E}_4(s,t)}{lE_4(x_0,y_0)}H_4(s,t)K^{\frac{r-l}{l}}dsdt, \tag{3.89}$$

$$G_4(u)=\overline{W}_2\left(\overline{W}_1(\frac{u}{D_6(M,N)})\right)-\overline{W}_2\left(\overline{W}_1(1+u)+A_3(M,N)\right), \tag{3.90}$$

where $G_4(u)$ is increasing on R_+.

4. Applications

In this section, we present some examples to show applications in the boundedness and uniqueness of a certain Gamidov type weakly singular integral equation with maxima. Consider the following weakly singular integral equations:

$$\begin{aligned}u^l(x,y)=&a(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\beta_1-1}s^{\gamma_{11}-1}(y-t)^{\beta_1-1}t^{\gamma_{21}-1}P_1(s,t,x,y,u(s,t))dsdt\\&+\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\beta_2-1}s^{\gamma_{12}-1}(y-t)^{\beta_2-1}t^{\gamma_{22}-1}P_2(s,t,x,y,\max_{\xi\in[\beta s,s]}u(\xi,t))dsdt\\&+\int_{x_0}^{M}\int_{y_0}^{N}(M-s)^{\beta_3-1}s^{\gamma_{13}-1}(N-t)^{\beta_3-1}t^{\gamma_{23}-1}P_3(s,t,x,y,u(s,t))dsdt\\&+\int_{x_0}^{M}\int_{y_0}^{N}(M-s)^{\beta_4-1}s^{\gamma_{14}-1}(N-t)^{\beta_4-1}t^{\gamma_{24}-1}P_4(s,t,x,y,\max_{\xi\in[\beta s,s]}u(\xi,t))dsdt,\\&(x,y)\in\Delta, \end{aligned}\tag{4.1}$$

$$u(x,y)=\phi(x,y),\quad (x,y)\in[\beta x_0,x_0]\times[y_0,N]\triangleq\Psi, \tag{4.2}$$

where $u(x,y)\in C(\Lambda,R)$, $a(x,y)\in C(\Delta,R)$, $P_i\in C(\Delta^2\times R,R)$ $(i=1,2,3,4)$, $\phi(x,y)\in C(\Psi,R)$, and $0<\beta<1$, $l\geqslant 1$ are constants.

First, we give the estimate for the solution of problem (4.1) (4.2). Suppose that the following conditions are satisfied:

$$\begin{aligned}&|P_1(s,t,x,y,u(s,t))|\leqslant f_1(x,y)h_1(s,t)\omega_1(|u(s,t)|),\\&|P_2(s,t,x,y,\max_{\xi\in[\beta s,s]}u(\xi,t))|\leqslant f_2(x,y)h_2(s,t)\omega_2(\max_{\xi\in[\beta s,s]}|u(\xi,t)|),\\&|P_3(s,t,x,y,u(s,t))|\leqslant f_3(x,y)h_3(s,t)|u(s,t)|^m,\\&|P_4(s,t,x,y,\max_{\xi\in[\beta s,s]}u(\xi,t))|\leqslant f_4(x,y)h_4(s,t)\max_{\xi\in[\beta s,s]}|u(\xi,t)|^r.\end{aligned}\tag{4.3}$$

THEOREM 4.1. *Let the following conditions be fulfilled:*
(i) The functions $f_i(x,y)$, $h_i(x,y)\in C(\Delta,R_+)$ $(i=1,2,3,4)$, and $f_i(x,y)$ are nondecreasing in each of the variables;
(ii) The functions $\omega_i(u)\in C(R_+,R_+)$ are nondecreasing on R_+ and positive on $(0,+\infty)$ such that $\omega_1\propto\omega_2$, and ω_i are subadditive and submultiplicative, that is $\omega_i(x+y)\leqslant\omega_i(x)+\omega_i(y)$, $\omega_i(tx)\geqslant t\omega_i(x)$ for $0\leqslant t\leqslant 1$;

(iii) $\beta_i \in (0,1)$ *and* $p(\gamma_{ki}-1)+1>0$, $p(\beta_i-1)+1>0$ *such that* $\frac{1}{p}+\beta_i+\gamma_{ki}-2 \geqslant 0$ $(p>1;\ k=1,2;\ i=1,2,3,4)$*;*

(iv) $\max_{(\iota,\tau)\in\Psi}|\phi(\iota,\tau)| \leqslant 5^{\frac{q-1}{ql}}|a(x_0,y_0)|^{\frac{1}{l}}$.

Then every solution $u(x,y)$ *of equations (4.1), (4.2) has the estimate*

$$|u(x,y)| \leqslant \Bigg\{5^{q-1}\overline{a}^q(x,y)+E_5(x,y)\overline{W}_1^{-1}\Bigg\{\overline{W}_2^{-1}\Bigg\{\overline{W}_2\Bigg[\overline{W}_1\Bigg(1+G_5^{-1}(B_4(M,N))\Bigg) + A_4(x,y)\Bigg]B_4(x,y)\Bigg\}\Bigg\}\Bigg\}^{\frac{1}{ql}}, \quad (x,y)\in\Delta, \tag{4.4}$$

where

$$\overline{a}(x,y)=\max_{(\iota,\tau)\in[x_0,x]\times[y_0,y]}|a(\iota,\tau)|, \tag{4.5}$$

$$\begin{aligned} E_5(x,y) =&1+\overline{F}_1(x,y)\int_{x_0}^{x}\int_{y_0}^{y}H_1(s,t)2^{q-1}\omega_1^q\left(\frac{1}{ql}K^{\frac{1-ql}{ql}}\widetilde{a}_2(s,t)+\frac{ql-1}{ql}K^{\frac{1}{ql}}\right)dsdt\\ &+\overline{F}_2(x,y)\int_{x_0}^{x}\int_{y_0}^{y}H_2(s,t)2^{q-1}\omega_2^q\left(\frac{1}{ql}K^{\frac{1-ql}{ql}}\widetilde{a}_2(s,t)+\frac{ql-1}{ql}K^{\frac{1}{ql}}\right)dsdt\\ &+\overline{F}_3(x,y)\int_{x_0}^{M}\int_{y_0}^{N}H_3(s,t)\left(\frac{m}{l}K^{\frac{m-l}{l}}\widetilde{a}_2(s,t)+\frac{l-m}{l}K^{\frac{m}{l}}\right)dsdt\\ &+\overline{F}_4(x,y)\int_{x_0}^{M}\int_{y_0}^{N}H_4(s,t)\left(\frac{r}{l}K^{\frac{r-l}{l}}\widetilde{a}_2(s,t)+\frac{l-r}{l}K^{\frac{r}{l}}\right)dsdt, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \overline{F}_i(x,y) =&5^{q-1}f_i^q(x,y)\overline{e}_i^q(x,y),\ (i=1,2), \quad \overline{F}_j(x,y)=5^{q-1}f_j^q(x,y)\overline{e}_j^q(M,N),\ (j=3,4).\\ H_i(x,y) =&h_i^q(x,y),\ (i=1,2,3,4). \end{aligned} \tag{4.7}$$

$$\begin{aligned} \overline{e}_i(x,y) =&(x^{\overline{\theta}_{1i}}y^{\overline{\theta}_{2i}})^{\frac{1}{p}}\Big[B(p(\gamma_{1i}-1)+1,p(\beta_i-1)+1)B(p(\gamma_{2i}-1)+1,p(\beta_i-1)+1)\Big]^{\frac{1}{p}},\\ \overline{\theta}_{ki} =&p[\beta_i+\gamma_{ki}-2]+1\geqslant 0,\ (k=1,2;\ i=1,2,3,4). \end{aligned} \tag{4.8}$$

$$A_4(x,y)=\overline{F}_1(x,y)\int_{x_0}^{x}\int_{y_0}^{y}H_1(s,t)2^{q-1}dsdt, \tag{4.9}$$

$$B_4(x,y)=\overline{F}_2(x,y)\int_{x_0}^{x}\int_{y_0}^{y}H_2(s,t)2^{q-1}dsdt, \tag{4.10}$$

$$\begin{aligned} D_7(M,N) =&\overline{F}_3(M,N)\int_{x_0}^{M}\int_{y_0}^{N}\frac{m\widetilde{E}_5(s,t)}{lE_5(x_0,y_0)}H_3(s,t)K^{\frac{m-l}{l}}dsdt\\ &+\overline{F}_4(M,N)\int_{x_0}^{M}\int_{y_0}^{N}\frac{r\widetilde{E}_5(s,t)}{lE_5(x_0,y_0)}H_4(s,t)K^{\frac{r-l}{l}}dsdt, \end{aligned} \tag{4.11}$$

$$\widetilde{a}_2(x,y)=\begin{cases}5^{q-1}\overline{a}^q(x,y), & (x,y)\in\Delta,\\ 5^{q-1}\overline{a}^q(x_0,y_0), & (x,y)\in\Psi,\end{cases} \qquad \widetilde{E}_5(x,y)=\begin{cases}E_5(x,y), & (x,y)\in\Delta,\\ E_5(x_0,y_0), & (x,y)\in\Psi.\end{cases} \tag{4.12}$$

$$G_5(u) = \overline{W}_2\left(\overline{W}_1\left(\frac{u}{D_7(M,N)}\right)\right) - \overline{W}_2\left(\overline{W}_1(1+u) + A_4(M,N)\right) \tag{4.13}$$

where $G_5(u)$ is increasing on R_+.

Proof. By (4.5), we obtain $\overline{a}(x,y) \in C(\Delta, R_+)$ is nondecreasing in each of the variables. From equation (4.1) and the condition (4.3), we get

$$\begin{aligned}
|u(x,y)|^l \leqslant& \overline{a}(x,y) \\
&+ f_1(x,y)\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\beta_1-1}s^{\gamma_{11}-1}(y-t)^{\beta_1-1}t^{\gamma_{21}-1}h_1(s,t)\omega_1(|u(s,t)|)dsdt \\
&+ f_2(x,y) \\
&\times \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\beta_2-1}s^{\gamma_{12}-1}(y-t)^{\beta_2-1}t^{\gamma_{22}-1}h_2(s,t)\omega_2\left(\max_{\xi\in[\beta s,s]}|u(\xi,t)|\right)dsdt \\
&+ f_3(x,y)\int_{x_0}^{M}\int_{y_0}^{N}(M-s)^{\beta_3-1}s^{\gamma_{13}-1}(N-t)^{\beta_3-1}t^{\gamma_{23}-1}h_3(s,t)|u(s,t)|^m dsdt \\
&+ f_4(x,y) \\
&\times \int_{x_0}^{M}\int_{y_0}^{N}(M-s)^{\beta_4-1}s^{\gamma_{14}-1}(N-t)^{\beta_4-1}t^{\gamma_{24}-1}h_4(s,t)\max_{\xi\in[\beta s,s]}|u(\xi,t)|^r dsdt, \\
&(x,y)\in\Delta.
\end{aligned} \tag{4.14}$$

From equation (4.2) and the condition (iv), we obtain

$$|u(x,y)| \leqslant |\phi(x,y)| \leqslant 5^{\frac{ql-1}{ql}}|a(x_0,y_0)|^{\frac{1}{l}}, \quad (x,y)\in\Psi. \tag{4.15}$$

Applying Theorem 3.5 to (4.14), (4.15) with $a(x,y) = \overline{a}(x,y)$, $b_i(x) = x$, $c_i(y) = y$, $\alpha_{ki} = 1$, $(k = 1,2;\ i = 1,2,3,4)$, we can obtain the estimation (4.4). □

Next, we give the conditions of the uniqueness of solutions for problem (4.1), (4.2). Suppose that the following conditions are satisfied:

$$\begin{aligned}
&|P_i(s,t,x,y,u) - P_i(s,t,x,y,v)| \leqslant f_i(x,y)h_i(s,t)\omega_i(|u-v|), \quad i = 1,2, \\
&|P_j(s,t,x,y,u) - P_j(s,t,x,y,v)| \leqslant f_j(x,y)h_j(s,t)|u-v|, \quad j = 3,4.
\end{aligned} \tag{4.16}$$

THEOREM 4.2. *Assume that the conditions (i-iii) of Theorem 4.1 are satisfied, and $l = 1$. Then the problem (4.1), (4.2) has at most one unique solution.*

Proof. Assume that there exist two different solutions $u(x,y)$ and $v(x,y)$ of the problem (4.1), (4.2) defined in Λ, then the functions $u(x,y)$ and $v(x,y)$ satisfy (4.1), (4.2) and

$$\begin{aligned}
v^l(x,y) =& a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\beta_1-1}s^{\gamma_{11}-1}(y-t)^{\beta_1-1}t^{\gamma_{21}-1}P_1(s,t,x,y,v(s,t))dsdt \\
&+ \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\beta_2-1}s^{\gamma_{12}-1}(y-t)^{\beta_2-1}t^{\gamma_{22}-1}P_2(s,t,x,y,\max_{\xi\in[\beta s,s]}v(\xi,t))dsdt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{x_0}^{M}\int_{y_0}^{N}(M-s)^{\beta_3-1}s^{\gamma_{13}-1}(N-t)^{\beta_3-1}t^{\gamma_{23}-1}P_3(s,t,x,y,v(s,t))dsdt\\
&+\int_{x_0}^{M}\int_{y_0}^{N}(M-s)^{\beta_4-1}s^{\gamma_{14}-1}(N-t)^{\beta_4-1}t^{\gamma_{24}-1}P_4(s,t,x,y,\max_{\xi\in[\beta s,s]}v(\xi,t))dsdt,\\
&(x,y)\in\Delta, \qquad (4.17)
\end{aligned}
$$

$$
v(x,y)=\phi(x,y), \quad (x,y)\in[\beta x_0,x_0]\times[y_0,N]\triangleq\Psi, \qquad (4.18)
$$

respectively. Then the norm of difference of the solutions $u(x,y)$ and $v(x,y)$ satisfies the inequalities

$$
\begin{aligned}
&|u(x,y)-v(x,y)|\\
\leqslant & f_1(x,y)\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\beta_1-1}s^{\gamma_{11}-1}(y-t)^{\beta_1-1}t^{\gamma_{21}-1}h_1(s,t)\omega_1(|u(s,t)-v(s,t)|)dsdt\\
&+f_2(x,y)\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\beta_2-1}s^{\gamma_{12}-1}(y-t)^{\beta_2-1}t^{\gamma_{22}-1}h_2(s,t)\\
&\quad\omega_2\left(|\max_{\xi\in[\beta s,s]}u(\xi,t)-\max_{\xi\in[\beta s,s]}v(\xi,t)|\right)dsdt\\
&+f_3(x,y)\int_{x_0}^{M}\int_{y_0}^{N}(M-s)^{\beta_3-1}s^{\gamma_{13}-1}(N-t)^{\beta_3-1}t^{\gamma_{23}-1}h_3(s,t)|u(s,t)-v(s,t)|dsdt\\
&+f_4(x,y)\int_{x_0}^{M}\int_{y_0}^{N}(M-s)^{\beta_4-1}s^{\gamma_{14}-1}(N-t)^{\beta_4-1}t^{\gamma_{24}-1}h_4(s,t)\\
&\quad|\max_{\xi\in[\beta s,s]}u(\xi,t)-\max_{\xi\in[\beta s,s]}v(\xi,t)|dsdt\\
\leqslant & f_1(x,y)\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\beta_1-1}s^{\gamma_{11}-1}(y-t)^{\beta_1-1}t^{\gamma_{21}-1}h_1(s,t)\omega_1(|u(s,t)-v(s,t)|)dsdt\\
&+f_2(x,y)\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\beta_2-1}s^{\gamma_{12}-1}(y-t)^{\beta_2-1}t^{\gamma_{22}-1}h_2(s,t)\\
&\quad\omega_2\left(\max_{\xi\in[\beta s,s]}|u(\xi,t)-v(\xi,t)|\right)dsdt\\
&+f_3(x,y)\int_{x_0}^{M}\int_{y_0}^{N}(M-s)^{\beta_3-1}s^{\gamma_{13}-1}(N-t)^{\beta_3-1}t^{\gamma_{23}-1}h_3(s,t)|u(s,t)-v(s,t)|dsdt\\
&+f_4(x,y)\int_{x_0}^{M}\int_{y_0}^{N}(M-s)^{\beta_4-1}s^{\gamma_{14}-1}(N-t)^{\beta_4-1}t^{\gamma_{24}-1}h_4(s,t)\\
&\quad\max_{\xi\in[\beta s,s]}|u(\xi,t)-v(\xi,t)|dsdt, \quad (x,y)\in\Delta, \qquad (4.19)
\end{aligned}
$$

$$
|u(x,y)-v(x,y)|\leqslant 0, \quad (x,y)\in\Psi. \qquad (4.20)
$$

Set $z(x,y)=|u(x,y)-v(x,y)|$ for $(x,y)\in\Lambda$. Applying Theorem 3.5 to (4.19), (4.20) with $a(x,y)=0$, $\phi(x,y)=0$, $b_i(x)=x$, $c_i(y)=y$, $\alpha_{ki}=1$ $(k=1,2;\ i=1,2,3,4)$. We obtain that $z(x,y)\leqslant 0$ for $(x,y)\in\Delta$, which implies the inequality $u(x,y)=v(x,y)$ for $(x,y)\in\Delta$. $\square$

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