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SOME NEW NONLINEAR POWERED GRONWALL–BELLMAN TYPE RETARDED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract. The purpose of the this paper is to establish some new retarded integral inequalities of Gronwall-Bellman type, which generalizes some known integral inequalities. The inequalities given here can be used in the analysis of the qualitative properties of certain classes of differential equations and integral equations. We apply our result to the boundedness of the solutions of integral equations.

1. Introduction

The integral inequalities provide explicit upper bound on unknown functions and play an important role in the study of qualitative properties of solutions of differential equations and integral equations, various generalizations of Gronwall-Bellman integral inequality and their applications have attracted great interests of many mathematicians (such as [1-11] and references therein). Gronwall [1] and Bellman [2] established the integral inequality

$$u(t) \leqslant c + \int_a^t f(s)u(s)ds, \quad t \in [a,b],$$

for some constant $c \geqslant 0$, obtained the estimation of unknown function

$$u(t) \leqslant c\exp\left(\int_a^t f(s)ds\right), \quad t \in [a,b].$$

Pachpatte [3] introduced the integral inequality

$$u(t) \leqslant a(t) + g(t)\int_0^t f(s)u(s)ds, \ t \in [0,\infty),$$

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where u, f, g are real-valued nonnegative continuous functions defined on $[0,\infty)$, $a(t)$ be a positive, monotonic nondecreasing continuous function defined on $[0,\infty)$, then the upper bound estimation of unknown function is that

$$u(t) \leqslant a(t)\big[1+g(t)\int_0^t f(s)\exp(\int_0^s g(\tau)f(\tau)d\tau)ds\big].$$

Pachpatte [4] studied the following integral inequality

$$u(t) \leqslant c+\int_0^t [f(s)u(s)+p(s)]ds+\int_0^t f(s)(\int_0^s g(\tau)u(\tau)d\tau)ds,\ t\in[0,\infty),$$

where $c>0$. Owaidy et al. [5] discussed the integral inequality

$$u(t) \leqslant u_0+\int_0^t f(s)\big[x^p(s)+\int_0^s g(\tau)u(\tau)d\tau\big]ds,\ t\in[0,\infty),$$

where $0\leqslant p<1$. Lipovan [6] studied the retarded integral inequality

$$u(t) \leqslant a+\int_{t_0}^t f(s)w(u(s))ds+\int_{\alpha(t_0)}^{\alpha(t)} g(s)w(u(s))ds,\ t_0\leqslant t<t_1,$$

where $u, f, g\in C([t_0,T),\mathbb{R}_+)$, and $w\in C(\mathbb{R}_+,\mathbb{R}_+)$ be nondecreasing with $w(u)>0$ for $u>0$. Agarwal et al. [7] investigated the inequality

$$u(t) \leqslant a(t)+\sum_{i=1}^n\int_{b_i(t_0)}^{b_i(t)} g_i(t,s)w_i(u(s))ds, \quad t_0\leqslant t<t_1,$$

where $w_i (i=1,\cdots,n)$ are continuous and nondecreasing functions on $[0,\infty)$ and are positive on $(0,\infty)$ such that $w_1 \propto w_2 \cdots \propto w_n$, $f_i(t,s)$ are continuous and nonnegative functions on $[t_0,t_1)\times[t_0,t_1)$. Lipovan [8] discussed the retarded integral inequality

$$u(t) \leqslant k(t)+a(t)\int_0^{\alpha(t)} b(s)w(u(s))ds,\ \ t\geqslant 0,$$

where $k, a, b\in C(\mathbb{R}_+,\mathbb{R}_+)$, $\alpha\in C^1(\mathbb{R}_+,\mathbb{R}_+)$, $w\in C(\mathbb{R}_+,\mathbb{R}_+)$ is a nondecreasing function with $w(t)>0$ for $t>0$. Agarwal et al. [9] discussed the retarded integral inequality

$$\varphi(u(t)) \leqslant c+\sum_{i=1}^n\int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)[f_i(s)\varphi(u(s))+g_i(s)]ds,$$

where c is a constant. Abdeldaim et al. [10] studied the inequality

$$u(t) \leqslant u_0+\int_0^t [f(s)u(s)+q(s)]ds+\int_0^t f(s)u(s)[u(s)+\int_0^s g(\lambda)u(\lambda)d\lambda]ds, t\in[0,\infty),$$

where u, f, q, g are nonnegative real valued continuous functions defined on $[0,\infty)$. Zhou et al. [11] studied the following retarded integral inequality

$$u(t) \leqslant a(t)+\sum_{i=1}^n\big\{\int_{b_i(t_0)}^{b_i(t)} g_i(t,s)w_i(u(s))ds\big\}^{p_i}, \quad t_0\leqslant t<\infty,$$

where $n \in \mathbb{N}, p_i \geqslant 1$, and all a, b_i, f_i, ϕ_i and u are nonnegative continuous functions for $i = 1, 2, \cdots, n$. Besides the results mentioned above, a lot of investigators have discovered many useful and new integral inequalities, mainly inspired by their applications in various branches of differential equations (see [12-30] and the references cited therein).

However, in some situations, such as some classes of delay differential equations with power and delay integral equations, it is desirable to find some new delay inequalities, in order to achieve a diversity of desired goals. In this paper, we discuss a class of retarded integral inequalities with power. We use some analysis techniques to get the explicit estimations of the unknown function in the inequality. Finally, we give one example to illustrate the application of our results.

2. Main results

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$ is the given subset of $\mathbb{R}$, and $C(M, S)$ denotes the class of all continuous functions defined on set M with range in the set S.

The following lemmas are very useful in the procedures of our proof in our main results.

LEMMA 1. *Let $a \geqslant 0, p \geqslant q \geqslant 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leqslant \frac{q}{p} a + \frac{p-q}{p}. \tag{2.1}$$

Proof. It is the case of $K = 1$ in [17]. $\square$

LEMMA 2. *Suppose that $a(t), f(t), g(t)$ are continuous positive functions on $[t_0, \infty)$, $\alpha(t)$ is differentiable, with $\alpha(t) \leqslant t, \alpha(t_0) = t_0$, $u(t)$ is a nonnegative continuous function on $[t_0, \infty)$, satisfying the integral inequality*

$$u(t) \leqslant a(t) + \int_{t_0}^{\alpha(t)} f(s)u(s)ds + \int_{t_0}^{\alpha(t)} g(s)u^2(s)ds, \tag{2.2}$$

then the following estimate hold

$$u(t) \leqslant \frac{1}{\exp(-a_1(t)) - \int_{t_0}^{\alpha(t)} g(s)ds}, \tag{2.3}$$

where $a_1(t) = \ln(a(t)) + \int_{t_0}^{\alpha(t)} f(s)ds$, $\exp(-a_1(t)) - \int_{t_0}^{\alpha(t)} g(s)ds > 0$.

Proof. Let

$$z(t) = a(t) + \int_{t_0}^{\alpha(t)} f(s)u(s)ds + \int_{t_0}^{\alpha(t)} g(s)u^2(s)ds, \tag{2.4}$$

then $z(t)$ is a nondecreasing function and $u(t) \leqslant z(t)$, $z(t_0) = a(t_0)$. Differentiating (2.4) with respect to t, we have

$$\begin{aligned} z'(t) &= a'(t) + \alpha'(t) f(\alpha(t)) u(\alpha(t)) + \alpha'(t) g(\alpha(t)) u^2(\alpha(t)) \\ &\leqslant a'(t) + \alpha'(t) f(\alpha(t)) z(\alpha(t)) + \alpha'(t) g(\alpha(t)) z^2(\alpha(t)), \end{aligned} \tag{2.5}$$

from (2.5), we have

$$\begin{aligned} \frac{z'(t)}{z(t)} &\leqslant \frac{a'(t)}{z(t)} + \frac{\alpha'(t) f(\alpha(t)) z(\alpha(t))}{z(t)} + \frac{\alpha'(t) g(\alpha(t)) z^2(\alpha(t))}{z(t)} \\ &\leqslant \frac{a'(t)}{a(t)} + \frac{\alpha'(t) f(\alpha(t)) z(\alpha(t))}{z(\alpha(t))} + \frac{\alpha'(t) g(\alpha(t)) z^2(\alpha(t))}{z(\alpha(t))} \\ &= \frac{a'(t)}{a(t)} + \alpha'(t) f(\alpha(t)) + \alpha'(t) g(\alpha(t)) z(\alpha(t)). \end{aligned} \tag{2.6}$$

Integrating (2.6) with respect to t from t_0 to t, we have

$$\begin{aligned} \ln(z(t)) &\leqslant \ln(a(t)) + \int_{t_0}^{\alpha(t)} f(s)ds + \int_{t_0}^{\alpha(t)} g(s)z(s)ds \\ &\leqslant a_1(t) + \int_{t_0}^{\alpha(t)} g(s)z(s)ds, \end{aligned}$$

where $a_1(t) = \ln(a(t)) + \int_{t_0}^{\alpha(t)} f(s)ds$. Let

$$\begin{aligned} \theta(t) &= \ln(z(t)), \\ z_1(t) &= a_1(t) + \int_{t_0}^{\alpha(t)} g(s)z(s)ds, \end{aligned} \tag{2.7}$$

then $z(t) = \ln(\theta(t)), \theta(t) \leqslant z_1(t), z(t) \leqslant \exp(z_1(t))$, $z_1(t)$ is a nondecreasing function. Differentiating (2.7) with respect to t, we obtain

$$\begin{aligned} z_1'(t) &\leqslant a_1'(t) + \alpha'(t) g(\alpha(t)) z(\alpha(t)) \\ &\leqslant a'(t) + \alpha'(t) g(\alpha(t)) \exp(z_1(\alpha(t))), \end{aligned} \tag{2.8}$$

from (2.8), we have

$$\begin{aligned} \frac{z_1'(t)}{\exp(z_1(t))} &\leqslant \frac{a_1'(t)}{\exp(z_1(t))} + \frac{\alpha'(t) g(\alpha(t)) z(\alpha(t))}{\exp(z_1(t))} \\ &\leqslant \frac{a_1'(t)}{\exp(a_1(t))} + \frac{\alpha'(t) g(\alpha(t)) \exp(z_1(\alpha(t)))}{\exp(z_1(\alpha(t)))} \\ &= \frac{a_1'(t)}{\exp(a_1(t))} + \alpha'(t) g(\alpha(t)). \end{aligned} \tag{2.9}$$

Integrating (2.9) with respect to t from t_0 to t, we have

$$z_1(t) \leqslant \ln\left(\frac{1}{\exp(-a_1(t)) - \int_{t_0}^{\alpha(t)} g(s)ds}\right), \tag{2.10}$$

by $u(t) \leqslant z(t), z(t) \leqslant \exp(z_1(t))$, from (2.10), we obtain the required estimation (2.3). □

THEOREM 2.1. *Let $a(t), f(t) \in C([t_0,\infty),\mathbb{R}_+)$, $u(t) \in C([t_0,\infty),\mathbb{R}_+)$ and let $\alpha(t)$ be a continuous, differentiable and increasing function on $[t_0,+\infty)$ with $\alpha(t) \leqslant t, \alpha(t_0) = t_0$. $p,m,n \in (0,1]$ are positive constants. If $u(t)$ satisfies the inequality*

$$u(t) \leqslant a(t) + \int_{t_0}^{\alpha(t)} f(s)\Big[u^m(s) + \int_{t_0}^{s} g(\tau)u^n(\tau)d\tau\Big]^p ds, \tag{2.11}$$

then, we have

$$u(t) \leqslant a(t) + A(t)\exp\Big(\int_{t_0}^{\alpha(t)} pmf(s)ds + \int_{t_0}^{\alpha(t)} pf(s)(\int_{t_0}^{s} ng(\tau)d\tau)ds\Big), \tag{2.12}$$

for $t \in \mathbb{R}_+$, where

$$\begin{aligned} A(t) = &\int_{t_0}^{\alpha(t)} f(s)\Big[(1-p)+p\Big(ma(s)+(1-m)\Big)\Big]ds \\ &+\int_{t_0}^{\alpha(t)} pf(s)\int_{t_0}^{s} g(\tau)\Big[na(\tau)+1-n\Big]d\tau ds. \end{aligned}$$

Proof. by Lemma 1, we obtain

$$\Big[u^m(s) + \int_{t_0}^{s} g(\tau)u^n(\tau)d\tau\Big]^p \leqslant p\Big[u^m(s) + \int_{t_0}^{s} g(\tau)u^n(\tau)d\tau\Big] + (1-p), \tag{2.13}$$

substituting (2.13) to (2.11), we have

$$u(t) \leqslant a(t) + \int_{t_0}^{\alpha(t)} f(s)\Big[p\Big(u^m(s) + \int_{t_0}^{s} g(\tau)u^n(\tau)d\tau\Big) + (1-p)\Big]ds. \tag{2.14}$$

Define a function $w(t)$ by

$$\begin{aligned} w(t) = &\int_{t_0}^{\alpha(t)} (1-p)f(s)ds + \int_{t_0}^{\alpha(t)} pf(s)u^m(s)ds \\ &+\int_{t_0}^{\alpha(t)} pf(s)\int_{t_0}^{s} g(\tau)u^n(\tau)d\tau ds. \end{aligned} \tag{2.15}$$

We can conclude that $w(t)$ is a nondecreasing function, from (2.14) and (2.15), we have

$$u(t) \leqslant a(t) + w(t). \tag{2.16}$$

By Lemma 1 and (2.16), we obtain

$$u^m(t) \leqslant \big(a(t)+w(t)\big)^m \leqslant m(a(t)+w(t)) + 1 - m, \tag{2.17}$$

$$u^n(t) \leqslant \big(a(t)+w(t)\big)^n \leqslant n(a(t)+w(t)) + 1 - n, \tag{2.18}$$

Substituting the inequality (2.17) and (2.18) into (2.15) we have

$$
\begin{aligned}
w(t) &\leqslant \int_{t_0}^{\alpha(t)}(1-p)f(s)ds+\int_{t_0}^{\alpha(t)} pf(s)\Big[m\big(a(s)+w(s)\big)+1-m\Big]ds \\
&\quad +\int_{t_0}^{\alpha(t)} pf(s)\int_{t_0}^{s} g(\tau)\Big[n(a(\tau)+w(\tau))+1-n\Big]d\tau ds \\
&\leqslant \int_{t_0}^{\alpha(t)} f(s)\Big[(1-p)+p\Big(ma(s)+(1-m)\Big)\Big]ds \\
&\quad +\int_{t_0}^{\alpha(t)} pf(s)\int_{t_0}^{s} g(\tau)\Big[na(\tau)+1-n\Big]d\tau ds \\
&\quad +\int_{t_0}^{\alpha(t)} pmf(s)w(s)ds+\int_{t_0}^{\alpha(t)} pf(s)\int_{t_0}^{s} ng(\tau)w(\tau)d\tau ds \\
&\leqslant A(t)+\int_{t_0}^{\alpha(t)} pmf(s)w(s)ds+\int_{t_0}^{\alpha(t)} pf(s)\int_{t_0}^{s} ng(\tau)w(\tau)d\tau ds, \\
&\leqslant A(T)+\int_{t_0}^{\alpha(t)} pmf(s)w(s)ds+\int_{t_0}^{\alpha(t)} pf(s)\int_{t_0}^{s} ng(\tau)w(\tau)d\tau ds, \quad (2.19)
\end{aligned}
$$

where $t\in[t_0,T], T\in\mathbb{R}_+$ and

$$
\begin{aligned}
A(t) &= \int_{t_0}^{\alpha(t)} f(s)\Big[(1-p)+p\Big(ma(s)+(1-m)\Big)\Big]ds \\
&\quad +\int_{t_0}^{\alpha(t)} pf(s)\int_{t_0}^{s} g(\tau)\Big[na(\tau)+1-n\Big]d\tau ds.
\end{aligned}
$$

Setting

$$
z(t) = A(T)+\int_{t_0}^{\alpha(t)} pmf(s)w(s)ds+\int_{t_0}^{\alpha(t)} pf(s)\int_{t_0}^{s} ng(\tau)w(\tau)d\tau ds, \quad (2.20)
$$

Then, $z(t)$ is a nondecreasing function, and $w(t)\leqslant z(t)$, $z(t_0)=A(T)$, from (2.20), we have

$$
\begin{aligned}
z'(t) &= \alpha'(t)pmf(\alpha(t))w(\alpha(t))+\alpha'(t)pf(\alpha(t))\int_{t_0}^{\alpha(t)} ng(s)w(s)ds \\
&\leqslant w(t)\big(\alpha'(t)pmf(\alpha(t))+\alpha'(t)pf(\alpha(t))\int_{t_0}^{\alpha(t)} ng(s)ds\big) \\
&\leqslant z(t)\big(\alpha'(t)pmf(\alpha(t))+\alpha'(t)pf(\alpha(t))\int_{t_0}^{\alpha(t)} ng(s)ds\big), \quad (2.21)
\end{aligned}
$$

from (2.21), we have

$$
\begin{aligned}
\frac{z'(t)}{z(t)} &= \alpha'(t)pmf(\alpha(t))w(\alpha(t))+\alpha'(t)pf(\alpha(t))\int_{t_0}^{\alpha(t)} ng(s)w(s)ds \\
&\leqslant \alpha'(t)pmf(\alpha(t))+\alpha'(t)pf(\alpha(t))\int_{t_0}^{\alpha(t)} ng(s)ds, \quad (2.22)
\end{aligned}
$$

Integrating the inequality (2.22) from t_0 to t, we obtain the estimation

$$z(t) \leqslant A(T)\exp\Big(\int_{t_0}^{\alpha(t)} pmf(s)ds + \int_{t_0}^{\alpha(t)} pf(s)(\int_{t_0}^{s} ng(\tau)d\tau)ds\Big). \tag{2.23}$$

From $w(t) \leqslant z(t)$ and (2.16), (2.23), we have

$$u(t) \leqslant a(t) + A(T)\exp\Big(\int_{t_0}^{\alpha(t)} pmf(s)ds + \int_{t_0}^{\alpha(t)} pf(s)(\int_{t_0}^{s} ng(\tau)d\tau)ds\Big),$$

then, we have

$$u(T) \leqslant a(T) + A(T)\exp\Big(\int_{t_0}^{\alpha(T)} pmf(s)ds + \int_{t_0}^{\alpha(T)} pf(s)(\int_{t_0}^{s} ng(\tau)d\tau)ds\Big),$$

by the arbitrariness of T, we obtain

$$u(t) \leqslant a(t) + A(t)\exp\Big(\int_{t_0}^{\alpha(t)} pmf(s)ds + \int_{t_0}^{\alpha(t)} pf(s)(\int_{t_0}^{s} ng(\tau)d\tau)ds\Big).$$

This completes the proof. □

REMARK 1. If $\alpha(t) = t, a(t) = c, p = n = 1$, Theorem 2.1 reduces to the Theorem 3 in [5], if $\alpha(t) = t, a(t) = c, p = 1$, Theorem 2.1 reduces to Theorem 4 in [5], if $\alpha(t) = t, m = p = 1$, Theorem 2.1 reduces to Theorem 5 in [5].

REMARK 2. If $a(t) = u_0 + \int_0^{\alpha(t)} f(s)p(s)ds$, $p = m = n = 1$, Theorem 2.1 reduces to Theorem 2.1 in [12].

THEOREM 2.2. *Let $a(t), f(t), h(t) \in C([t_0,\infty),\mathbb{R}_+)$, $u(t)$ is a nonnegative continuous function on $[t_0,\infty)$, $\alpha(t)$ is a continuous, differentiable and increasing function on $[t_0,+\infty)$ with $\alpha(t) \leqslant t, \alpha(t_0) = t_0$. Let $p \in (0,1]$ is positive constant. If $u(t)$ satisfies the inequality*

$$u(t) \leqslant a(t) + \int_{t_0}^{\alpha(t)} f(s)u(s)\Big[h(s) + u(s) + \int_{t_0}^{s} g(\tau)u(\tau)d\tau\Big]^p ds, \tag{2.24}$$

we have

$$u(t) \leqslant \frac{1}{\exp(-A(t)) - \int_{t_0}^{\alpha(t)} G(s)ds}, \tag{2.25}$$

where $A(t) = \ln(a_1(t)) + \int_{t_0}^{\alpha(t)} F(s)ds$, $\exp(-A(t)) - \int_{t_0}^{\alpha(t)} G(s)ds > 0$, *and*

$$\begin{aligned} a_1(t) &= \int_{t_0}^{\alpha(t)} f(s)a(s)\big[p(h(s)+a(s)) + \int_{t_0}^{s} g(\tau)a(\tau)d\tau + (1-p)\big]ds, \\ F(t) &= pf(t)a(t) + pf(t)a(t)(\int_{t_0}^{t} g(s)ds) + f(t)\big[p(h(t)+a(t)) \\ &\quad + \int_{t_0}^{t} g(s)a(s)ds + (1-p)\big], \\ G(t) &= pf(t) + pf(t)(\int_{t_0}^{t} g(s)ds). \end{aligned} \tag{2.26}$$

Proof. By Lemma 1, from (2.24), we have

$$[h(t)+u(t)+\int_{t_0}^{t} g(s)u(s)ds]^p \leqslant p\big(h(t)+u(t)+\int_{t_0}^{t} g(s)u(s)ds\big)+(1-p), \tag{2.27}$$

from (2.27), (2.24) can be rewritten as

$$u(t) \leqslant a(t)+\int_{t_0}^{\alpha(t)} f(s)u(s)\Big[p\big(h(s)+u(s)+\int_{t_0}^{s} g(\tau)u(\tau)d\tau\big) +(1-p)\Big]ds. \tag{2.28}$$

Denoting by

$$v(t)=\int_{t_0}^{\alpha(t)} f(s)u(s)\Big[p\big(h(s)+u(s)+\int_{t_0}^{s} g(\tau)u(\tau)d\tau\big)+(1-p)\Big], \tag{2.29}$$

then $v(t)$ is a nonnegative and nondecreasing continuous function, and

$$u(t) \leqslant a(t)+v(t),\ v(t_0)=0. \tag{2.30}$$

From (2.29) and (2.30), we have

$$\begin{aligned} v(t) \leqslant& \int_{t_0}^{\alpha(t)} f(s)(a(s)+v(s))\Big[p\big(h(s)+(a(s)+v(s))+\int_{t_0}^{s} g(\tau)(a(\tau)+v(\tau))d\tau\big) \\ &+(1-p)\Big] \\ \leqslant& \int_{t_0}^{\alpha(t)} f(s)a(s)\big[p(h(s)+a(s))+\int_{t_0}^{s} g(\tau)a(\tau)d\tau+(1-p)\big]ds \\ &+\int_{t_0}^{\alpha(t)} pf(s)a(s)v(s)+\int_{t_0}^{\alpha(t)} pf(s)a(s)(\int_{t_0}^{s} g(\tau)v(\tau)d\tau)ds \\ &+\int_{t_0}^{\alpha(t)} f(s)v(s)\big[p(h(s)+a(s))+\int_{t_0}^{s} g(\tau)a(\tau)d\tau+(1-p)\big]ds \\ &+\int_{t_0}^{\alpha(t)} pf(s)v^2(s)+\int_{t_0}^{\alpha(t)} pf(s)v(s)(\int_{t_0}^{s} g(\tau)v(\tau)d\tau)ds \\ \leqslant& \int_{t_0}^{\alpha(t)} f(s)a(s)\big[p(h(s)+a(s))+\int_{t_0}^{s} g(\tau)a(\tau)d\tau+(1-p)\big]ds \\ &+\int_{t_0}^{\alpha(t)} pf(s)a(s)v(s)+\int_{t_0}^{\alpha(t)} pf(s)a(s)(\int_{t_0}^{s} g(\tau)d\tau)v(s)ds \\ &+\int_{t_0}^{\alpha(t)} f(s)\big[p(h(s)+a(s))+\int_{t_0}^{s} g(\tau)a(\tau)d\tau+(1-p)\big]v(s)ds \\ &+\int_{t_0}^{\alpha(t)} pf(s)v^2(s)+\int_{t_0}^{\alpha(t)} pf(s)(\int_{t_0}^{s} g(\tau)d\tau)v^2(s)ds, \end{aligned} \tag{2.31}$$

then, (2.31) can be written as

$$v(t) \leqslant a_1(t)+\int_{t_0}^{\alpha(t)} F(s)v(s)ds+\int_{t_0}^{\alpha(t)} G(s)v^2(s)ds, \tag{2.32}$$

where $a_1(t), F(t), G(t)$ are defined in (2.26). Since (2.32) has a similar form of (2.2), and the functions of (2.2) satisfy the conditions of Lemma 2. Consequently, by using a similar procedure of Lemma 2, we can get the desired estimation (2.25). The proof is completed. □

3. Applications

In this section, we apply our result to study the boundedness of the solution of integral equations. We consider the following Volterra type retarded integral equation

$$x(t) - \int_{t_0}^{\alpha(t)} f(s)\left[x(s) + \int_{t_0}^{s} g(\tau)x(\tau)d\tau\right]^{\frac{1}{2}} ds = h(t), \tag{3.1}$$

which arises very often in various problems, especial describing physical processes with aftereffects.

COROLLARY 1. *Let $x(t), f(t), g(t)$ and $h(t)$ be continuous functions on $[0,+\infty)$, and let $\alpha(t)$ be continuous, differentiable and increasing functions on $[0,+\infty)$ with $\alpha(t) \leqslant t, \alpha(t_0) = t_0$. If $x(t)$ satisfies the equation (3.1), we have*

$$|x(t)| \leqslant u(t) \leqslant |h(t)| + B(t)\exp\Big(\frac{1}{2}\int_{t_0}^{\alpha(t)} |f(s)|ds + \frac{1}{2}\int_{t_0}^{\alpha(t)} |f(s)| \int_{t_0}^{s} |g(\tau)|d\tau ds\Big),$$

where

$$B(t) = \int_{t_0}^{\alpha(t)} \big(\frac{1}{2}|f(s)| + \frac{1}{2}|f(s)||h(s)|\big)ds + \frac{1}{2}\int_{t_0}^{\alpha(t)} |f(s)|\big(\int_{t_0}^{s} \big(|g(\tau)||h(\tau)|\big)d\tau\big)ds.$$

Proof. From (3.1), we obtain

$$|x(t)| \leqslant |h(t)| + \int_{t_0}^{\alpha(t)} |f(s)|\left[|x(s)| + \int_{t_0}^{s} |g(\tau)||x(\tau)|d\tau\right]^{\frac{1}{2}} ds. \tag{3.2}$$

Let $|x(t)| = u(t)$, then, (3.2) can be written as

$$u(t) \leqslant |h(t)| + \int_{t_0}^{\alpha(t)} |f(s)|\left[u(s) + \int_{t_0}^{s} |g(\tau)|u(\tau)d\tau\right]^{\frac{1}{2}} ds. \tag{3.3}$$

By Lemma 1, from (3.3), we obtain

$$\left[u(s) + \int_{t_0}^{s} |g(\tau)|u(\tau)d\tau\right]^{\frac{1}{2}} \leqslant \frac{1}{2}\left[u(s) + \int_{t_0}^{s} |g(\tau)|u(\tau)d\tau\right] + \frac{1}{2}, \tag{3.4}$$

from (3.2) and (3.4), we have

$$u(t) \leqslant |h(t)| + \int_{t_0}^{\alpha(t)} |f(s)|\left[\frac{1}{2}\Big(u(s) + \int_{t_0}^{s} |g(\tau)|u(\tau)d\tau\Big) + \frac{1}{2}\right] ds. \tag{3.5}$$

Setting

$$z(t)=\int_{t_0}^{\alpha(t)}\frac{1}{2}|f(s)|ds+\int_{t_0}^{\alpha(t)}\frac{1}{2}|f(s)|u(s)ds$$
$$+\int_{t_0}^{\alpha(t)}\frac{1}{2}|f(s)|\int_{t_0}^{s}|g(\tau)|u(\tau)d\tau ds. \tag{3.6}$$

Then, $z(t)$ is a nondecreasing function, from (3.5) and (3.6), we have

$$u(t)\leqslant|h(t)|+z(t). \tag{3.7}$$

Substituting the inequality (3.7) into (3.6), we have

$$\begin{aligned}
z(t)\leqslant&\int_{t_0}^{\alpha(t)}\frac{1}{2}|f(s)|ds+\int_{t_0}^{\alpha(t)}\frac{1}{2}|f(s)|\big(|h(s)|+z(s)\big)ds\\
&+\int_{t_0}^{\alpha(t)}\frac{1}{2}|f(s)|(\int_{t_0}^{s}|g(\tau)|\big(|h(\tau)|+z(\tau)\big)d\tau)ds\\
\leqslant&\int_{t_0}^{\alpha(t)}\big(\frac{1}{2}|f(s)|+\frac{1}{2}|f(s)||h(s)|\big)ds+\int_{t_0}^{\alpha(t)}\frac{1}{2}|f(s)|\int_{t_0}^{s}\big(|g(\tau)||h(\tau)|\big)d\tau ds\\
&+\int_{t_0}^{\alpha(t)}\frac{1}{2}|f(s)|z(s)ds+\int_{t_0}^{\alpha(t)}\frac{1}{2}|f(s)|\int_{t_0}^{s}|g(\tau)|z(\tau)d\tau ds\\
\leqslant&B(t)+\frac{1}{2}\int_{t_0}^{\alpha(t)}|f(s)|z(s)ds+\frac{1}{2}\int_{t_0}^{\alpha(t)}|f(s)|\int_{t_0}^{s}|g(\tau)|z(\tau)d\tau ds,\\
\leqslant&B(T)+\frac{1}{2}\int_{t_0}^{\alpha(t)}|f(s)|z(s)ds+\frac{1}{2}\int_{t_0}^{\alpha(t)}|f(s)|\int_{t_0}^{s}|g(\tau)|z(\tau)d\tau ds,
\end{aligned} \tag{3.8}$$

where $t\in[t_0,T], T\in\mathbb{R}_+$ and

$$B(t)=\int_{t_0}^{\alpha(t)}\big(\frac{1}{2}|f(s)|+\frac{1}{2}|f(s)||h(s)|\big)ds+\frac{1}{2}\int_{t_0}^{\alpha(t)}|f(s)|(\int_{t_0}^{s}\big(|g(\tau)||h(\tau)|\big)d\tau)ds.$$

(3.8) has the same form of (2.19), then, by the result of Theorem 2.1, we can obtain

$$u(t)\leqslant|h(t)|+B(t)\exp\Big(\frac{1}{2}\int_{t_0}^{\alpha(t)}|f(s)|ds+\frac{1}{2}\int_{t_0}^{\alpha(t)}|f(s)|\int_{t_0}^{s}|g(\tau)|d\tau ds\Big).$$

This completes the proof. □

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