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# EQUALITY CONDITIONS OF REVERSE SCHWARZ AND GRÜSS INEQUALITIES IN INNER PRODUCTSPACES

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*Abstract.* The cases of equality for reverse Schwarz inequalities and Grüss type inequalities are detailed studied. Necessary and sufficient conditions for them are given. Moreover, we introduce an unification of two reverse Schwarz inequalities obtained by S.S. Dragomir [S.S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Chap. I, th. 1, 6. Nova Science Publishers, New York 2005] and, as an application, we gain more general versions of Grüss inequalities.

## 1. Introduction and motivation

Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{F}$ ($\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$), where $\|v\|^2=\langle v,v\rangle$, $v\in V$. The inequality

$$|\langle x,y\rangle| \leqslant \|x\|\|y\|,\ x,y\in V, \tag{1}$$

is known in the literature as Schwarz's (or Cauchy - Schwarz or Cauchy - Bunyakovsky - Schwarz) inequality. The equality holds in (1) if and only if the vectors $x$ and $y$ are linearly dependent. Reverses of the Schwarz inequality usually establish upper bounds for one of the following nonnegative quantities

$$\|x\|\|y\|-|\langle x,y\rangle|,\ \|x\|^2\|y\|^2-|\langle x,y\rangle|^2,\ \frac{\|x\|\|y\|}{|\langle x,y\rangle|},\ \frac{\|x\|^2\|y\|^2}{|\langle x,y\rangle|^2}.$$

We recall few well known examples of such inequalities.

Let $x_i,y_i,w_i$ $i=1,\dots,n$ be positive numbers.

If $0<a\leqslant x_i\leqslant A<\infty$ and $0<b\leqslant y_i\leqslant B<\infty$ for some constants $a,A,b,B$, then the following inequalities are valid:

Pólya - Szegö [21] (probably the oldest reverse of Schwarz's inequality)

$$\sum_{i=1}^n x_i^2\sum_{i=1}^n y_i^2 \leqslant \frac{(ab+AB)^2}{4abAB}\left(\sum_{i=1}^n x_iy_i\right)^2,$$

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Shisha - Mond [22]

$$\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i y_i} - \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n y_i^2} \leqslant \left[\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}}\right]^2,$$

Ozeki [20]

$$\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 - \left(\sum_{i=1}^n x_i y_i\right)^2 \leqslant \frac{n^2}{4}(AB-ab)^2.$$

If $0 < a \leqslant \frac{x_i}{y_i} \leqslant A < \infty$, then we have next examples:
Cassel inequality [23]

$$\frac{(\sum_{i=1}^n w_i x_i^2)(\sum_{i=1}^n w_i y_i^2)}{\sum_{i=1}^n w_i x_i y_i} \leqslant \frac{(A+a)^2}{4aA},$$

Klamkin - McLenaghan inequality [16]

$$\frac{\sum_{i=1}^n w_i x_i^2}{\sum_{i=1}^n w_i x_i y_i} - \frac{\sum_{i=1}^n w_i x_i y_i}{\sum_{i=1}^n w_i y_i^2} \leqslant (\sqrt{A} - \sqrt{a})^2,$$

Further classical examples include the weighted version of Pólya-Szegö's inequality known as Greub - Reinboldt's inequality [13] and Diaz - Metcalf's inequalities [3].

In the present, there is a lot of reverses of the Schwarz inequality under various conditions in the literature. For discrete variants of these inequalities see [4]. Other results and supplements are accessible in monographs [17, chap.V], [6]. Counterparts for integrals, isotone functionals and other extensions in the context of inner product spaces are considered in [1],[2],[6],[7],[8], [9] and references therein. Moreover, there exist many generalizations of mentioned reverse Schwarz inequalities in more abstract structures, see [10], [11], [12], [14], [15], [18], [19] and references therein.

Recently, Dragomir (see e.g. [6, th. 1]) obtained the following result

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \leqslant \frac{|\Gamma-\gamma|^2}{4}\|y\|^4, \ x,y \in V, \tag{2}$$

if scalars $\Gamma, \gamma$ satisfy

$$\mathrm{Re}\,\langle \Gamma y - x, x - \gamma y\rangle \geqslant 0 \tag{3}$$

or, equivalently,

$$\left\| x - \frac{\Gamma+\gamma}{2} y \right\| \leqslant \frac{1}{2}|\Gamma-\gamma|\|y\|. \tag{4}$$

On the assumptions (3) or (4), the same author proved also that (see e.g. [6, th. 6])

$$\|x\|\|y\| \leqslant \frac{|\Gamma+\gamma|}{2\sqrt{\mathrm{Re}(\Gamma\overline{\gamma})}}|\langle x,y\rangle|, \ x,y \in V, \tag{5}$$

if, in addition, scalars $\Gamma,\gamma$ satisfy $\mathrm{Re}(\Gamma\overline{\gamma}) > 0$. The above result generalizes directly Pólya - Szegö's and Cassel's inequalities (see e.g. [9]). An equivalent reformulation of (5) is as follows

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \leqslant \left|\frac{\Gamma-\gamma}{\Gamma+\gamma}\right|^2 \|x\|^2\|y\|^2,\ x,y\in V. \tag{6}$$

The inequalities (2) and (6) have a plain geometric interpretation. According to (4), considering $r=\frac{1}{2}|\Gamma-\gamma|\|y\|$ as a circle radius the right hand side of (2) can be expressed as $r^2\|y\|^2$. For inequality (6), the quantity $\left|\frac{\Gamma-\gamma}{\Gamma+\gamma}\right|$ is the sine of an angle in a right triangle with the length of the hypotenuse equals $\frac{1}{2}|\Gamma+\gamma|\|y\|$ and the opposite side equal to $r$.

In the paper we obtain an inequality which simultaneously generalizes inequalities (2) and (6). The constants $1/4$ and $1/2$ apparent on the left in (2) and (5) are sharp, i.e. there exist nonzero vectors $x$ and $y$ that realize equalities in these inequalities (see [6, theorems 1, 6] and their proofs). However, general conditions for equality in (2) and (6) seem to be omitted. The paper fills this gap.

A brief outline of the paper is as follows. Theorem 1 presents our new reverse Schwarz inequality. In theorem 2 necessary and sufficient conditions for equality in this inequality are derived. As a consequence we complement Dragomir's results by adding conditions of equality for inequalities (2) and (6) (see corollary 1, 2). Applications to Grüss' inequality are included in theorem 3 and corollaries 3, 4. Conditions of equality for these new variants of Grüss' inequality are given as well (see proposition 1, 2). Generalizations and complements of known results are included in remark 3 and corollaries 5, 6.

## 2. Reverses of Schwarz's inequality and equality conditions

THEOREM 1. *Let $c_1,c_1',c_2,c_2'\in\mathbb{F}$ and $0\neq v\in V$.*
*If*

$$\|x-c_1v\|\leqslant|c_1'|\|v\| \text{ and } \|y-c_2v\|\leqslant|c_2'|\|v\|, \tag{7}$$

*then for $x,y\in V$ the following inequalities hold*

$$\begin{aligned} 0 &\leqslant \|x\|^2\|y\|^2-|\langle x,y\rangle|^2 \\ &\leqslant \begin{cases} \left(|c_1'|+\frac{|c_1|}{|c_2|}\min\{|c_2|,|c_2'|\}\right)^2\|y\|^2\|v\|^2, & c_2\neq 0 \quad (\mathrm{v1}) \\ \min\{|c_1'|^2\|y\|^2,|c_2'|^2\|x\|^2\}\,\|v\|^2, & c_1,c_2=0. \quad (\mathrm{v2}) \end{cases} \end{aligned} \tag{8}$$

*Proof.* If $x=0$ or $y=0$, the inequalities hold. Let $y\neq 0$. For any $c\in\mathbb{F}$ we have the estimate

$$\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|\leqslant\|x-cy\|=\|x-c_1v+c_1v-cy\|\leqslant\|x-c_1v\|+\|c_1v-cy\|.$$

Letting $c = \frac{c_1 \langle v,y\rangle}{\|y\|^2}$ and $c_2 \neq 0$, we obtain

$$\left\|x - \tfrac{\langle x,y\rangle}{\|y\|^2} y\right\| \leqslant \|x - c_1 v\| + \left\|c_1 v - \tfrac{c_1 \langle v,y\rangle}{\|y\|^2} y\right\| = \|x - c_1 v\| + \tfrac{|c_1|}{|c_2|} \left\|c_2 v - \tfrac{c_2 \langle v,y\rangle}{\|y\|^2} y\right\|.$$

Obviously, $\left\|c_2 v - \frac{c_2\langle v,y\rangle}{\|y\|^2} y\right\| \leqslant \|c_2 v - \tilde{c} y\|$ for any $\tilde{c} \in \mathbb{F}$. Next, substituting consecutively $\tilde{c} := 0$ and $\tilde{c} := 1$ gives

$$\left\|c_2 v - \frac{c_2 \langle v,y\rangle}{\|y\|^2} y\right\| \leqslant \min\{|c_2|\|v\|, \|c_2 v - y\|\}.$$

Hence

$$\begin{aligned}\left\|x - \tfrac{\langle x,y\rangle}{\|y\|^2} y\right\| &\leqslant \|x - c_1 v\| + \tfrac{|c_1|}{|c_2|} \min\{|c_2|\|v\|, \|c_2 v - y\|\} \\ &\leqslant \left(|c_1'| + \tfrac{|c_1|}{|c_2|} \min\{|c_2|, |c_2'|\}\right) \|v\|,\end{aligned}$$

by the hypothesis (7).

Finally,

$$\|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} = \left\|x - \frac{\langle x,y\rangle}{\|y\|^2} y\right\|^2 \leqslant \left(|c_1'| + \frac{|c_1|}{|c_2|} \min\{|c_2|, |c_2'|\}\right)^2 \|v\|^2.$$

Multiplying the both of sides by $\|y\|^2 > 0$ we get the first variant of the inequality.

If $c_1 = 0 = c_2$, then the hypothesis (7) takes the form $\|x\| \leqslant |c_1'|\|v\|$ and $\|y\| \leqslant |c_2'|\|v\|$. Hence, we have as follows

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \leqslant \|x\|^2\|y\|^2 \leqslant \begin{cases} |c_1'|^2\|y\|^2\|v\|^2 \\ |c_2'|^2\|x\|^2\|v\|^2. \end{cases}$$

This is nothing but the second variant of the inequality. □

The particular cases of the inequality presented in theorem 1 are well known.

REMARK 1. Setting in theorem 1 (v1):

A. $v = y$ and consequently $c_2 = 1$ and $c_2' = 0$ for arbitrary scalars $c_1, c_1' \in \mathbb{F}$ we obtain

$$\|x - c_1 y\| \leqslant |c_1'|\|y\| \implies 0 \leqslant \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \leqslant |c_1'|^2\|y\|^4,\ x,y \in V.$$

It is exactly Dragomir's result: (4) or, equivalently, (3) implies (2) for $c_1 = \frac{\Gamma+\gamma}{2}$ and $c_1' = \frac{\Gamma-\gamma}{2}$, where $\Gamma, \gamma \in \mathbb{F}$;

B. $v = x$ and consequently $c_1 = 1$ and $c_1' = 0$ for arbitrary scalars $c_2, c_2' \in \mathbb{F}$ such that $|c_2'| < |c_2|$ we get

$$\|y - c_2 x\| \leqslant |c_2'|\|x\| \implies 0 \leqslant \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \leqslant \frac{|c_2'|^2}{|c_2|^2}\|x\|^2\|y\|^2,\ x,y \in V.$$

This is inequality (6) under assumptions (4) or, equivalently, (3) for $c_2 = \frac{\Gamma+\gamma}{2}$ and $c_2' = \frac{\Gamma-\gamma}{2}$, where $\Gamma, \gamma \in \mathbb{F}$. Let us note, $|c_2'| < |c_2|$ is equivalent to $\mathrm{Re}(\Gamma\overline{\gamma}) > 0$. □

The example below illustrates how inequality (8) works.

EXAMPLE 1. Consider $l^2$, the space of all complex sequences $z=(z_1,z_2,\dots)$ such that $\sum|z_i|^2<\infty$, with the inner product $\langle x,y\rangle=\sum x_i\overline{y_i}$ and norm $\|x\|=\sqrt{\langle x,x\rangle}$, $x,y\in l^2$.

Fix $v=(v_1,v_2,\dots)\in l^2$ with nonzero entries, $c_1,c_2\in\mathbb{C}$ and choose $\rho_i\geqslant 0$, $i=1,2$. Let $r_i=\rho_i\|v\|$, $l_i=|c_i|\|v\|$, $i=1,2$.

Given $x,y\in l^2$, let

$$\left|\frac{x_i}{v_i}-c_1\right|\leqslant\rho_1,\ \text{ and }\ \left|\frac{y_i}{v_i}-c_2\right|\leqslant\rho_2,\ i=1,2,\dots.$$

Multiplying the above inequalities by $|v_i|>0$, $i=1,2$, respectively, taking the square, summing over $i$ and extracting the square root of both of sides gives

$$\|x-c_1v\|\leqslant r_1\ \text{ and }\ \|y-c_2v\|\leqslant r_2.$$

Applying theorem 1 we obtain

$$\begin{aligned}0&\leqslant\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\\&\leqslant\begin{cases}\left(r_1+\frac{|c_1|}{|c_2|}\min\{r_2,l_2\}\right)^2\|y\|^2\ ,\ c_2\neq 0\\[2ex]\qquad\min\{r_1^2\|y\|^2,r_2^2\|x\|^2\}\ ,\ c_1,c_2=0.\end{cases}\end{aligned}\tag{9}$$

Setting $v=y$ and consequently $c_2=1$ and $r_2=0$ we get (c.f. (2))

$$\|x-c_1y\|\leqslant r_1\implies 0\leqslant\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\leqslant r_1^2\|y\|^2.$$

If $r_2<l_2$, i.e. $\rho_2<|c_2|$, then inequality (9) can be rewritten as

$$0\leqslant\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\leqslant(r_1+l_1\sin\alpha)^2\|y\|^2,$$

where, for the interpretation, $0\leqslant\rho_2/|c_2|<1$ is assumed to be equal $sin\alpha$, $0\leqslant\alpha<\pi/2$.

Now, substituting $v=x$ and $c_1=1$, $l_1=\|x\|$ and $r_1=0$ we obtain (c.f. (6))

$$\|y-c_2x\|\leqslant r_2\implies 0\leqslant\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\leqslant\sin^2\alpha\|x|^2\|y\|^2.$$

We omit the details. □

The upper bounds of the quantity $\|x\|^2\|y\|^2-|\langle x,y\rangle|^2$, presented in theorem 1, are attainable. More precisely, there exist nonzero vectors $x$ and $y$ fulfilling (7) such that inequalities (8) become equalities. We will show even more.

THEOREM 2. *Let $c_1,c_1',c_2,c_2'\in\mathbb{F}$ and $x,y,v\in V$ with $v\neq 0$ satisfy (7).*

A. *If* $\begin{matrix} c_1 \neq 0, c_1' \in \mathbb{F} \\ c_2' \neq 0, c_2 \neq 0 \end{matrix}$ *and* $|c_2'| < |c_2|$, *then*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = \frac{(|c_1c_2'|+|c_1'c_2|)^2}{|c_2|^2}\|y\|^2\|v\|^2$$

$$\Updownarrow$$

$$\exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,y\rangle = 0, \\ y = c_2v - c_2'\|v\|u,\ x = c_1v + \frac{c_1c_2'|c_2|}{|c_1||c_2'|c_2}\cdot|c_1'|\|v\|u. \end{cases}$$

B. *If* $\begin{matrix} c_1 \neq 0, c_1' \in \mathbb{F} \\ c_2' \neq 0, c_2 \neq 0 \end{matrix}$ *and* $|c_2'| \geqslant |c_2|$, *then*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = (|c_1|+|c_1'|)^2\|y\|^2\|v\|^2$$

$$\Updownarrow$$

$$\langle y,v\rangle = 0,\ x = c_1\left(1+\tfrac{|c_1'|}{|c_1|}\right)v.$$

C. *If* $c_1c_2' = 0$ *and* $c_1'c_2 \neq 0$, *then*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = |c_1'|^2\|y\|^2\|v\|^2$$

$$\Updownarrow$$

$$\exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,y\rangle = 0, \\ y \in V,\ x = c_1v + c_1'\|v\|u. \end{cases}$$

D. *If* $c_1 = 0$ *and* $c_2 = 0$, *then*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = |c_1'|^2\|y\|^2\|v\|^2 \iff \exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,y\rangle = 0, \\ y \in V,\ x = c_1'\|v\|u, \end{cases}$$

*and*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = |c_2'|^2\|x\|^2\|v\|^2 \iff \exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,x\rangle = 0, \\ x \in V,\ y = c_2'\|v\|u. \end{cases}$$

*Proof.* A. Sufficiency. Since $y = c_2v - c_2'\|v\|u$ and $\langle u,y\rangle = 0$,

$$c_1v = \frac{c_1}{c_2}(y + c_2'\|v\|u),$$

$$x = \frac{c_1}{c_2}(y + c_2'\|v\|u) + \frac{c_1c_2'|c_2|}{|c_1||c_2'|c_2}\cdot|c_1'|\|v\|u = \frac{c_1}{c_2}y + \frac{c_1}{c_2}\left(1 + \frac{|c_1'||c_2|}{|c_1||c_2'|}\right)c_2'\|v\|u.$$

Consequently,

$$\|y\|^2 = (|c_2|^2 - |c_2'|^2)\|v\|^2 > 0, \|x\|^2 = \frac{|c_1|^2}{|c_2|^2}\|y\|^2 + \frac{(|c_1||c_2'|+|c_1'||c_2|)^2}{|c_2|^2}\|v\|^2,$$

$$|\langle x,y\rangle|^2 = \frac{|c_1|^2}{|c_2|^2}\|y\|^4.$$

Finally, usage of gathered above facts gives

$$\|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} = \frac{(|c_1||c_2'| + |c_1'||c_2|)^2}{|c_2|^2}\|v\|^2,$$

as required.

A. Necessity. Condition (7) and $|c_2'| < |c_2|$ ensure that $y \neq 0$. In this way

$$\|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} = \frac{(|c_1c_2'| + |c_1'c_2|)^2}{|c_2|^2}\|v\|^2,$$

or, equivalently,

$$\left\|x - \frac{\langle x,y\rangle}{\|y\|^2}y\right\| = \frac{|c_1c_2'| + |c_1'c_2|}{|c_2|}\|v\|.$$

Therefore, the all following obvious inequalities are, in fact, equalities:

$$\begin{aligned}\left\|x - \tfrac{\langle x,y\rangle}{\|y\|^2}y\right\| &\leqslant \|x - \tfrac{c_1}{c_2}y\| = \|x - c_1v + c_1v - \tfrac{c_1}{c_2}y\| \leqslant \|x - c_1v\| + \tfrac{|c_1|}{|c_2|}\|c_2v - y\| \\ &\leqslant |c_1'|\|v\| + \tfrac{|c_1|}{|c_2|}|c_2'|\|v\|.\end{aligned}$$

Moreover, the below identities must be met

$$\|x - c_1v\| = |c_1'|\|v\| \text{ and } \|y - c_2v\| = |c_2'|\|v\|. \tag{10}$$

Now, one can define $u = \frac{c_2v - y}{c_2'\|v\|}$. It is clear that $\|u\| = 1$. Equivalently

$$y = c_2v - c_2'\|v\|u. \tag{11}$$

If $\|(x - c_1v) + \frac{c_1}{c_2}(c_2v - y\| = \|x - c_1v\| + \|\frac{c_1}{c_2}(c_2v - y)\|$, then there exists $\lambda \geqslant 0$ such that

$$x - c_1v = \lambda\frac{c_1}{c_2}(c_2v - y). \tag{12}$$

Hence $\|x - c_1v\| = \lambda\frac{|c_1|}{|c_2|}\|c_2v - y\|$. Taking into account (10) we get

$$\lambda = \frac{|c_1'||c_2|}{|c_1||c_2'|}. \tag{13}$$

Linking (11), (12) and (13) gives

$$x = c_1v + \frac{c_1c_2'|c_2|}{|c_1||c_2'|c_2}\cdot|c_1'|\|v\|u. \tag{14}$$

By (11) and (14) we obtain

$$x - \tfrac{c_1}{c_2}y = c_1v + \tfrac{c_1c_2'|c_2|}{|c_1||c_2'|c_2}\cdot|c_1'|\|v\|u - \tfrac{c_1}{c_2}(c_2v - c_2'\|v\|u) = \tfrac{c_1}{c_2}\cdot\left(\tfrac{|c_1'||c_2|}{|c_1||c_2'|} + 1\right)c_2'\|v\|u.$$

On the other hand, since $\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\| = \|x-\frac{c_1}{c_2}y\|$, we have $\left\langle x-\frac{c_1}{c_2}y, y\right\rangle = 0$. Thus

$$\langle u,y\rangle = 0, \tag{15}$$

because $\frac{c_1}{c_2}\cdot\left(\frac{|c_1'||c_2|}{|c_1||c_2'|}+1\right)c_2'\|v\| \neq 0$. Conditions (11),(14) and (15) are exactly that what should be shown.

B. Sufficiency. At first we observe if $\langle y,v\rangle = 0$, then $\|y\|^2 \leqslant (|c_2'|^2-|c_2|^2)\|v\|^2$ is equivalent to $\|y-c_2v\| \leqslant |c_2'|\|v\|$. Moreover, if $|c_2|<|c_2'|$, then there exist vectors $y$ fulfil (7) which are nonzero.

Now, let $y$ be such as above. According to our assumptions, $\|x\|^2 = (|c_1|+|c_1'|)^2\|v\|^2$ and $\langle x,y\rangle = 0$. Therefore

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = (|c_1|+|c_1'|)^2\|v\|^2\|y\|^2,$$

as required.

B. Necessity. The case $y=0$ is trivial. Let $y\neq 0$ and $\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2} = (|c_1|+|c_1'|)^2\|v\|^2$. In this way, the following evident estimations become equalities

$$\begin{aligned}\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2} &= \left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2 \leqslant \|x\|^2 = \|(x-c_1v)+c_1v\|^2 \leqslant (\|x-c_1v\|+\|c_1v\|)^2\\ &\leqslant (|c_1|+|c_1'|)^2\|v\|^2.\end{aligned}$$

In particular, $\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2 = \|x\|^2$ gives $\langle x,y\rangle = 0$, $\|(x-c_1v)+c_1v\|^2 = (\|x-c_1v\|+\|c_1v\|)^2$ is equivalent to $x-c_1v = \lambda c_1v$ for any $\lambda \geqslant 0$, where $\|x-c_1v\|$ must be equal to $|c_1'|\|v\|$. Thus $\lambda = |c_1'|/|c_1|$. Finally, $x = c_1v+c_1\frac{|c_1'|}{|c_1|}v$. Hence $\langle y,v\rangle = 0$, since $\langle x,y\rangle = 0$.

C. Sufficiency. It is clear for $c_1 = 0$, because $\langle u,y\rangle = 0$ leads to $\langle x,y\rangle = 0$ and $\|x\|^2 = |c_1'|^2\|v\|^2$.

When $c_2' = 0$ we have $y = c_2v$ and then $\langle u,y\rangle = 0$ gives $\langle u,v\rangle = 0$. Hence

$$\begin{aligned}\|x\|^2 &= (|c_1|^2+|c_1'|^2)\|v\|^2, \|y\|^2 = |c_2|^2\|v\|^2,\\ \langle x,y\rangle &= \left\langle c_1v+|c_1'|\|v\|u, c_2v\right\rangle = c_1\overline{c_2}\|v\|^2, |\langle x,y\rangle|^2 = |c_1|^2|c_2|^2\|v\|^4.\end{aligned}$$

Thus $\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 = |c_1'|^2\|v\|^2\cdot|c_2|^2\|v\|^2 = |c_1'|^2\|v\|^2\cdot\|y\|^2$.

C. Necessity. Firstly, for $c_1 = 0$ we observe that, $\|x\|^2 \leqslant |c_1'|^2\|v\|^2$. So, if $y\neq 0$, then

$$\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2} \leqslant |c_1'|^2\|v\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2}.$$

Since $\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2} = |c_1'|^2\|v\|^2$, we get $\langle x,y\rangle = 0$ and, consequently, $\|x\| = |c_1'|\|v\|$. Hence one can set $u = x/c_1'\|v\|$. The case $y = 0$ is trivial.

Secondly, let $c_2' = 0$. Then by (7) we have $y = c_2v$. Since $\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 = |c_1'|^2\|y\|^2\|v\|^2$, the below evident inequalities

$$\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2} = \left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2 \leqslant \left\|x-\frac{c_1}{c_2}y\right\|^2 = \|x-c_1v\|^2 \leqslant |c_1'|^2\|v\|^2$$

are, in fact, equalities. Hence, in particular, $\langle x - c_1 v, y\rangle = 0$ and $\|x - c_1 v\| = |c_1'|\|v\|$. Setting $u = \frac{x - c_1 v}{c_1'\|v\|}$ we get what was required.

D. The proof of the first equivalence is analogous to the proof of the item C, the case $c_1 = 0$. The second one is a consequence of the first by changing $x \leftrightarrow y$ and $c_1' \leftrightarrow c_2'$.

The proof is complete. $\square$

REMARK 2. The initial assumptions in items A,B,C of theorem 2 guarantee that the quantity $|c_1'| + \frac{|c_1|}{|c_2|}\min\{|c_2|, |c_2'|\} \neq 0$, provided $c_2 \neq 0$.

Analogous variants for the case $c_1 \neq 0$ we obtain by the symmetric replacement $x \leftrightarrow y$, $c_1 \leftrightarrow c_2$ and $c_1' \leftrightarrow c_2'$ as follows.

A'. If $\begin{matrix} c_1 \neq 0, c_1' \neq 0 \\ c_2' \in \mathbb{F}, c_2 \neq 0 \end{matrix}$ and $|c_1'| < |c_1|$, then

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = \frac{(|c_1 c_2'| + |c_1' c_2|)^2}{|c_1|^2}\|x\|^2\|v\|^2$$

$$\Updownarrow$$

$$\exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,x\rangle = 0, \\ x = c_1 v - c_1'\|v\|u,\ y = c_2 v + \frac{c_2 c_1' |c_1|}{|c_2||c_1'|c_1} \cdot |c_2'|\|v\|u. \end{cases}$$

B'. If $\begin{matrix} c_1 \neq 0, c_1' \neq 0 \\ c_2' \in \mathbb{F}, c_2 \neq 0 \end{matrix}$ and $|c_1'| \geqslant |c_1|$, then

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = (|c_2| + |c_2'|)^2\|x\|^2\|v\|^2$$

$$\Updownarrow$$

$$\langle x,v\rangle = 0,\ y = c_2\left(1 + \frac{|c_2'|}{|c_2|}\right)v.$$

C'. If $c_1 c_2' \neq 0$ and $c_1' c_2 = 0$, then

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = |c_2'|^2\|x\|^2\|v\|^2$$

$$\Updownarrow$$

$$\exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,x\rangle = 0, \\ x \in V,\ y = c_2 v + c_2'\|v\|u. \end{cases}$$

Theorem 2 gives the following supplements of Dragomir's results [6, th. 1,6].

COROLLARY 1. *Let $c, c' \in \mathbb{F}$, $c' \neq 0$, $x, y \in V$ and $\|x - cy\| \leqslant |c'|\|y\|$. Then*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = |c'|^2\|y\|^4 \Leftrightarrow \exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,y\rangle = 0, \\ x = c_1 v + c_1'\|y\|u. \end{cases}$$

*Particularly, if $\Gamma, \gamma \in \mathbb{F}$, $\Gamma \neq \gamma$ and $x, y \in V$ satisfy (3) or (4), then*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = \frac{1}{4}|\Gamma - \gamma|^2\|y\|^4 \ \Leftrightarrow\ \exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,y\rangle = 0, \\ x = \frac{\Gamma+\gamma}{2}y + \frac{\Gamma-\gamma}{2}\|y\|u. \end{cases}$$

*Proof.* Make use of theorem 2 C with specifications $v := y$, $c_2 = 1$, $c_2' = 0$ and $c_1 := c$, $c_1' = c'$. Next substitute $c = \frac{\Gamma+\gamma}{2}$, $c' = \frac{\Gamma-\gamma}{2}$. $\square$

COROLLARY 2. *Let* $0 \neq c, c' \in \mathbb{F}$ *and* $x, y \in V$ *with* $|c'| < |c|$ *and* $\|x - cy\| \leqslant |c'|\|y\|$. *Then*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = \frac{|c'|^2}{|c|^2}\|x\|^2\|y\|^2 \Leftrightarrow \exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,x\rangle = 0, \\ x = cy - c'\|y\|u. \end{cases}$$

*Particularly, if* $\Gamma, \gamma \in \mathbb{F}$, $\mathrm{Re}\Gamma\overline{\gamma} > 0$ *and* $x, y \in V$ *satisfy (3) or, equivalently, (4), then*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = \left|\frac{\Gamma-\gamma}{\Gamma+\gamma}\right|^2 \|x\|^2\|y\|^2$$
$$\Updownarrow$$
$$\exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,x\rangle = 0, \\ x = \frac{\Gamma+\gamma}{2}y + \frac{\Gamma-\gamma}{2}\|y\|u. \end{cases}$$

*Proof.* Firstly, apply the statement A' (remark 2) for $v := y$, $c_2 = 1$, $c_2' = 0$ and $c_1 = c$, $c_1' = c'$. Secondly, set $c = \frac{\Gamma+\gamma}{2}$, $c' = \frac{\Gamma-\gamma}{2}$ and observe that $|c'| < |c|$ is equivalent to $\mathrm{Re}\Gamma\overline{\gamma} > 0$ what ensures $\Gamma + \gamma \neq 0$. At the end we note that the vector $u$ can always be replaced by $c_1 u$, where $|c_1| = 1$, $c_1 \in \mathbb{F}$, e.g. $c_1 = -1$. $\square$

## 3. Applications to Grüss' inequality

Let $x, y$ and $z \neq 0$ be vectors in an inner vectors space $V$. Grüss' type inequalities state upper bounds for the quantity $|\langle x,y\rangle - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}|$. Usually but not necessarily $\|z\| = 1$.

Applying classic Schwarz's inequality for the vectors $x - \frac{\langle x,z\rangle}{\|z\|^2}z$ and $y - \frac{\langle y,z\rangle}{\|z\|^2}z$ and taking into account that

$$\begin{aligned} \left\langle x - \frac{\langle x,z\rangle}{\|z\|^2}z, y - \frac{\langle y,z\rangle}{\|z\|^2}z \right\rangle &= \langle x,y\rangle - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}, \\ \left\| x - \frac{\langle x,z\rangle}{\|z\|^2}z \right\|^2 &= \frac{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2}{\|z\|^2}, \\ \left\| y - \frac{\langle y,z\rangle}{\|z\|^2}z \right\|^2 &= \frac{\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2}{\|z\|^2}, \end{aligned}$$

we have the initial estimate

$$\left| \langle x,y\rangle - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2} \right| \leqslant \frac{\sqrt{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2}\sqrt{\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2}}{\|z\|^2}. \tag{16}$$

By the equality condition for Schwarz's inequality, the equality holds in (16) if and only if $x - \frac{\langle x,z\rangle}{\|z\|^2}z$ and $y - \frac{\langle y,z\rangle}{\|z\|^2}z$ are linearly dependent. In the sequel, the following auxiliary lemma is useful. Its elementary proof we omit.

LEMMA 1. *Let* $x,y,z \in V,\ z \neq 0.$

$$\left|\langle x,y\rangle - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right| = \frac{\sqrt{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2}\sqrt{\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2}}{\|z\|^2} \tag{17}$$

$$\Updownarrow$$

$$\dim \operatorname{span}\{x,y,z\} \leqslant 2.$$

Beginning with (16), theorem 1 directly produces Grüss type inequalities.

THEOREM 3. *Let* $c_i, c_i' \in \mathbb{F},\ i = 1,2,3$ *and* $x,y,z,v \in V,\ z,v \neq 0.$
*If*

$$\|x - c_1 v\| \leqslant |c_1'|\|v\|,\ \|y - c_2 v\| \leqslant |c_2'|\|v\| \text{ and } \|z - c_3 v\| \leqslant |c_3'|\|v\|, \tag{18}$$

*then the inequalities*

$$\begin{aligned} &\left|\langle x,y\rangle - \tfrac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right| \\ &\leqslant \begin{cases} \left(|c_1'| + \frac{|c_1|}{|c_3|}\min\{|c_3|,|c_3'|\}\right)\left(|c_2'| + \frac{|c_2|}{|c_3|}\min\{|c_3|,|c_3'|\}\right)\|v\|^2, \\ \left(|c_3'| + \frac{|c_3|}{|c_1|}\min\{|c_1|,|c_1'|\}\right)\left(|c_3'| + \frac{|c_3|}{|c_2|}\min\{|c_2|,|c_2'|\}\right)\frac{\|x\|\|y\|\|v\|^2}{\|z\|^2}, \end{cases} \end{aligned} \tag{19}$$

*are valid, where* $c_3 \neq 0$ *or, respectively,* $c_1, c_2 \neq 0.$

*Proof.* Apply respectively inequality (8) (v1) to triples of vectors $x,z,v$ and $y,z,v$ and make use of the estimate (16). $\square$

Obvious specifications of theorem 3 give detailed variants of Grüss' inequality.

COROLLARY 3. *Let* $c_i, c_i' \in \mathbb{F},\ i = 1,2,3,\ c_3 \neq 0,\ x,y,z,v \in V,\ z,v \neq 0$ *and (18) is met.*

*Then*

$$\left|\langle x,y\rangle - \tfrac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right| \leqslant \begin{cases} \frac{(|c_1||c_3'|+|c_1'||c_3|)(|c_2||c_3'|+|c_2'||c_3|)}{|c_3|^2}\cdot\|v\|^2 & \text{if } |c_3'| < |c_3|,\ \text{(v1)} \\ (|c_1| + |c_1'|)(|c_2| + |c_2'|)\cdot\|v\|^2 & \text{if } |c_3'| \geqslant |c_3|,\ \text{(v2)} \\ |c_1'||c_2'|\cdot\|v\|^2 & \text{if } c_1, c_2 = 0.\ \text{(v3)} \end{cases} \tag{20}$$

COROLLARY 4. *Let* $c_i, c_i' \in \mathbb{F},\ i = 1,2,3,\ c_1, c_2 \neq 0,\ x,y,z,v \in V,\ z,v \neq 0$ *and (18) is met.*

*Then*

$$\left|\langle x,y\rangle - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right|$$
$$\leqslant \begin{cases} \frac{(|c_1||c_3'|+|c_1'||c_3|)(|c_2||c_3'|+|c_2'||c_3|)}{|c_1||c_2|}\cdot\frac{\|x\|\|y\|\|v\|^2}{\|z\|^2} & \text{if } \begin{cases}|c_1'|<|c_1|\\ |c_2'|<|c_2|\end{cases} \quad (\mathrm{v1})\\ \frac{(|c_1||c_3'|+|c_1'||c_3|)(|c_3|+|c_3'|)}{|c_1|}\cdot\frac{\|x\|\|y\|\|v\|^2}{\|z\|^2} & \text{if } \begin{cases}|c_1'|<|c_1|\\ |c_2'|\geqslant|c_2|\end{cases} \quad (\mathrm{v2})\\ (|c_3|+|c_3'|)^2\cdot\frac{\|x\|\|y\|\|v\|^2}{\|z\|^2} & \text{if } \begin{cases}|c_1'|\geqslant|c_1|\\ |c_2'|\geqslant|c_2|\end{cases} \quad (\mathrm{v3})\\ |c_3'|^2\cdot\frac{\|x\|\|y\|\|v\|^2}{\|z\|^2} & \text{if } \quad c_3=0. \quad (\mathrm{v4}) \end{cases} \tag{21}$$

REMARK 3. Let $\Gamma,\gamma,\Phi,\phi\in\mathbb{F}$ and $e\in V$ with $\|e\|=1$. Specifying $c_1=\frac{\Phi+\phi}{2}$, $c_1'=\frac{\Phi-\phi}{2}, c_2=\frac{\Gamma+\gamma}{2}, c_2'=\frac{\Gamma-\gamma}{2}$ and $z=v=e,\ c_3=1,\ c_3'=0$ in corollary 3 (v1) we get Dragomir's result [6, theorem 15,16]:

if

$$\left\|x-\frac{\Phi+\phi}{2}e\right\|\leqslant\frac{1}{2}|\Phi-\phi| \text{ and } \left\|y-\frac{\Gamma+\gamma}{2}e\right\|\leqslant\frac{1}{2}|\Gamma-\gamma|,$$

then

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leqslant\frac{1}{4}|\Phi-\phi||\Gamma-\gamma|.$$

Similarly, setting $z=v=e,\ c_3=1,\ c_3'=0$ and $|c_i'|<|c_i|,\ i=1,2$ in corollary 4 (v1) gives other Dragomir's result [6, theorem 21]:

$$\|x-c_1e\|\leqslant|c_1'| \text{ and } \|y-c_2e\|\leqslant|c_2'|$$

implies

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leqslant\left|\frac{c_1'}{c_1}\right|\left|\frac{c_2'}{c_2}\right|\|x\|\|y\|.$$

□

Theorem 2 yields conditions of equality for Grüss type inequalities (19). We start from such conditions for inequalities of the specific form (20).

PROPOSITION 1. *Let* $c_i,c_i'\in\mathbb{F},\ i=1,2,3,\ x,y,z,v\in V,\ z,v\neq0$ *and conditions (18) are met.*

A. *If* $c_1,c_2\neq0$ *and* $0<|c_3'|<|c_3|$, *then*

$$\left|\langle x,y\rangle-\frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right|=\frac{(|c_1||c_3'|+|c_1'||c_3|)(|c_2||c_3'|+|c_2'||c_3|)}{|c_3|^2}\cdot\|v\|^2 \tag{22}$$

$$\Updownarrow$$

$$\exists u\in V: \begin{cases}\|u\|=1,\ \langle u,z\rangle=0,\ z=c_3v-c_3'\|v\|u\\ x=c_1v+\frac{c_1c_3'|c_3|}{|c_1||c_3'|c_3}\cdot|c_1'|\|v\|u,\ y=c_2v+\frac{c_2c_3'|c_3|}{|c_2||c_3'|c_3}\cdot|c_2'|\|v\|u.\end{cases} \tag{23}$$

B. *If* $c_1, c_2 \neq 0$ *and* $|c_3'| > |c_3| > 0$*, then*

$$\left| \langle x,y\rangle - \frac{\langle x,z\rangle \langle z,y\rangle}{\|z\|^2} \right| = (|c_1|+|c_1'|)(|c_2|+|c_2'|)\cdot \|v\|^2 \tag{24}$$

$$\Updownarrow$$

$$\langle z,v\rangle = 0,\ x = c_1\left(1+\left|\frac{c_1'}{c_1}\right|\right)v,\ y = c_2\left(1+\left|\frac{c_2'}{c_2}\right|\right)v. \tag{25}$$

C. *If* $c_1c_3', c_2c_3' = 0$ *and* $c_1', c_2', c_3 \neq 0$*, then*

$$\left| \langle x,y\rangle - \frac{\langle x,z\rangle \langle z,y\rangle}{\|z\|^2} \right| = |c_1'||c_2'|\cdot \|v\|^2 \tag{26}$$

$$\Updownarrow$$

$$\exists u\in V \exists \alpha_1,\alpha_2 \in \mathbb{F} : \begin{cases} \|u\| = 1 = |\alpha_1| = |\alpha_2|,\ \langle u,z\rangle = 0, \\ x = c_1 v + c_1'\|v\|\alpha_1 u,\ y = c_2 + c_2'\|v\|\alpha_2 u,\ z\in V. \end{cases} \tag{27}$$

D. *If* $c_1 = 0, c_1', c_2 \neq 0$ *and* $0 < |c_3'| < |c_3|$*, then*

$$\left| \langle x,y\rangle - \frac{\langle x,z\rangle \langle z,y\rangle}{\|z\|^2} \right| = \frac{|c_1'|(|c_2||c_3'|+|c_2'||c_3|)}{|c_3|}\cdot \|v\|^2 \tag{28}$$

$$\Updownarrow$$

$$\exists u\in V \exists \alpha_1,\alpha_2 \in \mathbb{F} : \begin{cases} \|u\| = 1 = |\alpha_1| = |\alpha_2|,\ \langle u,z\rangle = 0,\ z = c_3 v - c_3'\|v\|\alpha_2 u, \\ x = c_1'\|v\|\alpha_1 u,\ y = c_2 v + \frac{c_2 c_3' |c_3|}{|c_2||c_3'|c_3}\cdot |c_2'|\|v\|\alpha_2 u. \end{cases} \tag{29}$$

E. *If* $c_1 = 0,\ c_1', c_2 \neq 0\ \ |c_3'| > |c_3| > 0$*, then*

$$\left| \langle x,y\rangle - \frac{\langle x,z\rangle \langle z,y\rangle}{\|z\|^2} \right| = |c_1'|(|c_2|+|c_2'|)\cdot \|v\|^2 \tag{30}$$

$$\Updownarrow$$

$$\exists \alpha\in\mathbb{F},\ |\alpha| = 1\ :\ x = \alpha c_1' v,\ y = c_2\left(1+\left|\frac{c_2'}{c_2}\right|\right)v,\ \langle v,z\rangle = 0. \tag{31}$$

*Proof.* At first we observe that in case of B or E, if $\langle z,v\rangle = 0$, then

$$\|z\|^2 \leqslant (|c_3'|^2 - |c_3|^2)\|v\|^2 \iff \|z - c_3 v\| \leqslant |c_3'|\|v\|.$$

Therefore, since $|c_3| < |c_3'|$, there exist nonzero vectors $z$ orthogonal to $v$ fulfilling (18).

Let

$$L_{1j} = \frac{|c_j c_3'| + |c_j' c_3|}{|c_3|},\ L_{2j} = (|c_j| + |c_j'|),\ L_{3j} = |c_j'|,\ j = 1,2,$$

$$L_{41} = L_{51} = L_{31},\ L_{42} = L_{12},\ L_{52} = L_{22}.$$

Firstly, utilizing the same scheme, we show the implications (n+1)$\Rightarrow$(n), where (n+1)=(23), (25), (27), (29), (31).

It is easily seen that dim span$\{x,y,z\}$ is not greater than 2, if one of the conditions (23), (25), (29), (31) holds. In case of C, if $c_3' = 0$, then $z = c_3 v$, by (18). Hence (27) ensures span$\{x,y,z\}$ = span$\{u,z\}$. If $c_1, c_2 = 0$, then (27) directly gives span$\{x,y,z\}$ = span$\{u,z\}$. Anyway, dim span$\{x,y,z\}$ = dim span$\{u,z\} = 2$, since $\langle u,z\rangle = 0$. Therefore, dim span$\{x,y,z\} \leqslant 2$ in any case and consequently (17) is fulfilled, by Lemma 1.

Applying twice theorem 2, firstly for vectors $x$ and $z$, secondly for $y$ and $z$ (more exactly: variants A and A, B and B, C and C, C and A, C and B to (23), (25), (27), (29), (31), resp.) we obtain the following identities:

$$\begin{aligned} \sqrt{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2} &= L_{i1}\|z\|\|v\| \\ \sqrt{\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2} &= L_{i2}\|z\|\|v\| \end{aligned},\ i = 1,2,3,4,5. \tag{32}$$

Now, linking (17) with (32) we get (22),(24),(26),(28),(30), respectively.

Now, we can use one pattern for converse implications (n)$\Rightarrow$(n+1), (n)=(22), (24), (26), (28), (30).

On the suitable assumptions of A, B, C, D and E, by (16) and theorem 1 (v1) employed to pairs of vectors $x,z$ and $y,z$ we have for $i = 1,2,3,4,5$

$$\begin{aligned} \sqrt{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2} &\leqslant L_{i1}\|z\|\|v\|, \\ \sqrt{\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2} &\leqslant L_{i2}\|z\|\|v\|, \\ \left|\langle x,y\rangle - \tfrac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right| &\leqslant \frac{\sqrt{\|x\|^2\|z\|^2-|\langle x,z\rangle|^2}\sqrt{\|y\|^2\|z\|^2-|\langle y,z\rangle|^2}}{\|z\|^2} \\ &\leqslant \tfrac{1}{\|z\|^2}\cdot(L_{i1}\|z\|\|v\|)\cdot(L_{i2}\|z\|\|v\|). \end{aligned}$$

On the assumption (n), the all above inequalities are, in fact, equalities. In particular, the equalities (32) are met. Hence, applying theorem 2 to vectors $x,z$ and to $y,z$ (more exactly: utilizing of variants A and A, B and B, C and C, C and A, C and B of the theorem, consecutively, ensures what follows.

In case of A: there exist unit vectors $u_1$ and $u_2$ such that

$$\begin{gathered} \langle u_i,z\rangle = 0,\ i = 1,2,\ z = c_3 v - c_3'\|v\|u_1 = c_3 v - c_3'\|v\|u_2 \\ x = c_1 v + \tfrac{c_1 c_3' |c_3|}{|c_1||c_3'|c_3}\cdot|c_1'|\|v\|u_1,\ y = c_2 v + \tfrac{c_2 c_3'|c_3|}{|c_2||c_3'|c_3}\cdot|c_2'|\|v\|u_2. \end{gathered}$$

Clearly, $u_1 = u_2$. This is exactly (23).

In case of B: obviously, (25) holds.

In case of C: there exist unit vectors $u_1$ and $u_2$ such that

$$\begin{gathered} \langle u_i,z\rangle = 0,\ i = 1,2, \\ x = c_1 v + c_1'\|v\|u_1,\ y = c_2 v + c_2'\|v\|u_2,\ z \in V. \end{gathered}$$

Moreover, dim span$\{u_1,u_2,z\} \leqslant 2$, by Lemma 1. Since $\langle u_1,z\rangle = 0$, it can be assumed that span$\{u_1,u_2,z\}$ = span$\{u,z\}$, where $\|u\| = 1$ and $\langle u,z\rangle = 0$. Hence, it easily follows $u_i = \alpha_i u$ for certain $\alpha_i \in \mathbb{F}$, $|\alpha_i| = 1$, $i = 1,2$.

In case of D: there exist unit vectors $u_1$ and $u_2$ such that

$$\langle u_i, z\rangle = 0,\ i = 1,2,$$
$$x = c_1'\|v\|u_1,\ y = c_2 v + \tfrac{c_2 c_3'|c_3|}{|c_2||c_3'|c_3}\cdot|c_2'|\|v\|u_2,\ z = c_3 v - c_3'\|v\|u_2.$$

Moreover, by Lemma 1, $\dim \operatorname{span}\{x,y,z\} \leqslant 2$ and the above forms of $x,y,z$ force that $\operatorname{span}\{x,y,z\} = \operatorname{span}\{u_1,u_2,z\}$. Hence, since $\langle u_1, z\rangle = 0$, $\dim \operatorname{span}\{u_1,u_2,z\} = 2$. One can assume that $\operatorname{span}\{u_1,u_2,z\} = \operatorname{span}\{u,z\}$, where $\|u\| = 1$ and $\langle u,z\rangle = 0$. In a consequence, $u_i = \alpha_i u$ for certain $\alpha_i \in \mathbb{F}$, $|\alpha_i| = 1$, $i = 1,2$, as required.

In case of E: there exists an unit vector $u$ such that

$$\langle u,z\rangle = 0,\ x = c_1'\|v\|u,\ y = c_2\left(1 + \left|\frac{c_2'}{c_2}\right|\right)v,\ \langle v,z\rangle = 0.$$

Moreover, by Lemma 1, $\dim \operatorname{span}\{x,y,z\} \leqslant 2$ and the above conditions on vectors $x,y,z$ gives $\operatorname{span}\{x,y,z\} = \operatorname{span}\{u,v,z\}$. Hence $\dim \operatorname{span}\{u,v,z\} = 2$, since $\langle v,z\rangle = 0$. Thus $\operatorname{span}\{u,v,z\} = \operatorname{span}\{v,z\}$ and furthermore, $u$ is linearly dependent of $v$, since $\langle u,z\rangle = 0$. Consequently, the unit vector $u$ has the form $u = \frac{\alpha}{\|v\|}v$ for certain $\alpha \in \mathbb{F}$, $|\alpha| = 1$. Finally, $x = \alpha c_1' v$, what is desired.

The proof is completed. □

The following corollary complements Dragomir's result [6, theorem 15,16].

COROLLARY 5. *Let* $\Gamma, \gamma, \Phi, \phi \in \mathbb{F}$ *and* $e \in V$ *with* $\|e\| = 1$.
*If*

$$\left\|x - \frac{\Phi+\phi}{2}e\right\| \leqslant \frac{1}{2}|\Phi-\phi| \text{ and } \left\|y - \frac{\Gamma+\gamma}{2}e\right\| \leqslant \frac{1}{2}|\Gamma-\gamma|,$$

*or, equivalently,*

$$\operatorname{Re}\langle \Phi e - x, x - \phi e\rangle \geqslant 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geqslant 0,$$

*then*

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| = \frac{1}{4}|\Phi-\phi||\Gamma-\gamma|$$
$$\Updownarrow$$
$$\exists u \in V \exists \alpha_1,\alpha_2 \in \mathbb{F} : \begin{cases} \|u\| = 1 = |\alpha_1| = |\alpha_2|,\ \langle u,e\rangle = 0, \\ x = \frac{\Phi+\phi}{2}e + \frac{\Phi-\phi}{2}\alpha_1 u,\ y = \frac{\Gamma+\gamma}{2}e + \frac{\Gamma-\gamma}{2}\alpha_2 u. \end{cases}$$

*Proof.* Use proposition 1 C for $c_1 = \frac{\Phi+\phi}{2}$, $c_1' = \frac{\Phi-\phi}{2}$, $c_2 = \frac{\Gamma+\gamma}{2}$, $c_2' = \frac{\Gamma+\gamma}{2}$ and $z = v = e$, $c_3 = 1$, $c_3' = 0$. □

Conditions of equality for Grüss type inequalities (21) are presented below.

PROPOSITION 2. *Let* $c_i, c_i' \in \mathbb{F}$, $i = 1,2,3$, $x,y,z,v \in V$, $z,v \neq 0$ *and conditions* (18) *are met.*

A. *If* $0<|c_i'|<|c_i|,\ i=1,2$ *and* $c_3\neq 0$, *then*

$$\left|\langle x,y\rangle-\frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right|=\frac{(|c_1||c_3'|+|c_1'||c_3|)(|c_2||c_3'|+|c_2'||c_3|)}{|c_1||c_2|}\cdot\frac{\|x\|\|y\|\|v\|^2}{\|z\|^2} \tag{33}$$

$$\Updownarrow$$

$$\exists u\in V:\begin{cases}\|u\|=1,\ \langle u,x\rangle=0=\langle u,y\rangle,\\ x=c_1v-c_1'\cdot\frac{c_2'|c_2|}{|c_2'|c_2}\cdot\|v\|u,\ y=c_2v-c_2'\cdot\frac{c_1'|c_1|}{|c_1'|c_1}\cdot\|v\|u\\ z=c_3v+\frac{c_3}{|c_3|}\cdot\frac{c_1'|c_1|}{|c_1'|c_1}\cdot\frac{c_2'|c_2|}{|c_2'|c_2}\cdot|c_3'|\|v\|u.\end{cases} \tag{34}$$

B. *If* $|c_i'|>|c_i|>0,\ i=1,2$ *and* $c_3\neq 0$, *then*

$$\left|\langle x,y\rangle-\frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right|=(|c_3|+|c_3'|)^2\cdot\frac{\|x\|\|y\|\|v\|^2}{\|z\|^2} \tag{35}$$

$$\Updownarrow$$

$$\begin{array}{l}x,y-\text{linearly dependent}\\ \langle x,v\rangle=0=\langle y,v\rangle,\ z=c_3\left(1+\left|\frac{c_3'}{c_3}\right|\right)v.\end{array} \tag{36}$$

C. *If* $c_1,c_2,c_3'\neq 0$ *and* $c_3=0$, *then*

$$\left|\langle x,y\rangle-\frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right|=|c_3'|^2\cdot\frac{\|x\|\|y\|\|v\|^2}{\|z\|^2} \tag{37}$$

$$\Updownarrow$$

$$\exists u\in V:\begin{cases}x,y-\text{linearly dependent}\\ \|u\|=1,\ \langle u,x\rangle=0=\langle u,y\rangle,\ z=c_3'\|v\|u.\end{cases} \tag{38}$$

*Proof.* At the beginning we observe that in case of B, if $\langle x,v\rangle=0$, then

$$\|x\|^2\leqslant(|c_1'|^2-|c_1|^2)\|v\|^2\iff\|x-c_1v\|\leqslant|c_1'|\|v\|.$$

In this way, since $|c_1|<|c_1'|$, there exist nonzero vectors $x$ orthogonal to $v$ fulfilling (18). The same holds for $y$.

Let

$$M_{1j}=\frac{|c_jc_3'|+|c_j'c_3|}{|c_j|},\ M_{2j}=(|c_3|+|c_3'|),\ M_{3j}=|c_3'|,\ j=1,2.$$

(n+1)$\Rightarrow$(n), (n+1)=(34), (36), (38).

The same argumentation proves these all implications.

Namely, if (n+1) holds, then dim span$\{x,y,z\}\leqslant 2$ and consequently (17) is fulfilled by Lemma 1. Applying theorem 2 A',B',C' (see remark 2) (more exactly: A',B',C' for (34), (36), (38), resp.), firstly for vectors $x$ and $z$, secondly for $y$ and $z$ we obtain the following identities:

$$\begin{matrix} \sqrt{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2} = M_{i1}\|x\|\|v\| \\ \sqrt{\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2} = M_{i2}\|y\|\|v\| \end{matrix}, \ i = 1,2,3\,. \tag{39}$$

Now, linking (17) with (39) we get (33), (35), (37), respectively.

(n)$\Rightarrow$(n+1), (n)=(33), (35), (37).

As before, the proofs run in the similar manner.

Under suitable assumptions of A, B, C, by (16) and theorem 1 (v1) employed to pairs of vectors $z,x$ and $z,y$ we have for $i = 1,2,3$

$$\begin{aligned} \sqrt{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2} &\leqslant M_{i1}\|x\|\|v\|, \\ \sqrt{\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2} &\leqslant M_{i2}\|y\|\|v\|, \\ \left|\langle x,y\rangle - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right| &\leqslant \frac{\sqrt{\|x\|^2\|z\|^2-|\langle x,z\rangle|^2}\sqrt{\|y\|^2\|z\|^2-|\langle y,z\rangle|^2}}{\|z\|^2} \\ &\leqslant \frac{1}{\|z\|^2}\cdot(M_{i1}\|x\|\|v\|)\cdot(M_{i2}\|y\|\|v\|)\,. \end{aligned}$$

On the assumption (n), the all above inequalities are, in fact, equalities. In particular, the equalities (39) are met. Now, theorem 2, A',B',C' (see remark 2) ensures what follows.

In case of A: there exist unit vectors $u_1$ and $u_2$ such that $\langle u_1,x\rangle = 0 = \langle u_2,y\rangle = 0$ and

$$x = c_1 v - c_1'\|v\|u_1, \ y = c_2 v - c_2'\|v\|u_2,$$
$$c_3 v + \frac{c_3}{|c_3|}\cdot\frac{c_1'|c_1|}{|c_1'|c_1}\cdot|c_3'|\|v\|u_1 = z = c_3 v + \frac{c_3}{|c_3|}\cdot\frac{c_2'|c_2|}{|c_2'|c_2}\cdot|c_3'|\|v\|u_2.$$

Hence $\frac{c_1'|c_1|}{|c_1'|c_1}\cdot u_1 = \frac{c_2'|c_2|}{|c_2'|c_2}\cdot u_2$. At last, setting $u := u_1/\frac{c_2'|c_2|}{|c_2'|c_2} = u_2/\frac{c_1'|c_1|}{|c_1'|c_1}$ we come to (34).

In case of B:

$$\langle x,v\rangle = 0 = \langle y,v\rangle\,, \ z = c_3\left(1 + \left|\frac{c_3'}{c_3}\right|\right)v.$$

Moreover, by Lemma 1, dim span$\{x,y,z\} \leqslant 2$. Hence we conclude that $x$ and $y$ are linearly dependent, since $\langle x,v\rangle = 0 = \langle y,v\rangle$.

In case of C: there exist unit vectors $u_1$ and $u_2$ such that

$$\langle u_1,x\rangle = 0 = \langle u_2,y\rangle\,, \ c_3'\|v\|u_1 = z = c_3'\|v\|u_2.$$

Clearly, $u_1 = u_2$. Furthermore, by Lemma 1, dim span$\{x,y,z\} \leqslant 2$. Hence we infer that $x$ and $y$ are linearly dependent, since $\langle x,z\rangle = 0 = \langle y,z\rangle$.

The proof is entirely finished. □

REMARK 4. If (33) holds in proposition 2 A, then necessarily $\frac{|c_1'|}{|c_1|} = \frac{|c_2'|}{|c_2|}$. This is a consequence of the condition $\langle x,u\rangle = 0 = \langle y,u\rangle$. Furthermore, since $x = c_1(v - \frac{c_1'c_2'}{c_1c_2}\frac{|c_2|}{|c_2'|}\|v\|u)$ and $y = c_2(v - \frac{c_1'c_2'}{c_1c_2}\frac{|c_1|}{|c_1'|}\|v\|u)$, the vectors $x$ and $y$ are then linearly dependent. □

Using of proposition 2 A for $z = v = e,\ c_3 = 1,\ c_3' = 0$ and remark 4 leads to the following supplement of Dragomir's result [6, theorem 21].

COROLLARY 6. *Given* $0 \neq c_i, c_i' \in \mathbb{F}$ *with* $0 < r = |c_i'|/|c_i| < 1,\ i = 1,2$ *and* $x, y, e \in V,\ \|e\| = 1$. *If* $\|x - c_1 e\| < |c_1'|$ *and* $\|y - c_2 e\| < |c_2'|$, *then*

$$|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| = r^2 \|x\| \|y\|$$

*if and only if*

$$\exists u \in V : \begin{cases} \|u\| = 1,\ \langle u,x\rangle = 0 = \langle u,y\rangle, \\ x = c_1(e - r\cdot u),\ y = c_2(e - r\cdot u). \end{cases}$$

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