

## BUZANO'S INEQUALITY IN ALGEBRAIC PROBABILITY SPACES

J. AGREDO\*, Y. LEON, J. OSORIO AND A. PEÑA

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*Abstract.* We obtain a generalization of Buzano's inequality of vectors in Hilbert spaces, using the theory of algebraic probability spaces. In particular, we extend a result of Dragomir given in [7]. Applications for numerical inequalities for  $n$ -tuples of bounded linear operators and functions of operators defined by double power series are also generalized.

### 1. Introduction

Quantum probability theory traces back to the famous book *Mathematical foundations of quantum mechanics* by J. von Neumann, where a new structure containing the classical probability theory was proposed, though in a different language, on the basis of an algebraic probability space. This language is typically used in open quantum systems and quantum Markov semigroups (see, e.g., Refs. [1], [14],[15],[16], and [17], for results of the same flavour), many of his elements are used in Noncommutative Geometry [5].

During the recent development, quantum probability theory and algebraic probability spaces, has been related to various fields of mathematical sciences beyond the original purposes.

Hora and Obata focus, in [23], on the spectral analysis of a large graph (or of a growing graph) and show how the quantum probabilistic techniques are applied, especially, for the study of asymptotics of spectral distributions in terms of quantum central limit theorem.

Salimi studied continuous-time quantum and classical random walk on spider network lattices in [27], quantum central limit theorem for continuous-time quantum walks on odd graphs in [28], and continuous-time quantum walks on quotient graphs (see [29]) via quantum probability theory.

In [6] the author uses the theory of Noncommutative geometry to explore connections with the Pythagorean theorem in this context.

On the other hand, in [4], Buzano obtained the following extension of the celebrated Cauchy-Schwarz inequality in a complex Hilbert space  $\mathfrak{h}$ :

$$|\langle a, x \rangle \langle x, b \rangle| \leq \frac{1}{2} [ ( \|a\| \|b\| + |\langle a, b \rangle| ) \|x\|^2 ]$$

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\* Corresponding author.

for all  $a, b, x \in \mathfrak{h}$ . When  $a = b$  this inequality becomes the Cauchy-Schwarz inequality

$$|\langle a, x \rangle|^2 \leq \|a\|^2 \|x\|^2.$$

For a real inner product space, Richard [25] independently obtained the following stronger inequality:

$$\left| \langle a, x \rangle \langle x, b \rangle - \frac{1}{2} \langle a, x \rangle \|x\|^2 \right| \leq \frac{1}{2} \|a\| \|b\| \|x\|^2$$

for all  $a, b, x \in \mathfrak{h}$ .

Dragomir [11] showed that this inequality (for real or complex case) is valid with coefficients  $\frac{1}{|\alpha|}$  instead of  $\frac{1}{2}$ , where a non-zero number  $\alpha$  satisfies the equality  $|1 - \alpha| = 1$ . As an application of this inequality, Fujii and Kubo [20] found a bound for roots of algebraic equations. During developing the operator theory and its applications, the authors of [8] have recently extended some numerical inequalities to operator inequalities. Some mathematicians have also investigated the operator versions of the Cauchy-Schwarz inequality or its reverse (see [19],[18],[22],[30]).

In this work we are interested in to establish a generalization of Buzano's inequality, using the theory of algebraic probability spaces. Our interest in this inequality is motivated by [7],[12],[13],[10], and [9], where several techniques of Hilbert space are used and different applications are established.

The paper is organized as follows. In Section 2 we show an introduction of basic notions in quantum probability theory and algebraic probability spaces. In Section 3 we show how to generalize the Buzano's inequality using algebraic probability spaces. Finally, we present two applications: application for numerical radius inequalities of bounded operators, and application of inequalities of double power series with bounded operators in section 4.

## 2. Algebraic probability spaces

We begin by reviewing basic notions and notations in quantum probability theory.

All the vector spaces, Hilbert spaces, algebras, are here supposed to be defined on the field of complex numbers  $\mathbb{C}$ .

On a Hilbert space the scalar product  $\langle \cdot, \cdot \rangle$  is linear in the right variable and anti-linear in the left one. The notation for the scalar product does not refer to the underlying space, there should not be any possible confusion.

In the same way, the norm of any normed space is denoted by  $\| \cdot \|$ , unless it is necessary for the comprehension to specify the underlying space.

On any Hilbert space, the identity operator is denoted by  $I$ , without precisising the associated space. The algebra of bounded operators on a Hilbert space  $\mathfrak{h}$  is denoted by  $\mathfrak{B}(\mathfrak{h})$ .

**DEFINITION 1.** An algebra  $\mathcal{A}$  is a vector space over  $\mathbb{C}$ , in which a binary operation  $((a, b) \mapsto ab \in \mathcal{A}$ , for all  $a, b \in \mathcal{A}$ ), called multiplication, is defined. The

multiplication satisfies the bilinearity:

$$(a+b)c = ac + bc, \quad a(b+c) = ab + ac, \quad (\lambda a)b = a(\lambda b) = \lambda(ab),$$

for all  $a, b \in \mathcal{A}$ , and  $\lambda \in \mathbb{C}$ .

We assume in this paper that for all algebra  $\mathcal{A}$  there exists an element  $1_{\mathcal{A}} \in \mathcal{A}$  such that  $a1_{\mathcal{A}} = 1_{\mathcal{A}}a = a$ ,  $\forall a \in \mathcal{A}$ .

DEFINITION 2.  $1_{\mathcal{A}}$  is called the *identity* of  $\mathcal{A}$ .

It's easy to see that  $1_{\mathcal{A}}$  is unique. In many literatures an algebra is defined over an arbitrary field and does not necessarily possess the identity. In Noncommutative Geometry it is useful to work with  $C^*$ -algebras, however in this work it will be sufficient to work with  $*$ -algebras.

DEFINITION 3. A  $*$ -algebra is an algebra  $\mathcal{A}$  equipped with an involution defined on  $\mathcal{A}$ . An *involution* is a map  $a \mapsto a^*$  defined on  $\mathcal{A}$  such that:  $(a+b)^* = a^* + b^*$ ,  $(\lambda a)^* = \overline{\lambda}a^*$ ,  $(ab)^* = b^*a^*$ ,  $(a^*)^* = a$ , with  $a, b \in \mathcal{A}$ , and  $\lambda \in \mathbb{C}$ .

$\mathcal{A} = \mathbb{C}$  is the most basic example of  $*$ -algebra where the complex conjugation is the involution. A key tool for our work is the notion of states.

DEFINITION 4. A linear function  $\varphi$  defined on a  $*$ -algebra  $\mathcal{A}$  with values in  $\mathbb{C}$  is called

- (i) *positive* if  $\varphi(a^*a) \geq 0$  for all  $a \in \mathcal{A}$ ;
- (ii) *normalized* if  $\varphi(1_{\mathcal{A}}) = 1$ ; and
- (iii) a *state* if  $\varphi$  is positive and normalized.

With these terminologies we give the following:

DEFINITION 5. An *algebraic probability space* is a pair  $(\mathcal{A}, \varphi)$  of a  $*$ -algebra  $\mathcal{A}$  and a state  $\varphi$  on it.

### 3. Generalized Buzano's inequality

We show how to generalize the Buzano's inequality using algebraic probability spaces.

LEMMA 1. Let  $(\mathcal{A}, \varphi)$  an algebraic probability space. Then

$$\varphi(a^*) = \overline{\varphi(a)}, \quad a \in \mathcal{A}.$$

*Proof.* Given that  $\varphi((a + \lambda b)^*(a + \lambda b)) \geq 0$  for all  $\lambda \in \mathbb{C}$  then

$$\varphi(a^*a) + \overline{\lambda}\varphi(a) + \lambda\varphi(a^*) + |\lambda|^2 \geq 0.$$

In particular  $\overline{\lambda}\varphi(a) + \lambda\varphi(a^*) \in \mathbb{R}$ , i.e.,

$$\overline{\lambda}\varphi(a) + \lambda\varphi(a^*) = \lambda\overline{\varphi(a)} + \overline{\lambda\varphi(a^*)}, \quad \lambda \in \mathbb{C}$$

so

$$\overline{\lambda}\varphi(a) - \overline{\lambda\varphi(a^*)} = \lambda\overline{\varphi(a^*)} - \lambda\varphi(a^*), \quad \lambda \in \mathbb{C}.$$

Hence  $\varphi(a) - \overline{\varphi(a^*)} = \frac{\lambda^2}{|\lambda|^2} (\overline{\varphi(a)} - \varphi(a^*))$  for all  $\lambda \in \mathbb{C} - \{0\}$ .

Given that  $\varphi(a) - \overline{\varphi(a^*)}$  is independent of  $\lambda$ , we obtain  $\varphi(a) = \overline{\varphi(a^*)}$ .

LEMMA 2. *Let  $(\mathcal{A}, \varphi)$  an algebraic probability space. Then*

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b), \quad a, b \in \mathcal{A}.$$

*Proof.* If  $\varphi(a^*b) = 0$  then the statement follows trivially. Hence, we suppose  $\varphi(a^*b) \neq 0$ . Given that  $\varphi((a + \lambda b)^*(a + \lambda b)) \geq 0$  for all  $\lambda \in \mathbb{C}$ , using Lemma 1, we obtain

$$0 \leq \varphi(a^*a) + \overline{\lambda}\varphi(b^*a) + \lambda\varphi(a^*b) + |\lambda|^2\varphi(b^*b) \tag{1}$$

$$= \varphi(a^*a) + \overline{\lambda\varphi(b^*a)} + \lambda\varphi(a^*b) + |\lambda|^2\varphi(b^*b). \tag{2}$$

By hypothesis,  $\varphi(a^*b) \neq 0$  then exists  $\alpha \in \mathbb{R}$  such that  $\varphi(a^*b) = e^{i\alpha}|\varphi(a^*b)|$ . Letting  $\lambda = te^{i\alpha}$  with  $t \in \mathbb{R}$  in (1), we see that

$$\varphi(a^*a) + 2t|\varphi(b^*a)| + t^2\varphi(b^*b) \geq 0 \quad \text{for all } t \in \mathbb{R}. \tag{3}$$

In particular  $\varphi(b^*b) \neq 0$  (if  $\varphi(b^*b) = 0$  then (3) does not hold). Then (3) is an quadratic inequality equivalent true if and only if

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b).$$

THEOREM 1. *Let  $(\mathcal{A}, \varphi)$  an algebraic probability space. Then*

$$|\varphi(a^*b)| \leq |\varphi(a^*b) - \varphi(a^*d)\varphi(d^*b)| + |\varphi(a^*d)\varphi(d^*b)| \leq \varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2}, \tag{4}$$

$a, b, d \in \mathcal{A}$  with  $\varphi(d^*d) = 1$ .

*Proof.* We take  $a, b \in \mathcal{A}$ . Using triangle inequality for modulus we have

$$|\varphi(a^*b)| \leq |\varphi(a^*b) - \varphi(a^*d)\varphi(d^*b)| + |\varphi(a^*d)\varphi(d^*b)|.$$

Moreover,

$$|\varphi(a^*b) - \varphi(a^*d)\varphi(d^*b)| + |\varphi(a^*d)\varphi(d^*b)| = |\varphi(a^*(b - d\varphi(d^*b)))| + |\varphi(a^*d\varphi(d^*b))|.$$

We take  $\alpha_1, \alpha_2 \in \mathbb{R}$  such that

$$|\varphi(a^*(b - d\varphi(d^*b)))| = e^{i\alpha_1} \varphi(a^*(b - d\varphi(d^*b)))$$

and

$$|\varphi(a^*d\varphi(d^*b))| = e^{i\alpha_2} \varphi(a^*d\varphi(d^*b)).$$

Letting  $h = e^{i\alpha_1}(b - d\varphi(d^*b)) + e^{i\alpha_2}d\varphi(d^*b)$ , we obtain that

$$\begin{aligned} & |\varphi(a^*b) - \varphi(a^*d)\varphi(d^*b)| + |\varphi(a^*d)\varphi(d^*b)| \\ &= e^{i\alpha_1} \varphi(a^*(b - d\varphi(d^*b))) + e^{i\alpha_2} \varphi(a^*d\varphi(d^*b)) \\ &= \varphi(a^*e^{i\alpha_1}(b - d\varphi(d^*b))) + \varphi(a^*e^{i\alpha_2}d\varphi(d^*b)) = \varphi(a^*h). \end{aligned}$$

By Lemma 2, we have

$$|\varphi(a^*b) - \varphi(a^*d)\varphi(d^*b)| + |\varphi(a^*d)\varphi(d^*b)| \leq \varphi(a^*a)^{1/2} \varphi(h^*h)^{1/2}. \tag{5}$$

Where

$$\begin{aligned} \varphi(h^*h) &= \varphi((b - d\varphi(d^*b))^*(b - d\varphi(d^*b))) + \varphi((d\varphi(d^*b))^*d\varphi(d^*b)) \\ &\quad + e^{i(\alpha_2 - \alpha_1)} \varphi(d^*b)\varphi((b - d\varphi(d^*b))^*d) \\ &\quad + e^{-i(\alpha_2 - \alpha_1)} \overline{\varphi(d^*b)\varphi((b - d\varphi(d^*b))^*d)} \\ &= \varphi((b^* - d^*\overline{\varphi(d^*b)})(b - d\varphi(d^*b))) + |\varphi(d^*b)|^2 \varphi(d^*d) \\ &\quad + 2\Re[e^{i(\alpha_2 - \alpha_1)} \varphi(d^*b)\varphi((b - d\varphi(d^*b))^*d)] \\ &= \varphi(b^*b - b^*d\varphi(d^*b) - d^*b\overline{\varphi(d^*b)} + d^*d|\varphi(d^*b)|^2) + |\varphi(d^*b)|^2 \varphi(d^*d) \\ &\quad + 2\Re[e^{i(\alpha_2 - \alpha_1)} \varphi(d^*b)\varphi((b^* - d^*\overline{\varphi(d^*b)})d)] \\ &= \varphi(b^*b) - 2|\varphi(d^*b)|^2 + 2\varphi(d^*d)|\varphi(d^*b)|^2 \\ &\quad + 2\Re[e^{i(\alpha_2 - \alpha_1)} \varphi(d^*b)(\varphi(b^*d) - \varphi(d^*d)\overline{\varphi(d^*b)})]. \end{aligned}$$

We recall that  $\varphi(d^*d) = 1$  and  $\varphi(b^*d) = \overline{\varphi(d^*b)}$  then

$$\varphi(h^*h) = \varphi(b^*b) - 2|\varphi(d^*b)|^2 + 2|\varphi(d^*b)|^2 \tag{6}$$

$$+ 2\Re[e^{i(\alpha_2 - \alpha_1)} \varphi(d^*b)(\varphi(b^*d) - \varphi(b^*d))] = \varphi(b^*b). \tag{7}$$

By (5) and (6), we obtain the sentence.

Using Theorem 1 and triangle inequality for modulus we have

$$2|\varphi(a^*d)\varphi(d^*b)| - |\varphi(a^*b)| \leq |\varphi(a^*d)\varphi(d^*b) - \varphi(a^*b)| + |\varphi(a^*d)\varphi(d^*b)| \leq \varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2}.$$

This implies a generalization of Buzano’s inequality:

COROLLARY 1. *Let  $(\mathcal{A}, \varphi)$  an algebraic probability space. Then*

$$|\varphi(a^*d)\varphi(d^*b)| \leq \frac{1}{2} [|\varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2} + |\varphi(a^*b)|] \tag{8}$$

for all  $a, b, d \in \mathcal{A}$  with  $\varphi(d^*d) = 1$ .

We use in this paper the following algebraic probability space: Let  $\mathcal{A} = M(n, \mathbb{C})$  be the set of  $n \times n$  complex matrices. Equipped with the usual addition, multiplication and involution (defined by complex conjugation and transposition),  $M(n, \mathbb{C})$  becomes a  $*$ -algebra. It is noncommutative if  $n \geq 2$ . The normalized trace

$$\varphi_{tr} a = \frac{1}{n} tr a = \frac{1}{n} \sum_{i=1}^n a_{ii}, \quad a = (a_{ij}) \in M(n, \mathbb{C}),$$

is a state on  $M(n, \mathbb{C})$ . We preserve the symbol  $tr a$  for the usual trace. This algebraic probability space is denoted by  $(M(n, \mathbb{C}), \varphi_{tr})$ .

On the other hand, let  $\mathfrak{h}$  be a complex vectorial space with inner product  $\langle \cdot, \cdot \rangle$ .

DEFINITION 6. Given  $u, v \in \mathfrak{h}$  we denote by  $|u\rangle\langle v|$  the linear map such that for all  $w \in \mathfrak{h}$

$$|u\rangle\langle v|(w) = \langle v, w \rangle u.$$

This operator is called a *projector*. In particular if  $v = u$ , we obtain a projection.

When  $\mathfrak{h}$  is finite dimensional, we abuse of language and use the notation  $|u\rangle\langle v|$  for operator and his associated matrix.

REMARK 1. The theory of orthonormal basis of Hilbert space (see chapter 4 of [26]) supports the following properties:

- (i)  $tr |u\rangle\langle v| = \langle v, u \rangle$  for all  $u, v \in \mathfrak{h}$ , where  $\mathfrak{h}$  is a complex Hilbert space;
- (ii)  $tr(|u\rangle\langle v||y\rangle\langle x|) = \langle x, u \rangle \langle v, y \rangle$  for all  $u, v, x, y \in \mathfrak{h}$ , where  $\mathfrak{h}$  is a complex Hilbert space.

If  $\mathfrak{h}$  is a  $n$ -dimensional space then using inequality (8), the algebraic probability space  $(M(n, \mathbb{C}), \varphi_{tr})$ , with  $a = |x\rangle\langle e|$ ,  $b = |y\rangle\langle e|$ ,  $d = n|e\rangle\langle e|$ , and remark 1, we obtain the classical Buzano’s inequality:

$$|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} [||x|| ||y|| + |\langle x, y \rangle|] \tag{9}$$

for all  $x, y, e \in \mathfrak{h}$  with  $||e|| = 1$ , i.e., inequality (8) is a generalization of the classical Buzano’s inequality.

DEFINITION 7. A matrix  $\rho \in M(n, \mathbb{C})$  is called a *density matrix* if

- (i)  $\rho = \rho^*$ ;
- (ii) all eigenvalues of  $\rho$  are non-negative; and
- (iii)  $tr \rho = 1$ .

A density matrix  $\rho$  gives rise to a state  $\varphi_\rho$  on  $M(n, \mathbb{C})$  defined by

$$\varphi_\rho(a) = tr(\rho a), \quad \forall a \in M(n, \mathbb{C}).$$

Conversely, any state on  $M(n, \mathbb{C})$  is of this form. Moreover, there is a one-to-one correspondence between the set of states and the set of density matrices. This algebraic probability space is denoted by  $(M(n, \mathbb{C}), \rho)$ .

The generalization of Buzano's inequality obtained in [7] (Theorem 2.1) its follow as corollary of theorem 1:

COROLLARY 2. Let  $\mathbf{p} = (p_1, \dots, p_n)$  be a probability distribution, such that  $\sum_{i=1}^n p_i = 1$ . For any  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathfrak{h}^n$  we have

$$\left| \left\langle \sum_{i=1}^n p_i x_i, e \right\rangle \left\langle e, \sum_{i=1}^n p_i y_i \right\rangle \right| \tag{10}$$

$$\leq \frac{1}{2} \left[ \left( \sum_{i=1}^n p_i |x_i|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i |y_i|^2 \right)^{1/2} + \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \right] \tag{11}$$

for all  $e \in \mathfrak{h}$  with  $\|e\| = 1$ , where  $\mathfrak{h}$  is a Hilbert space.

*Proof.* For a probability distribution  $\mathbf{p} = (p_1, \dots, p_n)$ ,  $\mathbf{x} = (x_1, \dots, x_n)$ , and  $\mathbf{y} = (y_1, \dots, y_n)$  in  $\mathfrak{h}^n$  we distinguish two cases:

- (i)  $n \leq dim(\mathfrak{h})$ : We consider the density matrix defined as:

$$\sigma_{\mathbf{p}} := \sum_{i=1}^n p_i |v_i\rangle\langle v_i|$$

and  $X, Y \in M(n, \mathbb{C})$  define as:

$$X := \sum_{i=1}^n |x_i\rangle\langle v_i|, \quad Y := \sum_{i=1}^n |y_i\rangle\langle v_i|,$$

where each  $v_i$  is a vector belongs to an orthonormal basis of  $\mathfrak{h}$ , in other words, we are using the algebraic probability space  $(M(n, \mathbb{C}), \varphi_{\sigma_{\mathbf{p}}})$ . Then  $X^* = \sum_{i=1}^n |v_i\rangle\langle x_i|$ ,

by remark 1, properties (i), (ii) it follows that

$$\varphi_{\sigma_{\mathbf{p}}}(X^*Y) = tr(\sigma_{\mathbf{p}}X^*Y) = \sum_{i,j,k} p_i tr(|v_i\rangle\langle v_i||v_j\rangle\langle x_j||y_k\rangle\langle v_k|) \stackrel{(i),(ii)}{=} \sum_{i=1}^n p_i \langle x_i, y_i \rangle.$$

Given  $e \in \mathfrak{h}$  with  $\|e\| = 1$ , we take  $E := \sum_{i=1}^n |e\rangle\langle v_i| \in M(n, \mathbb{C})$  then

$$\varphi_{\sigma_p}(E^*E) = \left( \sum_{i=1}^n p_i \|e\|^2 \right)^{1/2} = 1.$$

Making use of Theorem 1, we obtain the inequality (10).

- (ii)  $n > \dim(\mathfrak{h}) = m$ : We consider  $\mathfrak{h}_c = \mathfrak{h} \times \mathbb{C}^n$ , where  $\mathfrak{h}$  is gifted with orthonormal basis  $(v_i)_{i=1}^m$ .  $\mathbb{C}^n$  is gifted with the canonical basis  $(e_k)_{k \in \{1, \dots, n\}}$  and canonical inner product. Moreover,  $\mathfrak{h}_c$  is the cartesian product between  $\mathfrak{h}$  and  $\mathbb{C}^n$  with standard inner product.

It follows an analogous procedure to first case replacing  $\mathfrak{h}_c$  by  $\mathfrak{h}$ , taking

$$\sigma_p := \sum_{i=1}^n p_i |(0, e_i)\rangle\langle(0, e_i)| + \sum_{i=1}^m \frac{1}{m} |(v_i, 0)\rangle\langle(v_i, 0)|$$

and  $X, Y, E \in M(n + m, \mathbb{C})$  define as:

$$X = \sum_{i=1}^n |(x_i, 0)\rangle\langle(0, e_i)|,$$

$$Y := \sum_{i=1}^n |(y_i, 0)\rangle\langle(0, e_i)|,$$

$$E := \sum_{i=1}^n |(e, 0)\rangle\langle(0, e_i)|.$$

### 4. Applications

#### 4.1. Numerical radius inequalities of bounded operators

Let  $\mathfrak{h}$  be a complex Hilbert space.

DEFINITION 8. The *numerical range* of an operator  $T$  on  $\mathfrak{h}$  is the set given by

$$W(T) := \{ \langle Tx, x \rangle; x \in \mathfrak{h}, \|x\| = 1 \}$$

and the *numerical radius* of an operator  $T$  on  $\mathfrak{h}$  is defined by

$$w(T) := \sup\{ |\alpha|; \alpha \in W(T) \} = \sup\{ \langle Tx, x \rangle; x \in \mathfrak{h}, \|x\| = 1 \}.$$

It is well known the following inequality

$$w(T) \leq \|T\| \leq 2w(T) \quad \text{for any } T \in \mathfrak{B}(\mathfrak{h}).$$

Using inequality (9) it is possible to obtain the following inequality for the numerical radius (see [9] or [10]):



**THEOREM 2.** *Let  $\mathfrak{h}$  be a Hilbert space and  $T : \mathfrak{h} \mapsto \mathfrak{h}$  a bounded operator on  $\mathfrak{h}$ . Then*

$$w^2(T) \leq \frac{1}{2} [w(T^2) + \|T\|^2].$$

*The constant 1/2 is sharp in the inequality.*

We use the algebraic probability space  $(M(n, \mathbb{C}), \rho)$  to obtain a generalization of Theorem 2.

**THEOREM 3.** *Let  $\rho = (\rho_{ij}) \in M(n, \mathbb{C})$  be a density matrix and  $(A_1, \dots, A_n)$  an  $n$ -tuples of bounded operators on  $\mathfrak{h}$ . Then we have*

$$w^2 \left( \sum_{i,j=1}^n \rho_{ij} A_i \right) \tag{12}$$

$$\leq \frac{1}{2} \left[ \left\| \sum_{i,j=1}^n \rho_{ij} A_j^* A_i \right\|^{1/2} \left\| \sum_{i,j=1}^n \rho_{ij} A_j A_i^* \right\|^{1/2} + w \left( \sum_{i,j=1}^n \rho_{i,j} A_j A_i \right) \right] \tag{13}$$

*Proof.* For a density matrix  $\rho = (\rho_{ij}) \in M(n, \mathbb{C})$ ,  $e \in \mathfrak{h}$  with  $\|e\| = 1$ , and  $(A_1, \dots, A_n)$  an  $n$ -tuples of bounded operators on  $\mathfrak{h}$  we distinguish two cases:

(i)  $n \leq \dim(\mathfrak{h})$ : We consider  $X, Y \in M(n, \mathbb{C})$  define as:

$$X := \sum_{i=1}^n |A_i e\rangle \langle v_i|, \quad Y := \sum_{i=1}^n |B_i e\rangle \langle v_i|, \quad \sigma_\rho := \sum_{i,j=1}^n \rho_{ij} |v_i\rangle \langle v_j|$$

$E := \sum_{i=1}^n |e\rangle \langle v_i| \in M(n, \mathbb{C})$  where each  $v_i$  is a vector belongs to an orthonormal basis of  $\mathfrak{h}$ , in other words, we are using the algebraic probability space  $(M(n, \mathbb{C}), \varphi_{\sigma_\rho})$ . Then  $X^* = \sum_{i=1}^n |v_i\rangle \langle A_i e|$ , by remark 1, properties (i), (ii) it follows that

$$\begin{aligned} \varphi_{\sigma_\rho}(X^*Y) &= \text{tr}(\sigma_\rho X^*Y) = \sum_{i,j,k,l} \rho_{ij} \text{tr}(|v_j\rangle \langle v_i| |v_k\rangle \langle A_k e| |B_l e\rangle \langle v_l|) \\ &\stackrel{(i),(ii)}{=} \sum_{i,j=1}^n \rho_{ij} \langle A_i e, B_j e \rangle \end{aligned}$$

and  $\varphi_{\sigma_\rho}(E^*E) = \left( \sum_{i=1}^n \rho_{ii} \|e\|^2 \right)^{1/2} = 1$ . Making use of Theorem 1, we obtain

the inequality

$$\begin{aligned} & \left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_i e, e \right\rangle \left\langle \sum_{i,j=1}^n \rho_{ij} B_i e, e \right\rangle \right| \\ & \leq \frac{1}{2} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_j^* A_i e, e \right\rangle \right|^{1/2} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} B_j^* B_i e, e \right\rangle \right|^{1/2} \\ & \quad + \frac{1}{2} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} B_j^* A_i e, e \right\rangle \right|. \end{aligned}$$

We take  $B_i = A_i^*$  in the last inequality, then we obtain

$$\left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_i e, e \right\rangle \right|^2 \tag{14}$$

$$\leq \frac{1}{2} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_j^* A_i e, e \right\rangle \right|^{1/2} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_j A_i^* e, e \right\rangle \right|^{1/2} \tag{15}$$

$$+ \frac{1}{2} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_j A_i e, e \right\rangle \right| \tag{16}$$

for all  $e \in \mathfrak{h}$  with  $\|e\| = 1$ .

By taking the supremum over  $\|e\| = 1$  in (14), we get

$$\begin{aligned} w^2 \left( \sum_{i,j=1}^n \rho_{ij} A_i \right) &= \sup_{\|e\|=1} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_i e, e \right\rangle \right|^2 \\ &\leq \frac{1}{2} \sup_{\|e\|=1} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_j^* A_i e, e \right\rangle \right|^{1/2} \sup_{\|e\|=1} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} B_j^* B_i e, e \right\rangle \right|^{1/2} \\ &\quad + \frac{1}{2} \sup_{\|e\|=1} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_j A_i e, e \right\rangle \right| \\ &\leq \frac{1}{2} \left[ \left\| \sum_{i,j=1}^n \rho_{ij} A_j^* A_i \right\|^{1/2} \left\| \sum_{i,j=1}^n \rho_{ij} A_j A_i^* \right\|^{1/2} + w \left( \sum_{i,j=1}^n \rho_{i,j} A_j A_i \right) \right] \end{aligned}$$

and the theorem is proved.

- (ii)  $n > \dim(\mathfrak{h}) = m$ : We consider  $\mathfrak{h}_c = \mathfrak{h} \times \mathbb{C}^n$ , where  $\mathfrak{h}$  is gifted with orthonormal basis  $(v_i)_{i=1}^m$ .  $\mathbb{C}^n$  is gifted with the canonical basis  $(e_k)_{k \in \{1, \dots, n\}}$  and canonical inner product. Moreover,  $\mathfrak{h}_c$  is the cartesian product between  $\mathfrak{h}$  and  $\mathbb{C}^n$  with standard inner product.

It follows an analogous procedure to first case replacing  $\mathfrak{h}_c$  by  $\mathfrak{h}$ , taking

$$\sigma_\rho := \sum_{i,j=1}^n \rho_{ij} |(0, e_i)\rangle\langle(0, e_j)| + \sum_{i,j=1}^m \frac{1}{m} |(v_i, 0)\rangle\langle(v_j, 0)|$$

and  $X, Y, E \in M(n + m, \mathbb{C})$  define as:

$$X := \sum_{i=1}^n |(A_i e, 0)\rangle\langle(0, e_i)|,$$

$$Y := \sum_{i=1}^n |(B_i e, 0)\rangle\langle(0, e_i)|,$$

and  $E := \sum_{i=1}^n |(e, 0)\rangle\langle(0, e_i)|.$

**4.2. Double power series of bounded operators**

The procedure to proof the last theorem is similar to the procedure that it follow in the corollary 2. Using this procedure with  $X, Y \in M(n, \mathbb{C})$  define as:

$$X := \sum_{i=1}^n |A_i e\rangle\langle v_i|, \quad Y := \sum_{i=1}^n |B_i e\rangle\langle v_i|, \quad \sigma_\rho := \sum_{i,j=1}^n \rho_{ij} |v_i\rangle\langle v_j|$$

$E := \sum_{i=1}^n |e\rangle\langle v_i| \in M(n, \mathbb{C})$ . where each  $v_i$  is a vector belongs to an orthonormal basis of  $\mathfrak{h}$ , and  $x, y, e \in \mathfrak{h}$  with  $\|e\| = 1$ , if  $n \leq \dim(\mathfrak{h})$ , or

$$\sigma_\rho := \sum_{i,j=1}^n \rho_{ij} |(0, e_i)\rangle\langle(0, e_j)| + \sum_{i,j=1}^m \frac{1}{m} |(v_i, 0)\rangle\langle(v_j, 0)|$$

and  $X, Y, E \in M(n + m, \mathbb{C})$  define as:

$$X := \sum_{i=1}^n |(A_i x, 0)\rangle\langle(0, e_i)|,$$

$$Y := \sum_{i=1}^n |(B_i y, 0)\rangle\langle(0, e_i)|,$$

and  $E := \sum_{i=1}^n |(e, 0)\rangle\langle(0, e_i)|$  defined on  $\mathfrak{h}_c$  cartesian product between  $\mathfrak{h}$  and  $\mathbb{C}^n$  with standard inner product, and  $x, y, e \in \mathfrak{h}$  with  $\|e\| = 1$ , if  $n > \dim(\mathfrak{h})$ , we obtain the following result:

PROPOSITION 4. Let  $\rho = (\rho_{ij}) \in M(n, \mathbb{C})$  be a density matrix, and  $(A_1, \dots, A_n)$  an  $n$ -tuples of bounded operators on  $\mathfrak{h}$ . Then we have

$$\begin{aligned} & \left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_i e, x \right\rangle \left\langle \sum_{i,j=1}^n \rho_{ij} B_j e, y \right\rangle \right| \\ & \leq \frac{1}{2} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} A_j^* A_i x, x \right\rangle \right|^{1/2} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} B_j^* B_i y, y \right\rangle \right|^{1/2} \\ & \quad + \frac{1}{2} \left| \left\langle \sum_{i,j=1}^n \rho_{ij} B_j^* A_i x, y \right\rangle \right|. \end{aligned}$$

for all  $x, y, e \in \mathfrak{h}$  with  $\|e\| = 1$ .

This proposition is useful to establish the next inequality for double power series:

COROLLARY 3. Let  $f(z, w) = \sum_{i,j=0}^\infty a_{ij} z^i w^j$  be a double power series with non-negative coefficients  $a_{ij} \geq 0$  for  $i, j \in \mathbb{N}$  and convergence for all  $z, w$  such that  $|z| \in [0, R)$  and  $|w| \in [0, R)$ , where  $R \in (0, \infty]$  and  $z, w \in \mathbb{C}$ . If  $A, B$  are operators on the Hilbert space  $\mathfrak{h}$ ,  $(\alpha, \beta) \in (0, R) \times (0, R)$  such that  $\rho_n := (a_{ij} \alpha^i \beta^j)_{i,j=1}^n$  is a density matrix for all  $n \in \mathbb{N}$ , and  $\|A\|, \|B\| \leq 1$ , then

$$\begin{aligned} & |\langle f(\alpha A, \beta I) e, x \rangle \langle f(\alpha B, \beta I) e, y \rangle| \\ & \leq \frac{1}{2} |\langle f(\alpha A, \beta A^*) x, x \rangle|^{1/2} |\langle f(\alpha B, \beta B^*) y, y \rangle|^{1/2} \\ & \quad + \frac{1}{2} |\langle f(\alpha A, \beta B^*) x, y \rangle| \end{aligned}$$

for any  $x, y, e \in \mathfrak{h}$  with  $\|e\| = 1$ , where  $I$  is identity operator on  $\mathfrak{h}$ .

If we consider in the corollary 3,  $f(z, 0) = \sum_{i=0}^\infty a_{i0} z^i$  (with convention  $0^0 = 1$ ) then recover the theorem 3.1 of [7]:

COROLLARY 4. Let  $f(z) = \sum_{i=0}^\infty a_i z^i$  be a power series with nonnegative coefficients  $a_i \geq 0$  for  $i \in \mathbb{N}$  and having radius of convergence  $R > 0$  or  $R = \infty$ . If  $A, B$  are normal operators on the Hilbert space  $\mathfrak{h}$ ,  $A^* B = B A^*$ ,  $\alpha \in (0, R)$ , and  $\|A\|, \|B\| \leq 1$ , then

$$\begin{aligned} & |\langle f(\alpha A) e, x \rangle \langle f(\alpha B) e, y \rangle| \\ & \leq \frac{1}{2} |\langle f(\alpha A^* A) x, x \rangle| |\langle f(\alpha B^* B) y, y \rangle| f(\alpha) \\ & \quad + \frac{1}{2} |\langle f(\alpha B^*) x, y \rangle| f(\alpha) \end{aligned}$$

for any  $x, y, e \in \mathfrak{h}$  with  $\|e\| = 1$ , where  $I$  is identity operator on  $\mathfrak{h}$ .

*Proof.* By corollary 3 we obtain

$$\left| \left\langle \sum_{i=0}^n a_i \alpha^i A^i e, x \right\rangle \left\langle \sum_{i=0}^n a_i \alpha^i B^i e, y \right\rangle \right| \tag{17}$$

$$\leq \frac{1}{2} \left\langle \sum_{i=0}^n a_i \alpha^i (A^i)^* A^i x, x \right\rangle^{1/2} \left\langle \sum_{i=0}^n a_i \alpha^i (B^i)^* B^i y, y \right\rangle^{1/2} \sum_{i=0}^n a_i \alpha_i \tag{18}$$

$$+ \frac{1}{2} \left\langle \sum_{i=0}^n a_i \alpha^i (B^*)^i A^i x, y \right\rangle^{1/2} \sum_{i=0}^n a_i \alpha_i \tag{19}$$

for any  $x, y, e \in \mathfrak{h}$  with  $\|e\| = 1$ . Since  $A, B$  are normal operators, then for  $i \geq 1$   $(A^i)^* A^i = (A^*)^i A^i = (A^* A)^i$  and  $(B^i)^* B^i = (B^* B)^i$ .

Also, since  $B^* A = AB^*$ , then  $(B^*)^i A^i = (B^* A)^i$  for all  $i \geq 1$ . Then from (17) we have

$$\left| \left\langle \sum_{i=0}^n a_i \alpha^i A^i e, x \right\rangle \left\langle \sum_{i=0}^n a_i \alpha^i B^i e, y \right\rangle \right| \tag{20}$$

$$\leq \frac{1}{2} \left\langle \sum_{i=0}^n a_i \alpha^i (A^* A)^i x, x \right\rangle^{1/2} \left\langle \sum_{i=0}^n a_i \alpha^i (B^* B)^i y, y \right\rangle^{1/2} \sum_{i=0}^n a_i \alpha_i \tag{21}$$

$$+ \frac{1}{2} \left\langle \sum_{i=0}^n a_i \alpha^i (B^* A)^i x, y \right\rangle^{1/2} \sum_{i=0}^n a_i \alpha_i \tag{22}$$

for all  $x, y, e \in \mathfrak{h}$  with  $\|e\| = 1$ .

Since

$$\|\alpha A^* A\| = \alpha \|A\|^2 < R, \|\alpha B^* B\| = \alpha \|B\|^2 < R,$$

$$\|\alpha B^* A\| \leq \alpha \|B\| \|A\| < R, \|\alpha B\| < R, \|\alpha A\| < R,$$

then the series

$$\sum_{i=0}^{\infty} a_i \alpha^i B^i, \sum_{i=0}^{\infty} a_i \alpha^i A^i, \sum_{i=0}^{\infty} a_i \alpha^i (A^* A)^i, \sum_{i=0}^{\infty} a_i \alpha^i (B^* B)^i, \sum_{i=0}^{\infty} a_i \alpha^i (B^* A)^i$$

are convergent in  $\mathfrak{B}(\mathfrak{h})$  and  $\sum_{i=0}^{\infty} a_i \alpha^i$  in  $\mathbb{R}$ . Taking limits in (20) we get the proof.

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*J. Agredo*  
*Department of Mathematics*  
*Colombian School of Engineering “Julio Garavito”*  
*Bogotá, Colombia*  
*e-mail: julian.agredo@escuelaing.edu.co*

*Y. Leon*  
*Department of Mathematics*  
*Colombian School of Engineering “Julio Garavito”*  
*Bogotá, Colombia*  
*e-mail: yessica.leon@mail.escuelaing.edu.co*

*J. Osorio*  
*Department of Mathematics*  
*Colombian School of Engineering “Julio Garavito”*  
*Bogotá, Colombia*  
*e-mail: juan.osorio-r@mail.escuelaing.edu.co*

*A. Peña*  
*Department of Mathematics*  
*Colombian School of Engineering “Julio Garavito”*  
*Bogotá, Colombia*  
*e-mail: alvaro.pena-m@mail.escuelaing.edu.co*