

ON INNER AND OUTER RADII IN MINKOWSKI SPACES

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(Communicated by J. Pečarić)

Abstract. Some sharp bounds for the inner radius and the outer radius of the unit ball of a Minkowski space with respect to its isoperimetrix are known. To find more such bounds is a challenging problem. Related to this, we derive new relations between inner and outer radii as well as cross-section measures for the Holmes-Thompson and Busemann measures.

1. Introduction

This paper refers to the geometry of finite dimensional real Banach spaces, also called *Minkowski spaces*, and to classical convexity. More precisely, the notions of different cross-section measures from classical convexity are used to obtain new results on unit balls and isoperimetrices of Minkowski spaces.

We recall that a *convex body* K in \mathbb{R}^d , $d \geq 2$, is a compact, convex set with nonempty interior, and that K is said to be *centered* if it is symmetric with respect to the origin o of \mathbb{R}^d . As usual, S^{d-1} denotes the standard Euclidean unit sphere in \mathbb{R}^d . We write λ_i for the *i -dimensional Lebesgue measure (volume)* in \mathbb{R}^d , where $1 \leq i \leq d$, and instead of λ_d we simply write λ . We denote by u^\perp the $(d-1)$ -dimensional subspace orthogonal to $u \in S^{d-1}$, and by l_u the 1-subspace parallel to u .

For a convex body $K \subset \mathbb{R}^d$ we denote by $\lambda_{d-1}(K, u^\perp)$ and $\lambda_1(K, u)$ the $(d-1)$ -dimensional and 1-dimensional inner cross-section measures of K , i.e., the maximal measure of a hyperplane section of K normal to $u \in S^{d-1}$, and the maximal chord length of K in the direction u , respectively. Furthermore, $\lambda_1(K|l_u)$ denotes the width of K at u , and $\lambda_{d-1}(K|u^\perp)$ the $(d-1)$ -dimensional outer cross-section measure or brightness of K at $u \in S^{d-1}$, where $K|u^\perp$ is the orthogonal projection of K onto u^\perp . These notions can be found in the monograph [3]. In [11] and [16] the following results for cross-section measures were derived.

For a convex body K in \mathbb{R}^d , $d \geq 2$, and every direction $u \in S^{d-1}$ we have

$$\lambda(K) \leq \lambda_{d-1}(K|u^\perp)\lambda_1(K, u) \leq d\lambda(K),$$

Mathematics subject classification (2010): 46B20, 52A20, 52A21, 52A40.

Keywords and phrases: Affine isoperimetric inequalities, Busemann measure, cross-section measures, Holmes-Thompson measure, inner radius, intersection body, isoperimetrix, Minkowski geometry, outer radius, projection body.

and both sides are sharp.

On the other hand, for each $u \in S^{d-1}$ a convex body K in \mathbb{R}^d , $d \geq 2$, satisfies

$$\lambda(K) \leq \lambda_{d-1}(K, u^\perp) \lambda_1(K|_{l_u}) \leq d\lambda(K),$$

again with sharpness on both sides.

Our main purpose is to establish connections between cross-section measures and inner/outer radii for centered convex bodies. The inner radius and outer radius of the unit ball with respect to its isoperimetrix in Minkowski spaces (for Holmes-Thompson and Busemann measures) will be used to obtain these connections. Thus, our main results will be related to finite dimensional real Banach spaces as well.

For a convex body K in \mathbb{R}^d , the *polar body* K° of K is defined by

$$K^\circ = \{y \in \mathbb{R}^d : \langle x, y \rangle \leq 1, x \in K\}.$$

We identify \mathbb{R}^d and its *dual space* \mathbb{R}^{d*} by using the standard basis. In that case, λ_i and λ_i^* coincide in \mathbb{R}^d . The symbol ε_i stands for the volume of the standard Euclidean unit ball in \mathbb{R}^i .

For a convex body K in \mathbb{R}^d and $u \in S^{d-1}$, the *support function* of K is defined by

$$h_K(u) = \sup\{\langle u, y \rangle : y \in K\},$$

and with o as an interior point of K its *radial function* $\rho_K(u)$ is defined by

$$\rho_K(u) = \max\{\alpha \geq 0 : \alpha u \in K\}.$$

It is well known that

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)}, u \in S^{d-1}.$$

If K is a centered convex body, then $2\rho_K(u) = \lambda_1(K \cap l_u)$, and $2h_K(u) = \lambda_1(K|_{l_u})$ for any $u \in S^{d-1}$.

The *projection body* ΠK of a convex body K in \mathbb{R}^d is defined by $h_{\Pi K}(u) = \lambda_{d-1}(K|_{u^\perp})$ for each $u \in S^{d-1}$ (see [3, Chapter 4]). Note that any projection body is a *zonoid* (i.e., a limit of vector sums of segments) centered at the origin. In particular, if K is a polytope, then its projection body is a *zonotope* centered at the origin (see [15] and [4] for many properties and applications of this interesting class of convex bodies). We also refer to [1], [6], [7], and [12] for affine isoperimetric inequalities related to projection bodies. The *intersection body* IK of a convex body $K \subset \mathbb{R}^d$ is defined by $\rho_{IK}(u) = \lambda_{d-1}(K \cap u^\perp)$ for each $u \in S^{d-1}$ (cf. [5] and [3, Chapter 8]). If K is a centered convex body, then IK is also a centered convex body (see [2]).

We write $(\mathbb{R}^d, \|\cdot\|) = \mathbb{M}^d$ for a *d-dimensional real Banach space*, i.e., a *Minkowski space* with *unit ball* B which is a centered convex body; see [17]. The *unit sphere* of \mathbb{M}^d is the boundary ∂B of the unit ball.

2. Isoperimetrices and inner/outer radii in Minkowski spaces

A Minkowski space \mathbb{M}^d possesses a Haar measure μ , and this measure is unique up to multiplying the Lebesgue measure by a constant, i.e., $\mu = \sigma_B \lambda$.

The following notions are well known; see [17, Chapter 5]. The *d-dimensional Holmes-Thompson volume* of a convex body K in \mathbb{M}^d is defined by

$$\mu_B^{HT}(K) = \frac{\lambda(K)\lambda(B^\circ)}{\varepsilon_d}, \text{ i.e., } \sigma_B = \frac{\lambda(B^\circ)}{\varepsilon_d},$$

and the *d-dimensional Busemann volume* of K is defined by

$$\mu_B^{Bus}(K) = \frac{\varepsilon_d}{\lambda(B)}\lambda(K), \text{ i.e., } \sigma_B = \frac{\varepsilon_d}{\lambda(B)} \text{ (and } \mu_B^{Bus}(B) = \varepsilon_d).$$

In order to define the Minkowski surface area of a convex body, one has to define σ_B similarly in \mathbb{M}^{d-1} . That is, for the Holmes-Thompson measure we have $\sigma_B(u) = \lambda_{d-1}((B \cap u^\perp)^\circ) / \varepsilon_{d-1}$, and for the Busemann measure $\sigma_B(u) = \varepsilon_{d-1} / \lambda_{d-1}(B \cap u^\perp)$ (see [17, pp. 150-151]). The *Minkowski surface area* of K can be also defined in terms of mixed volumes (see [14] for notation and more about mixed volumes) by

$$\mu_B(\partial K) = dV(K[d-1], I_B),$$

where I_B is that convex body whose support function is $\sigma_B(u)$. For the Holmes-Thompson measure, I_B is given by $I_B^{HT} = \Pi(B^\circ) / \varepsilon_{d-1}$ (cf. [17, p. 150 and p. 157] for detailed explanation), and therefore it is a centered zonoid. For the Busemann measure we have $I_B^{Bus} = \varepsilon_{d-1}(IB)^\circ$ (see again [17, pp. 150-151]). Among all homothetic images of I_B a unique one is specified, which is called the *isoperimetrix* \hat{I}_B and is determined by $\mu_B(\partial \hat{I}_B) = d\mu_B(\hat{I}_B)$. The *isoperimetrix for the Holmes-Thompson measure* is defined by

$$\hat{I}_B^{HT} = \frac{\varepsilon_d}{\lambda(B^\circ)} I_B^{HT} = \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{1}{\lambda(B^\circ)} \Pi B^\circ, \tag{1}$$

and the *isoperimetrix for the Busemann measure* by

$$\hat{I}_B^{Bus} = \frac{\lambda(B)}{\varepsilon_d} I_B^{Bus} = \frac{\varepsilon_{d-1}}{\varepsilon_d} \lambda(B)(IB)^\circ; \tag{2}$$

see [17, Chapter 5].

If K and L are convex bodies in \mathbb{M}^d , the *inner radius* of K with respect to L is defined by $r(K, L) := \max\{\alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha L \subseteq K + x\}$, and the *outer radius* of K with respect to L is defined by $R(K, L) := \min\{\alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha L \supseteq K + x\}$.

One should notice that $r(K, \hat{I}_B)$ and $R(K, \hat{I}_B)$ can also be defined in terms of the support functions of the involved sets. In particular, if K is a centered convex body, then $r(K, \hat{I}_B)$ is the maximum value of α such that $\alpha \leq h_K(u) / h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$. Similarly, $R(K, \hat{I}_B)$ is the minimum value of α such that $\alpha \geq h_K(u) / h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$ (see [13] and [18]).

3. Cross-section measures and inner/outer radii

For a convex body K , we denote by $w_B(K)$ and $D_B(K)$ the *Minkowskian thickness* (i.e., $w_B(K) = \min_{u \in S^{d-1}} \frac{2w(K, u)}{w(B, u)}$, where $w(K, u)$ is the Euclidean width of K in the direction u) and the *Minkowskian diameter* (i.e., the maximum of this Minkowskian width function of K), respectively.

One can easily see that $r(\hat{I}_B^{HT}, B) = \frac{1}{R(B, \hat{I}_B^{HT})}$ and $R(\hat{I}_B^{HT}, B) = \frac{1}{r(B, \hat{I}_B^{HT})}$. Also, it is easy to establish that if K is a centered convex body in \mathbb{M}^d , then $r(K, B) = \frac{w_B(K)}{2}$ and $R(K, B) = \frac{D_B(K)}{2}$.

We recall that some sharp bounds for $r(B, \hat{I}_B^{HT})$ and $R(B, \hat{I}_B^{HT})$ are known (see [8], [9], and also the next section). Below we show the connection between cross-section measures and the outer radius $R(B, \hat{I}_B^{HT})$.

Our first theorem refers to cross-section measures of polars of unit balls and outer radii of isoperimetrices for the Holmes-Thompson measure.

THEOREM 1. *Let B be the unit ball of \mathbb{M}^d . Then*

- a) $R(B, \hat{I}_B^{HT}) \geq 1$ if and only if $\min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)} \leq \frac{2\varepsilon_{d-1}}{\varepsilon_d}$.
- b) $R(B, \hat{I}_B^{HT}) \leq 1$ if and only if $\frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d}$ for all $u \in S^{d-1}$.

Proof. As we know,

$$\frac{2}{R(B, \hat{I}_B^{HT})} = 2r(\hat{I}_B^{HT}, B) = w_B(\hat{I}_B^{HT}).$$

We can expand $w_B(\hat{I}_B^{HT})$ as follows:

$$\begin{aligned} w_B(\hat{I}_B^{HT}) &= \min_{u \in S^{d-1}} \frac{2w(\hat{I}_B^{HT}, u)}{w(B, u)} = \min_{u \in S^{d-1}} \frac{2h_{\hat{I}_B^{HT}}(u)}{h_B(u)} \\ &= \min_{u \in S^{d-1}} \frac{2\varepsilon_d}{\lambda(B^\circ)} h_{\hat{I}_B^{HT}}(u) \rho_{B^\circ}(u) = \min_{u \in S^{d-1}} \frac{2\varepsilon_d}{\varepsilon_{d-1}} \frac{h_{\Pi B^\circ}(u) \rho_{B^\circ}(u)}{\lambda(B^\circ)} \\ &= \min_{u \in S^{d-1}} \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)}. \end{aligned}$$

Therefore,

$$\frac{2\varepsilon_{d-1}}{\varepsilon_d} = R(B, \hat{I}_B^{HT}) \min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)}.$$

Hence, the results follow. \square

From Theorem 1 the following result can be easily deduced:

COROLLARY 2. Let B be the unit ball of \mathbb{M}^d . Then $\frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d}$ for all $u \in S^{d-1}$ if and only if $B \subseteq \hat{I}_B^{HT}$.

We can also use Theorem 1 to get a characterization of ellipsoids.

THEOREM 3. Let B be the unit ball of \mathbb{M}^d . Then

$$\max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)} = \frac{2\varepsilon_{d-1}}{\varepsilon_d}$$

if and only if B is an ellipsoid.

Proof. As mentioned above,

$$\frac{2}{r(B, \hat{I}_B^{HT})} = 2R(\hat{I}_B^{HT}, B) = D_B(\hat{I}_B^{HT}).$$

Then, from the expansion of $D_B(\hat{I}_B^{HT})$ similar to $w_B(\hat{I}_B^{HT})$, one gets

$$D_B(\hat{I}_B^{HT}) = \max_{u \in S^{d-1}} \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)}.$$

Thus,

$$\frac{2\varepsilon_{d-1}}{\varepsilon_d} = r(B, \hat{I}_B^{HT}) \max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)}.$$

It is known that $r(B, \hat{I}_B^{HT}) \leq 1$ with equality if and only if B is an ellipsoid. Hence we have the result. \square

It is known that there exists $u \in S^{d-1}$ such that

$$\frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B \cap l_u)}{\lambda(B)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d},$$

with equality for all $u \in S^{d-1}$ if and only if B is an ellipsoid (cf. [8]).

Also, some sharp bounds for $r(B, \hat{I}_B^{Bus})$ and $R(B, \hat{I}_B^{Bus})$ are known (see [8], [9], or our next section). Below we discuss the connection between cross-section measures and $r(B, \hat{I}_B^{Bus})$ as well as $R(B, \hat{I}_B^{Bus})$.

THEOREM 4. Let B be the unit ball of \mathbb{M}^d . Then

- a) $R(B, \hat{I}_B^{Bus}) \geq 1$ if and only if $\max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d}$.
- b) $R(B, \hat{I}_B^{Bus}) \leq 1$ if and only if $\frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \leq \frac{2\varepsilon_{d-1}}{\varepsilon_d}$ for all $u \in S^{d-1}$.

Proof. As we know,

$$\frac{2}{R(B, \hat{I}_B^{Bus})} = 2r(\hat{I}_B^{Bus}, B) = w_B(\hat{I}_B^{Bus}),$$

and $w_B(\hat{I}_B^{Bus})$ can be expanded as follows:

$$\begin{aligned} w_B(\hat{I}_B^{Bus}) &= \min_{u \in S^{d-1}} \frac{2h_{\hat{I}_B^{Bus}}(u)}{h_B(u)} = \min_{u \in S^{d-1}} \frac{2\lambda(B)\varepsilon_{d-1}h_{(IB)^\circ}(u)}{\varepsilon_d h_B(u)} \\ &= \frac{2\varepsilon_{d-1}}{\varepsilon_d} \min_{u \in S^{d-1}} \frac{\lambda(B)}{\rho_{IB}(u)h_B(u)} = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \min_{u \in S^{d-1}} \frac{2\lambda(B)}{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\varepsilon_d}{2\varepsilon_{d-1}} &= R(B, \hat{I}_B^{Bus}) \min_{u \in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)} \\ &= R(B, \hat{I}_B^{Bus}) \frac{\lambda(B)}{\max_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}, \end{aligned}$$

and hence

$$\max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} = R(B, \hat{I}_B^{Bus}) \frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Thus, our results are confirmed. \square

The next result can be easily deduced from Theorem 4.

COROLLARY 5. *Let B be the unit ball of \mathbb{M}^d . Then $\frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \leq \frac{2\varepsilon_{d-1}}{\varepsilon_d}$ for all $u \in S^{d-1}$ if and only if $B \subseteq \hat{I}_B^{Bus}$.*

Now we combine inner radii of isoperimetrices for the Busemann measure with cross-section measures.

THEOREM 6. *If B is the unit ball of \mathbb{M}^d , then*

- a) $r(B, \hat{I}_B^{Bus}) \leq 1$ if and only if $\min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \leq \frac{2\varepsilon_{d-1}}{\varepsilon_d}$,
- b) $r(B, \hat{I}_B^{Bus}) \geq 1$ if and only if $\frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d}$ for all $u \in S^{d-1}$.

Proof. By expanding $D_B(\hat{I}_B^{Bus})$ (similar to $w_B(\hat{I}_B^{Bus})$), we obtain

$$D_B(\hat{I}_B^{Bus}) = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \max_{u \in S^{d-1}} \frac{2\lambda(B)}{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}.$$

Since $D_B(\hat{I}_B^{Bus}) = \frac{2}{r(B, \hat{I}_B^{Bus})}$, we get

$$\frac{\varepsilon_d}{2\varepsilon_{d-1}} = \max_{u \in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)} r(B, \hat{I}_B^{Bus}).$$

Hence, the results follow from the following equality:

$$\min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u)}{\lambda(B)} = r(B, \hat{I}_B^{Bus}) \frac{2\varepsilon_{d-1}}{\varepsilon_d}. \quad \square$$

COROLLARY 7. Let B be the unit ball of \mathbb{M}^d . Then $\frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u)}{\lambda(B)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d}$ for all $u \in S^{d-1}$ if and only if $\hat{I}_B^{Bus} \subseteq B$.

The following proposition refers to cross-section measures of centered convex bodies.

PROPOSITION 8. Let B be a centered convex body in \mathbb{R}^d . Then

$$a) \max_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u) \leq \max_{u \in S^{d-1}} \lambda_{d-1}(B|u^\perp) \lambda_1(B \cap l_u),$$

$$b) \min_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u) \leq \min_{u \in S^{d-1}} \lambda_{d-1}(B|u^\perp) \lambda_1(B \cap l_u).$$

Proof. a) In [10] it was proved that $R(B, \hat{I}_B^{Bus}) \cdot r(B^\circ, \hat{I}_{B^\circ}^{HT}) \leq 1$. Thus, by using the equalities

$$R(B, \hat{I}_B^{Bus}) = \frac{\varepsilon_d}{2\varepsilon_{d-1}} \max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u)}{\lambda(B)},$$

$$r(B^\circ, \hat{I}_{B^\circ}^{HT}) = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \frac{\lambda(B)}{\max_{u \in S^{d-1}} \lambda_{d-1}(B|u^\perp) \lambda_1(B \cap l_u)},$$

the result follows.

b) In [10] it was also proved that $R(B^\circ, \hat{I}_{B^\circ}^{HT}) \cdot r(B, \hat{I}_B^{Bus}) \leq 1$. Thus, by

$$R(B^\circ, \hat{I}_{B^\circ}^{HT}) = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \min_{u \in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B|u^\perp) \lambda_1(B \cap l_u)}$$

and

$$r(B, \hat{I}_B^{Bus}) = \frac{\varepsilon_d}{2\varepsilon_{d-1}} \frac{\min_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u)}{\lambda(B)},$$

we have the result. \square

4. Inclusion results

One of the challenging open problems of Minkowski Geometry is the question whether for $d \geq 3$ the homothety of the unit ball and the normalized solution of the isoperimetric problem implies that the unit ball must be an ellipsoid (see [3], [14], and [17]). For Minkowski planes, apart from ellipses, other curves (such as Radon curves) have this property as well. Related to this, we clarify now the inclusions between the unit ball B and ΠB° , and between B and $(IB)^\circ$. We start with the inclusions referring to the projection body.

PROPOSITION 9. *If B is a Minkowskian ball of \mathbb{M}^d with $\lambda(B) = 1$, then the following exact inclusions hold:*

$$B^\circ \subseteq 2\Pi B \subseteq dB^\circ.$$

Proof. By the definitions of the inner and outer radii we have

$$r(B, \hat{I}_B^{HT}) \hat{I}_B^{HT} \subseteq B \subseteq R(B, \hat{I}_B^{HT}) \hat{I}_B^{HT}.$$

We recall the following exact bounds for the inner radius and the outer radius:

$$\frac{2\varepsilon_{d-1}}{d\varepsilon_d} \leq r(B, \hat{I}_B^{HT}) \leq 1,$$

$$R(B, \hat{I}_B^{HT}) \leq \frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Therefore, using (1) we get

$$\frac{2\varepsilon_{d-1}}{d\varepsilon_d} \frac{\varepsilon_d}{\lambda(B^\circ)} \frac{\Pi B^\circ}{\varepsilon_{d-1}} \subseteq B \subseteq \frac{2\varepsilon_{d-1}}{\varepsilon_d} \frac{\varepsilon_d}{\lambda(B^\circ)} \frac{\Pi B^\circ}{\varepsilon_{d-1}}.$$

Thus

$$\frac{2}{d} \Pi B^\circ \subseteq \lambda(B^\circ) B \subseteq 2\Pi B^\circ.$$

Hence, the results follow by setting B to be B° , and $\lambda(B) = 1$. \square

Analogously, we obtain for the intersection body

PROPOSITION 10. *Let B be a Minkowskian ball of \mathbb{M}^d with $\lambda(B) = 1$. Then the following exact inclusions hold:*

$$B^\circ \subseteq 2IB \subseteq dB^\circ.$$

Proof. Again we have

$$r(B, \hat{I}_B^{Bus}) \hat{I}_B^{Bus} \subseteq B \subseteq R(B, \hat{I}_B^{Bus}) \hat{I}_B^{Bus}.$$

From the exact bounds

$$\frac{\varepsilon_d}{2\varepsilon_{d-1}} \leq r(B, \hat{I}_B^{Bus}), \quad R(B, \hat{I}_B^{Bus}) \leq \frac{d\varepsilon_d}{2\varepsilon_{d-1}}$$

and (2) we have

$$\frac{\varepsilon_d}{2\varepsilon_{d-1}} \frac{\lambda(B)}{\varepsilon_d} \varepsilon_{d-1} (IB)^\circ \subseteq B \subseteq \frac{d\varepsilon_d}{2\varepsilon_{d-1}} \frac{\lambda(B)}{\varepsilon_d} \varepsilon_{d-1} (IB)^\circ.$$

Thus

$$\frac{\lambda(B)}{2}(IB)^\circ \subseteq B \subseteq \frac{d\lambda(B)}{2}(IB)^\circ.$$

Setting $\lambda(B) = 1$, we have

$$\frac{2}{d}IB \subseteq B^\circ \subseteq 2IB,$$

and the results follow. \square

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(Received February 12, 2018)

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