LYAPUNOV–TYPE INEQUALITIES FOR FRACTIONAL DIFFERENTIAL EQUATIONS UNDER MULTI–POINT BOUNDARY CONDITIONS

YOUYU WANG* AND QICHAO WANG

(*Communicated by H. M. Srivastava*)

Abstract. In this work, we establish new Lyapunov-type inequalities for fractional differential equations under multi-point boundary conditions.

1. Introduction

The well-known result of Lyapunov [9] states that if $u(t)$ is a nontrivial solution of the differential system

$$
\begin{aligned}
u''(t) + r(t)u(t) = 0, \quad t \in (a,b), \\
u(a) = 0 = u(b),
\end{aligned}
\tag{1.1}
$$

where $r(t)$ is a continuous function defined in $[a,b]$, then

$$
\int_a^b |r(t)|dt > \frac{4}{b-a},
\tag{1.2}
$$

and the constant 4 cannot be replaced by a larger number.

Lyapunov inequality (1.2) is a useful tool in various branches of mathematics including disconjugacy, oscillation theory, and eigenvalue problems. Many improvements and generalizations of the inequality (1.2) have appeared in the literature. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey articles by Cheng [3], Brown and Hinton [1] and Tiryaki [12].

The study of Lyapunov-type inequalities for the differential equation depends on a fractional differential operator was initiated by Rui A. C. Ferreira [4]. He first obtained a Lyapunov-type inequality when the differential equation depends on the Riemann-Liouville fractional derivative, the main result is as follows.

THEOREM 1.1. *If the following fractional boundary value problem*

$$(D_{a^+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \leqslant 2, \tag{1.3}$$
$$u(a) = 0 = u(b), \tag{1.4}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1}. \tag{1.5}$$

Recently, some Lyapunov-type inequalities were obtained for different fractional boundary value problems. In this direction, we refer to Ferreira [5], Jleli and Samet [6,7], O'Regan and Samet [10], Rong and Bai [11], Wang, Liang and Xia [13] and Cabrera, Sadarangani, and Samet [2].

For example, Cabrera, Sadarangani, and Samet [2] obtain some Lyapunov-type inequalities for a higher-order nonlocal fractional boundary value problem, they give the following Lyapunov inequalities.

THEOREM 1.2. *If the fractional boundary value problem*

$$(D_{a^+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 2 < \alpha \leqslant 3, \tag{1.6}$$
$$u(a) = u'(a) = 0, \quad u'(b) = \beta u(\xi), \tag{1.7}$$

has a nontrivial solution, where q is a real and continuous function, $a < \xi < b, 0 \leqslant \beta(\xi - a)^{\alpha-1} < (\alpha - 1)(b - a)^{\alpha-2}$, then

$$\int_a^b (b-s)^{\alpha-2}(s-a)|q(s)|ds \geqslant \left(1 + \frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2} - \beta(\xi-a)^{\alpha-1}} \right)^{-1} \Gamma(\alpha). \tag{1.8}$$

THEOREM 1.3. *If the fractional boundary value problem*

$$(D_{a^+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 2 < \alpha \leqslant 3, \tag{1.9}$$
$$u(a) = u'(a) = 0, \quad u'(b) = \beta u(\xi), \tag{1.10}$$

has a nontrivial solution, where q is a real and continuous function, $a < \xi < b, 0 \leqslant \beta(\xi - a)^{\alpha-1} < (\alpha - 1)(b - a)^{\alpha-2}$, then

$$\int_a^b |q(s)|ds \geqslant \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}} \left(1 + \frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2} - \beta(\xi-a)^{\alpha-1}} \right)^{-1}. \tag{1.11}$$

Motivated by [2], in this paper, we study the problem of finding some Lyapunov-type inequalities for the fractional differential equations with multi-point boundary conditions.

$$(D_{a^+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \leqslant 2, \tag{1.12}$$

$$u(a) = 0, \quad (D_{a^+}^{\beta} u)(b) = \sum_{i=1}^{m-2} b_i (D_{a^+}^{\beta} u)(\xi_i), \tag{1.13}$$

where D_{a+}^{α} denotes the standard Riemann-Liouville fractional derivative of order α, $\alpha > \beta + 1$, $0 \leqslant \beta < 1$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $b_i \geqslant 0 (i = 1, 2, \cdots, m-2), 0 \leqslant \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha - \beta - 1} < (b-a)^{\alpha - \beta - 1}$ and $q : [a,b] \to \mathbb{R}$ is a continuous function.

2. Preliminaries

In this section, we recall the concepts of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative of order $\alpha \geqslant 0$.

DEFINITION 2.1. [8] Let $\alpha \geqslant 0$ and f be a real function defined on $[a,b]$. The Riemann-Liouville fractional integral of order α is defined by $(I_{a+}^0 f) \equiv f$ and

$$(I_{a+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha - 1} f(s) ds, \quad \alpha > 0, \ t \in [a,b].$$

DEFINITION 2.2. [8] The Riemann-Liouville fractional derivative of order $\alpha \geqslant 0$ is defined by $(D_{a+}^0 f) \equiv f$ and

$$(D_{a+}^{\alpha} f)(t) = (D^m I_{a+}^{m-\alpha} f)(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} f(s) ds,$$

for $\alpha > 0$, where m is the smallest integer greater or equal to α.

LEMMA 2.3. [8] Assume that $u \in C(a,b) \cap L(a,b)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(a,b) \cap L(a,b)$. Then

$$I_{a+}^{\alpha}(D_{a+}^{\alpha} u)(t) = u(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n},$$

where $c_i \in \mathbb{R}, i = 1, 2, \cdots, n$, and $n = [\alpha] + 1$.

LEMMA 2.4. For $1 < \alpha \leqslant 2, 0 \leqslant \beta < 1$, we have

$$(D_{a+}^{\beta}(s-a)^{\alpha-1})(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1}.$$

3. Main results

We begin by writing problems (1.12)-(1.13) in its equivalent integral form.

LEMMA 3.1. We have that $u \in C[a,b]$ is a solution to the boundary value problem (1.12)-(1.13) if and only if u satisfies the integral equation

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds + T(t) \int_a^b \left(\sum_{i=1}^{m-2} b_i G(\xi,s)q(s)u(s)\right) ds, \qquad (3.1)$$

where Green's function $G(t,s)$ is defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} - (t-s)^{\alpha-1}, & a \leqslant s \leqslant t \leqslant b, \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}, & a \leqslant t \leqslant s \leqslant b. \end{cases}$$

$$T(t) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}, \quad a \leqslant t \leqslant b.$$

Proof. From Lemma 2.3, $u \in C[a,b]$ is a solution to the boundary value problem (1.12)-(1.13) if and only if

$$u(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} - (I_{a^+}^{\alpha} qu)(t),$$

for some real constants c_1, c_2. Using the boundary condition $u(a) = 0$, we obtain $c_2 = 0$. Therefore

$$u(t) = c_1(t-a)^{\alpha-1} - (I_{a^+}^{\alpha} qu)(t).$$

We apply the operator $D_{a^+}^{\beta}$ to both side of above equation, we obtain

$$(D_{a^+}^{\beta} u)(t) = c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1} - (I_{a^+}^{\alpha-\beta} qu)(t)$$

$$= c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1} - \frac{1}{\Gamma(\alpha-\beta)} \int_a^t (t-s)^{\alpha-\beta-1} q(s)u(s)ds,$$

the boundary condition $(D_{a^+}^{\beta} u)(b) = \sum_{i=1}^{m-2} b_i (D_{a^+}^{\beta} u)(\xi_i)$ imply that

$$c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1} - \frac{1}{\Gamma(\alpha-\beta)} \int_a^b (b-s)^{\alpha-\beta-1} q(s)u(s)ds$$

$$= \sum_{i=1}^{m-2} b_i \left[c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\xi_i-a)^{\alpha-\beta-1} - \frac{1}{\Gamma(\alpha-\beta)} \int_a^{\xi_i} (\xi_i-s)^{\alpha-\beta-1} q(s)u(s)ds \right],$$

thus

$$c_1 = \frac{1}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-\beta-1} q(s)u(s)ds$$

$$- \frac{1}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \int_a^{\xi_i} (\xi_i-s)^{\alpha-\beta-1} q(s)u(s)ds.$$

By the relation

$$\frac{1}{(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}$$

$$= \frac{1}{(b-a)^{\alpha-\beta-1}} + \frac{\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]},$$

we obtain

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_a^b \frac{(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} q(s)u(s)ds + \frac{\sum_{i=1}^{m-2} b_i \int_a^b \frac{(\xi_i-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} q(s)u(s)ds}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}$$

$$- \frac{\sum_{i=1}^{m-2} b_i \int_a^{\xi_i} (\xi_i-s)^{\alpha-\beta-1} q(s)u(s)ds}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)},$$

therefore

$$u(t) = c_1(t-a)^{\alpha-1} - (I_{a^+}^{\alpha} qu)(t)$$

$$= \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b \frac{(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} q(s)u(s)ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)u(s)ds$$

$$+ \frac{(t-a)^{\alpha-1} \sum_{i=1}^{m-2} b_i \int_a^b \frac{(\xi_i-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} q(s)u(s)ds}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}$$

$$- \frac{(t-a)^{\alpha-1} \sum_{i=1}^{m-2} b_i \int_a^{\xi_i} (\xi_i-s)^{\alpha-\beta-1} q(s)u(s)ds}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}$$

$$= \int_a^b G(t,s)q(s)u(s)ds + T(t) \int_a^b \left(\sum_{i=1}^{m-2} b_i G(\xi,s)q(s)u(s) \right) ds,$$

which concludes the proof. $\quad\square$

LEMMA 3.2. *The Green's function G defined in Lemma 3.1 satisfies the following properties:*

(**i**) $0 \leqslant G(t,s) \leqslant G(s,s) = \dfrac{(s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}\Gamma(\alpha)},$

(**ii**) *For any $s \in [a,b]$,*

$$\max_{s \in [a,b]} G(s,s) = G(s^*,s^*) = (\alpha-\beta-1)^{\alpha-\beta-1} \frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}{(2\alpha-\beta-2)^{2\alpha-\beta-2}\Gamma(\alpha)},$$

where $s^* = \dfrac{\alpha-\beta-1}{2\alpha-\beta-2}a + \dfrac{\alpha-1}{2\alpha-\beta-2}b$.

Proof. (**i**) Let us define two functions

$$g_1(t,s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} - (t-s)^{\alpha-1}, \ a \leqslant s \leqslant t \leqslant b,$$

$$g_2(t,s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}, \qquad\qquad a \leqslant t \leqslant s \leqslant b.$$

Obviously, $g_2(t,s)$ is an increasing function in t and $0 \leqslant g_2(t,s) \leqslant g_2(s,s)$. Now we turn our attention to the function $g_1(t,s)$. By the relation $\alpha > \beta+1, 2-\alpha \geqslant 0$, we

have $0 < \left(\frac{b-s}{b-a}\right)^{\alpha-\beta-1} \leqslant 1$, $0 < \frac{1}{(t-a)^{2-\alpha}} \leqslant \frac{1}{(t-s)^{2-\alpha}}$, so we obtain

$$\frac{\partial g_1(t,s)}{\partial t} = (\alpha-1)\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-1}\frac{1}{(t-a)^{2-\alpha}} - \frac{1}{(t-s)^{2-\alpha}}\right] \leqslant 0.$$

Hence, for a given $s \in [a,b]$, $g_1(t,s)$ is an non-increasing function of $t \in [s,b]$. Therefore, we have

$$g_1(b,s) \leqslant g_1(t,s) \leqslant g_1(s,s).$$

As

$$\begin{aligned}
g_1(b,s) &= \frac{(b-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} - (b-s)^{\alpha-1}\\
&= (b-a)^{\beta}(b-s)^{\alpha-\beta-1} - (b-s)^{\alpha-1}\\
&= (b-a)^{\beta}(b-s)^{\alpha-1}\left[\frac{1}{(b-s)^{\beta}} - \frac{1}{(b-a)^{\beta}}\right]\\
&\geqslant 0,
\end{aligned}$$

so we get

$$0 \leqslant g_1(t,s) \leqslant g_1(s,s),$$

thus

$$0 \leqslant G(t,s) \leqslant G(s,s).$$

(ii) Let $\varphi(s) = (s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}$, $s \in [a,b]$, then

$$\varphi'(s) = (s-a)^{\alpha-2}(b-s)^{\alpha-\beta-2}[(\alpha-1)(b-s) - (\alpha-\beta-1)(s-a)], \ s \in (a,b),$$

moreover,

$$\varphi'(s) = 0, \ s \in (a,b) \Leftrightarrow s = s^* = \frac{\alpha-\beta-1}{2\alpha-\beta-2}a + \frac{\alpha-1}{2\alpha-\beta-2}b.$$

It is easy to check that $\varphi''(s) < 0, s \in (a,b)$, therefore,

$$\max_{s\in[a,b]} \varphi(s) = \varphi(s^*) = (\alpha-\beta-1)^{\alpha-\beta-1}\frac{(\alpha-1)^{\alpha-1}(b-a)^{2\alpha-\beta-2}}{(2\alpha-\beta-2)^{2\alpha-\beta-2}},$$

hence

$$\max_{s\in[a,b]} G(s,s) = G(s^*,s^*) = (\alpha-\beta-1)^{\alpha-\beta-1}\frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}{(2\alpha-\beta-2)^{2\alpha-\beta-2}\Gamma(\alpha)}. \quad \square$$

Now, we are ready to prove our first Lyapunov-type inequality.

THEOREM 3.3. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D_{a^+}^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \leqslant 2,$$

$$u(a) = 0, \quad (D_{a^+}^\beta u)(b) = \sum_{i=1}^{m-2} b_i (D_{a^+}^\beta u)(\xi_i),$$

exists, then

$$\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}|q(s)|ds$$

$$\geqslant (b-a)^{\alpha-\beta-1} \cdot \frac{(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1} + (b-a)^{\alpha-1}\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}} \Gamma(\alpha).$$

Proof. Let $B = C[a,b]$ be the Banach space endowed with norm $\|u\| = \sup\limits_{t\in[a,b]} |u(t)|$. It follows from Lemma 3.1 that a solution u to the boundary value problem satisfies the integral equation

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds + T(t)\int_a^b \left(\sum_{i=1}^{m-2} b_i G(\xi,s)q(s)u(s)\right)ds.$$

Now, using Lemma 3.2 (i), we obtain

$$\|u\| \leqslant \|u\| \int_a^b |G(s,s)||q(s)|ds + \|u\| \sum_{i=1}^{m-2} b_i T(b) \int_a^b |G(s,s)||q(s)|ds,$$

which yields

$$\|u\| \leqslant \|u\| \int_a^b \left(1 + \sum_{i=1}^{m-2} b_i T(b)\right) G(s,s)|q(s)|ds.$$

Therefore, if u is a nontrivial continuous solution to (1.12)-(1.13), we have

$$\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}|q(s)|ds \geqslant \frac{(b-a)^{\alpha-\beta-1}\Gamma(\alpha)}{1 + \sum_{i=1}^{m-2} b_i T(b)}$$

$$= (b-a)^{\alpha-\beta-1} \cdot \frac{(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1} + (b-a)^{\alpha-1}\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}} \Gamma(\alpha). \quad \square$$

Now, from Theorem 3.3 and Lemma 3.2 (ii), if problem (1.12)-(1.13) has a nontrivial continuous solution, then we have the following result.

COROLLARY 3.4. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D_{a^+}^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \leqslant 2,$$

$$u(a) = 0, \quad (D_{a^+}^\beta u)(b) = \sum_{i=1}^{m-2} b_i (D_{a^+}^\beta u)(\xi_i),$$

exists, then

$$\int_a^b |q(s)|ds$$

$$\geqslant \frac{\Gamma(\alpha)}{[(\alpha-1)(b-a)]^{\alpha-1}} \cdot \frac{(2\alpha-\beta-2)^{2\alpha-\beta-2}}{(\alpha-\beta-1)^{\alpha-\beta-1}}$$

$$\cdot \frac{(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1} + (b-a)^{\alpha-1}\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}.$$

Let $\beta = 0$ in Theorem 3.3, we obtain

COROLLARY 3.5. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D_{a^+}^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leqslant 2,$$

$$u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

exists, then

$$\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-1}|q(s)|ds$$

$$\geqslant (b-a)^{\alpha-1} \cdot \frac{(b-a)^{\alpha-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-1}}{(b-a)^{\alpha-1}(1 + \sum_{i=1}^{m-2} b_i) - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-1}} \Gamma(\alpha).$$

Let $\beta = 0$ in Corollary 3.4, we have the following result.

COROLLARY 3.6. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D_{a^+}^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leqslant 2,$$

$$u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

exists, then

$$\int_a^b |q(s)|ds \geqslant \left(\frac{4}{b-a}\right)^{\alpha-1} \cdot \frac{(b-a)^{\alpha-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-1}}{(b-a)^{\alpha-1}(1 + \sum_{i=1}^{m-2} b_i) - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-1}} \Gamma(\alpha).$$

REMARK 3.7. Let $b_1 = b_2 = \cdots = b_{m-2} = 0$ in Corollary 3.6, then we obtain (1.5).

REFERENCES

[1] R. C. Brown, D. B. Hinton, *Lyapunov inequalities and their applications*, in Survey on Classical Inequalities, T. M. Rassias, Ed. Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000, 1–25.

[2] I. Cabrera, K. Sadarangani and B. Samet, *Hartman-Wintner-type inequalities for a class of nonlocal fractional boundary value problems*, Math. Meth. Appl. Sci., **40**, (2017) 129–136.

[3] S. Cheng, *Lyapunov inequalities for differential and difference equations*, Fasc. Math. **23** (1991) 25–41.

[4] R. A. C. Ferreira, *A Lyapunov-type inequality for a fractional boundary value problem*, Fract. Calc. Appl. Anal. **16**, No 4 (2013), 978–984.

[5] R. A. C. Ferreira, *On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function*, J. Math. Anal. Appl. **412**, No 2 (2014), 1058–1063.

[6] M. Jleli and B. Samet, *Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions*, Math. Inequal. Appl. **18**, No 2 (2015), 443–451.

[7] M. Jleli, L. Ragoub and B. Samet, *Lyapunov-type inequality for a fractional differential equation under a Robin boundary conditions*, J. Func. Spaces. **2015**, Article ID 468536, 5 pages.

[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies 204 Elsevier, Amsterdam, The Netherlands, 2006.

[9] A. M. Lyapunov, *Probleme général de la stabilité du mouvement*, (French Translation of a Russian paper dated 1893), Ann. Fac. Sci. Univ. Toulouse 2 (1907)27-247 (Reprinted as Ann. Math. Studies, No. 17, Princeton Univ. Press, Princeton, NJ, USA, 1947).

[10] D. O'Regan, B. Samet, *Lyapunov-type inequalities for a class of fractional differential equations*, Journal of Inequalities and Applications, **2015** 2015(247):1–10.

[11] J. Rong, C. Bai, *Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions*, Advances in Difference Equations **2015** 2015 (82): 1–10.

[12] A. Tiryaki, *Recent development of Lyapunov-type inequalities*, Adv. Dyn. Syst. Appl., **5** No 2 (2010), 231–248.

[13] Y. Wang, S. Liang and C. Xia, *A Lyapunov-type inequality for a fractional differential equation under Sturm-Liouville boundary conditions*, Math. Inequal. Appl., **20** No 1 (2017), 139–148.

(Received March 14, 2018)

Youyu Wang
Department of Mathematics
Tianjin University of Finance and Economics
Tianjin 300222, P. R. China
e-mail: `wang_youyu@163.com`

Qichao Wang
Department of Mathematics
Tianjin University of Finance and Economics
Tianjin 300222, P. R. China

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com