QUANTUM HERMITE–HADAMARD INEQUALITIES FOR DOUBLE INTEGRAL AND q–DIFFERENTIABLE CONVEX FUNCTIONS

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Abstract. In this paper, we establish some new quantum analogue of Hermite-Hadamard inequalities for double integral and refinements of Hermite-Hadamard inequality for q-differentiable convex functions.

1. Introduction

In mathematics, the quantum calculus is the study of calculus without limits and is sometimes called the q-calculus. In quantum calculus, we obtain q-analogues of mathematical objects that can be recaptured as $q \to 1$. The history of quantum calculus can be traced back to Euler (1707-1783), who first introduced the q-calculus in the tracks of Newton's work of infinite series. In the early twentieth century, Jackson [13] first defined and studied the q-integral in a systematic way. Later, the integral representations of q-gamma and q-beta functions were proposed by De Sole and Kac [5]. In recent years, the topic of q-calculus have been studied by several researchers and variety of new results can be found in the literature [1, 2, 4, 8, 9, 10, 12, 14, 16, 17, 19, 20, 23] and the references cited therein.

In 1893, Hadamard [11] investigated one of the fundamental inequalities in analysis, that is,

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which is now known as Hermite-Hadamard inequality. In 2014, Tariboon and Ntouyas [21] studied the extension to q-calculus of several important integral inequalities, from which they obtained the q-Hölder, q-Hermite-Hadamard, q-trapezoid, q-Ostrowski, q-Cauchy-Bunyakovsky-Schwarz, q-Grüss and q-Grüss-Čebyšev integral inequalities. In 2016, Alp et al. [3] proved the correct q-Hermite-Hadamard inequality, and then obtained some new q-Hermite-Hadamard inequalities and generalized q-Hermite-Hadamard inequalities. Using the left hand part of the correct q-Hermite-Hadamard inequality, they also obtained a new equality. Furthermore, they used the new equality to obtain q-midpoint type integral inequalities through q-differentiable convex and q-differentiable quasi-convex functions.

In 1990, Dragomir [6] gave the following refinements of Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx+(1-t)y)dxdy$$

$$\leqslant \frac{1}{b-a} \int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Since then, many researchers have developed various extensions and refinements of Hermite-Hadamard inequality.

The purpose of this paper is to present the q-calculus of Hermite-Hadamard inequalities for double integrals and refinements of Hermite-Hadamard inequality, obtained as special cases when $q \to 1$.

2. Preliminaries

In this section, we recall some previously known concepts and basic results. Throughout this section, we let $J = [a,b] \subset \mathbb{R}$ be an interval and q be a constant with $0 < q < 1$.

DEFINITION 2.1. Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then the q-derivative of f on J at x is defined as

$$_aD_qf(x) = \frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)}, \text{ for } x \neq a. \tag{2.1}$$

For $x = a$, we define $_aD_qf(a) = \lim_{x \to a} {_aD_qf(x)}$.

A function f is q-differentiable on J if $_aD_qf(x)$ exists for all $x \in J$. Moreover, if $a = 0$ in (2.1), then $_0D_qf = D_qf$, where D_q is the well-known q-derivative of the function $f(x)$, which is defined by

$$D_qf(x) = \frac{f(x)-f(qx)}{(1-q)x}.$$

For more details, see [15].

In addition, we shall define higher-order q-derivatives of functions on J.

DEFINITION 2.2. Let $f : J \to \mathbb{R}$ be a continuous function. The second-order q-derivative of f on J, denote by $_aD_q^2f$ (provided that $_aD_qf$ is q-differentiable on J), is the function from $J \to \mathbb{R}$ defined by

$$_aD_q^2f = {_aD_q}({_aD_qf}).$$

Similarly, provided that $_aD_q^{n-1}f$ is q-differentiable on J for some integer $n > 2$, the nth-order q-derivative of f on J is the function from $J \to \mathbb{R}$ defined by

$$_aD_q^nf = {_aD_q}({_aD_q^{n-1}f}).$$

EXAMPLE 2.1. Define function $f : J \to \mathbb{R}$ by $f(x) = x^2 + 1$. Let $0 < q < 1$. Then for $x \neq a$, we have

$$_aD_q(x^2 + 1) = \frac{(x^2 + 1) - [(qx + (1-q)a)^2 + 1]}{(1-q)(x-a)} = \frac{(1+q)x^2 - 2qax - (1-q)a^2}{(x-a)}$$
$$= (1+q)x + (1-q)a. \tag{2.2}$$

For $x = a$, we have $_aD_qf(a) = \lim_{x \to a} {_aD_qf(x)} = 2a$.

DEFINITION 2.3. Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the q-integral on J is defined by

$$\int_a^x f(t) \, _ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \tag{2.3}$$

for $x \in J$.

If $a = 0$ in (2.3), then we have the classical q-integral of the function $f(x)$, which is defined by

$$\int_0^x f(t) \, _0d_qt = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x)$$

for $x \in [0, \infty)$; see [15] for more details.

EXAMPLE 2.2. Define function $f : J \to \mathbb{R}$ by $f(x) = 2x$. Let $0 < q < 1$. Then we have

$$\int_a^b f(x) \, _ad_qx = \int_a^b 2x \, _ad_qx = 2(1-q)(b-a)\sum_{n=0}^{\infty} q^n(q^n b + (1-q^n)a)$$
$$= \frac{2(b-a)(b+qa)}{1+q}.$$

Note that if $q \to 1$, then we have the classical integration

$$\int_a^b f(x)dx = \int_a^b 2xdx = b^2 - a^2.$$

THEOREM 2.1. Let $f : J \to \mathbb{R}$ be a continuous function. Then we have the following:

(i) $_aD_q \int_a^x f(t) \, _ad_qt = f(x)$;

(ii) $\int_c^x {_aD_qf(t)} \, _ad_qt = f(x) - f(c)$ for $c \in (a,x)$.

THEOREM 2.2. Let $f, g : J \to \mathbb{R}$ be continuous functions and $\alpha \in \mathbb{R}$. Then we have the following:

(i) $\int_a^x [f(t) + g(t)] \, _ad_qt = \int_a^x f(t) \, _ad_qt + \int_a^x g(t) \, _ad_qt$;

(ii) $\int_a^x (\alpha f)(t) \, _a d_q t = \alpha \int_a^x f(t) \, _a d_q t$;

(iii) $\int_c^x f(t) _a D_q g(t) \, _a d_q t = (fg)|_c^x - \int_c^x g(qt + (1-q)a)_a D_q f(t) \, _a d_q t$ for $c \in (a,x)$.

For the proofs of theorem 2.1 and theorem 2.2, see [22].

THEOREM 2.3. Let $f : J \to \mathbb{R}$ be a convex differentiable function on (a,b) and $0 < q < 1$. Then we have

$$f\left(\frac{qa+b}{1+q}\right) \leqslant \frac{1}{b-a} \int_a^b f(x) \, _a d_q x \leqslant \frac{qf(a)+f(b)}{1+q}. \qquad (2.4)$$

THEOREM 2.4. Let $f : J \to \mathbb{R}$ be a convex differentiable function on (a,b) and $0 < q < 1$. Then we have

$$f\left(\frac{a+qb}{1+q}\right) + \frac{(1-q)(b-a)}{1+q} f'\left(\frac{a+qb}{1+q}\right) \leqslant \frac{1}{b-a} \int_a^b f(x) \, _a d_q x \leqslant \frac{qf(a)+f(b)}{1+q}. \qquad (2.5)$$

THEOREM 2.5. Let $f : J \to \mathbb{R}$ be a convex differentiable function on (a,b) and $0 < q < 1$. Then we have

$$f\left(\frac{a+b}{2}\right) + \frac{(1-q)(b-a)}{2(1+q)} f'\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x) \, _a d_q x \leqslant \frac{qf(a)+f(b)}{1+q}. \quad (2.6)$$

For the proof of theorem 2.3, theorem 2.4, and theorem 2.5, see [3].

LEMMA 2.1. Let $f : J \to \mathbb{R}$ be a convex continuous function on J and $0 < q < 1$. Then we have

$$f\left(\frac{1}{(b-a)^2} \int_a^b \int_a^b (tx + (1-t)y) \, _a d_q x _a d_q y\right)$$

$$\leqslant \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y) \, _a d_q x _a d_q y. \qquad (2.7)$$

Proof. The lemma 2.1 follows directly from definition 2.3 and Jensen's inequality. □

3. Main results

In this section, we present the q-Hermite-Hadamard double integral inequality and refinements of q-Hermite-Hadamard inequalities on the interval $J = [a,b]$.

THEOREM 3.1. *Let $f : J \to \mathbb{R}$ be a convex continuous function on J and $0 < q < 1$. Then we have*

$$f\left(\frac{qa+b}{1+q}\right) \leqslant \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx+(1-t)y) \, _a d_q x_a d_q y \leqslant \frac{1}{b-a} \int_a^b f(x)_a d_q x$$

$$\leqslant \frac{qf(a)+f(b)}{1+q} \tag{3.1}$$

for all $t \in [0,1]$.

Proof. Since f is convex on J, it follows that

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y) \tag{3.2}$$

for all $x,y \in J$ and $t \in [0,1]$. Taking double q-integration on both sides of (3.2) on $J \times J$, we obtain

$$\int_a^b \int_a^b f(tx+(1-t)y) \, _a d_q x_a d_q y \leqslant \int_a^b \int_a^b [tf(x)+(1-t)f(y)] \, _a d_q x_a d_q y$$

$$= (b-a) \int_a^b f(x)_a d_q x, \tag{3.3}$$

which proves the second part of (3.1) by using the right hand side of q-Hermite-Hadamard's inequality.

On the other hand, by lemma 2.1, we have

$$f\left(\frac{1}{(b-a)^2} \int_a^b \int_a^b (tx+(1-t)y) \, _a d_q x_a d_q y\right)$$

$$\leqslant \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx+(1-t)y) \, _a d_q x_a d_q y.$$

Since

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b (tx+(1-t)y)_a d_q x_a d_q y = \frac{qa+b}{1+q},$$

this yields the first part of (3.1). $\square$

REMARK 3.1. If $q \to 1$, then (3.1) reduces to (1.2), that is,

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx+(1-t)y)dxdy \leqslant \frac{1}{b-a} \int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}.$$

COROLLARY 3.1. *Let $f : J \to \mathbb{R}$ be a convex continuous function on $[a,b]$ and $0 < q < 1$. Then we have*

$$f\left(\frac{qa+b}{1+q}\right) \leqslant \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) \, _a d_q x_a d_q y \leqslant \frac{1}{b-a} \int_a^b f(x) \, _a d_q x$$

$$\leqslant \frac{qf(a)+f(b)}{1+q}. \tag{3.4}$$

REMARK 3.2. If $q \to 1$, then (3.4) reduces to

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \leqslant \frac{1}{b-a} \int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2},$$

which readily appeared in [18].

THEOREM 3.2. *Let* $f : J \to \mathbb{R}$ *be a convex continuous function on* J *and* $0 < q < 1$. *Then we have*

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+qy}{1+q}\right) {}_a d_q x {}_a d_q y$$

$$\leqslant \frac{1}{(b-a)^2} \int_0^1 \int_a^b \int_a^b f(tx+(1-t)y) {}_a d_q x {}_a d_q y {}_a d_q t \leqslant \frac{1}{b-a} \int_a^b f(x) {}_a d_q x. \quad (3.5)$$

Proof. Consider the mapping $g : J \to \mathbb{R}$ given by

$$g(t) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx+(1-t)y) {}_a d_q x {}_a d_q y.$$

For all $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$, we have

$$g(\alpha t_1 + \beta t_2) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f((\alpha t_1 + \beta t_2)x + (1-(\alpha t_1 + \beta t_2))y) {}_a d_q x {}_a d_q y$$

$$\leqslant \frac{\alpha}{(b-a)^2} \int_a^b \int_a^b f(t_1 x + (1-t_1)y) {}_a d_q x {}_a d_q y$$

$$+ \frac{\beta}{(b-a)^2} \int_a^b \int_a^b f(t_2 x + (1-t_2)y) {}_a d_q x {}_a d_q y$$

$$= \alpha g(t_1) + \beta g(t_2),$$

which proves that g is convex on $[0,1]$. Using theorem 2.3 for the convex function g, we have

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+qy}{1+q}\right) {}_a d_q x {}_a d_q y$$

$$= g\left(\frac{1}{1+q}\right) \leqslant \int_0^1 g(t) {}_a d_q t = \frac{1}{(b-a)^2} \int_0^1 \int_a^b \int_a^b f(tx+(1-t)y) {}_a d_q x {}_a d_q y {}_a d_q t$$

$$\leqslant \frac{qg(0)+g(1)}{1+q} = \frac{1}{b-a} \int_a^b f(x) {}_a d_q x.$$

This completes the proof. $\square$

REMARK 3.3. If $q \to 1$, then (3.5) reduces to

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \leqslant \frac{1}{(b-a)^2} \int_0^1 \int_a^b \int_a^b f(tx+(1-t)y)dxdydt$$

$$\leqslant \frac{1}{b-a} \int_a^b f(x)dx,$$

which readily appeared in [18].

THEOREM 3.3. *Let $f : J \to \mathbb{R}$ be a q-differentiable convex continuous function and $0 < q < 1$. Then the inequalities*

$$0 \leqslant \frac{1}{b-a} \int_a^b f(x) \,_a d_q x - \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y) \,_a d_q x \,_a d_q y$$

$$\leqslant t \left[\frac{f(a) + qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(qx + (1-q)a) \,_a d_q x \right] \qquad (3.6)$$

are valid for all $t \in [0,1]$.

Proof. Since f is convex on J, it follows that

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

for all $x, y \in J$ and $t \in [0,1]$. Taking double q-integration on both sides of the above inequality on $J \times J$, we obtain

$$\int_a^b \int_a^b f(tx + (1-t)y) \,_a d_q x \,_a d_q y \leqslant \int_a^b \int_a^b [tf(x) + (1-t)f(y)] \,_a d_q x \,_a d_q y$$

$$= (b-a) \int_a^b f(x) \,_a d_q x,$$

which yields the first part of (3.6).

On the other hand, since f is q-differentiable convex on J and $f' \geqslant \,_a D_q f$, we have

$$f(tx + (1-t)y) - f(y) \geqslant t(x-y) \,_a D_q f(y)$$

for all $x, y \in J$ and $t \in [0,1]$. Taking double q-integration on both sides of the above inequality on $J \times J$, we obtain

$$\int_a^b \int_a^b f(tx + (1-t)y) \,_a d_q x \,_a d_q y - (b-a) \int_a^b f(x) \,_a d_q x$$

$$\geqslant t \int_a^b \int_a^b (x-y) \,_a D_q f(y) \,_a d_q x \,_a d_q y. \qquad (3.7)$$

Since

$$\int_a^b \int_a^b (x-y) \,_a D_q f(y) \,_a d_q x \,_a d_q y$$

$$= (b-a) \int_a^b f(qx + (1-q)a) \,_a d_q x - (b-a)^2 \frac{f(a) + qf(b)}{1+q},$$

it follows from (3.7) that

$$(b-a)\int_a^b f(x) - \int_a^b \int_a^b f(tx+(1-t)y) \,_a d_q x_a d_q y$$

$$\leqslant t\left[(b-a)^2 \frac{f(a)+qf(b)}{1+q} - (b-a)\int_a^b f(qx+(1-q)a) \,_a d_q x\right]$$

for all $t \in [0,1]$, which is the second part of (3.6).   □

REMARK 3.4. If $q \to 1$, then (3.6) reduces to

$$0 \leqslant \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{(b-a)^2}\int_a^b \int_a^b f(tx+(1-t)y)dxdy$$

$$\leqslant t\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right],$$

which readily appeared in [7, 18].

COROLLARY 3.2. *Let $f : J \to \mathbb{R}$ be a q-differentiable convex continuous function and $0 < q < 1$. Then we have*

$$0 \leqslant \frac{1}{b-a}\int_a^b f(x) \,_a d_q x - \frac{1}{(b-a)^2}\int_a^b \int_a^b f\left(\frac{x+y}{2}\right) \,_a d_q x_a d_q y$$

$$\leqslant \frac{1}{2}\left[\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a) \,_a d_q x\right]. \tag{3.8}$$

REMARK 3.5. If $q \to 1$, then (3.5) reduces to

$$0 \leqslant \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{(b-a)^2}\int_a^b \int_a^b f\left(\frac{x+y}{2}\right)dxdy$$

$$\leqslant \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right],$$

which readily appeared in [18].

THEOREM 3.4. *Let $f : J \to \mathbb{R}$ be a q-differentiable convex continuous function which is define at the point $\frac{qa+b}{1+q} \in (a,b)$ and $0 < q < 1$. Then the inequalities*

$$0 \leqslant \frac{1}{b-a}\int_a^b f(x) \,_a d_q x - \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{qa+b}{1+q}\right) \,_a d_q x$$

$$\leqslant (1-t)\left[\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a) \,_a d_q x\right] \tag{3.9}$$

are valid for all $t \in [0,1]$.

Proof. Since f is convex on J, it follow from theorem 2.3 that

$$\frac{1}{b-a}\int_a^b f(x)\,_ad_qx = \frac{t}{b-a}\int_a^b f(x)\,_ad_qx + \frac{1-t}{b-a}\int_a^b f(x)\,_ad_qx$$

$$\geqslant \frac{t}{b-a}\int_a^b f(x)\,_ad_qx + (1-t)f\left(\frac{qa+b}{1+q}\right)$$

$$\geqslant \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{qa+b}{1+q}\right)\,_ad_qx$$

for all $t \in [0,1]$, which the first part of (3.9).

On the other hand, since f is q-differentiable convex on J, we have

$$f\left(tx+(1-t)\frac{qa+b}{1+q}\right) - f(x) \geqslant (1-t)\left(\frac{qa+b}{1+q}-x\right)\,_aD_qf(x).$$

Taking q-integration on both sides of the above inequality on J, we obtain

$$\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{qa+b}{1+q}\right)\,_ad_qx - \frac{1}{b-a}\int_a^b f(x)\,_ad_qx$$

$$\geqslant \frac{1}{b-a}\int_a^b (1-t)\left(\frac{qa+b}{1+q}-x\right)\,_aD_qf(x)_ad_qx. \tag{3.10}$$

Since

$$\int_a^b \left(\frac{qa+b}{1+q}-x\right)\,_aD_qf(x)_ad_qx = \int_a^b f(qx+(1-q)a)_ad_qx - (b-a)\frac{f(a)+qf(b)}{1+q}. \tag{3.11}$$

Using (3.10) and (3.11), we get the second part of (3.9). $\square$

COROLLARY 3.3. *Let* $f : J \to \mathbb{R}$ *be a q-differentiable convex continuous function and* $0 < q < 1$. *Then we have*

$$0 \leqslant \frac{1}{b-a}\int_a^b f(x)\,_ad_qx - \frac{2}{b-a}\int_{\frac{a(1+2q)+b}{2(1+q)}}^{\frac{qa+b(2+q)}{2(1+q)}} f(x)\,_ad_qx$$

$$\leqslant \frac{1}{2}\left[\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a)\,_ad_qx\right]. \tag{3.12}$$

THEOREM 3.5. *Let* $f : J \to \mathbb{R}$ *be a q-differentiable convex continuous function which is define at the point* $\frac{a+qb}{1+q} \in (a,b)$ *and* $0 < q < 1$. *Then the inequalities*

$$(1-t)\frac{(1-q)(b-a)}{1+q}f'\left(\frac{a+qb}{1+q}\right)$$

$$\leqslant \frac{1}{b-a}\int_a^b f(x)\,_ad_qx - \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+qb}{1+q}\right)\,_ad_qx$$

$$\leqslant (1-t)\left[\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a)\,_ad_qx\right] \tag{3.13}$$

are valid for all $t \in [0,1]$.

Proof. The proof uses theorem 2.4 and is similar to that of theorem 3.4.   □

COROLLARY 3.4. *Let* $f : J \to \mathbb{R}$ *be a q-differentiable convex continuous function and* $0 < q < 1$. *Then we have*

$$\frac{(1-q)(b-a)}{2(1+q)} f'\left(\frac{a+qb}{1+q}\right)$$

$$\leqslant \frac{1}{b-a} \int_a^b f(x) \, _a d_q x - \frac{2}{b-a} \int_{\frac{2a+q(a+b)}{2(1+q)}}^{\frac{a+b(2q+1)}{2(1+q)}} f(x) \, _a d_q x$$

$$\leqslant \frac{1}{2} \left[\frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(qx + (1-q)a) \, _a d_q x \right]. \tag{3.14}$$

THEOREM 3.6. *Let* $f : J \to \mathbb{R}$ *be a q-differentiable convex continuous function which is define at the point* $\frac{a+b}{2} \in (a,b)$ *and* $0 < q < 1$. *Then the inequalities*

$$(1-t)\frac{(1-q)(b-a)}{2(1+q)} f'\left(\frac{a+b}{2}\right)$$

$$\leqslant \frac{1}{b-a} \int_a^b f(x) \, _a d_q x - \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) \, _a d_q x$$

$$\leqslant (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(qx + (1-q)a) \, _a d_q x\right] \tag{3.15}$$

are valid for all $t \in [0,1]$.

Proof. The proof uses theorem 2.5 and is similar to that of theorem 3.4.   □

COROLLARY 3.5. *Let* $f : J \to \mathbb{R}$ *be a q-differentiable convex continuous function and* $0 < q < 1$. *Then we have*

$$\frac{(1-q)(b-a)}{4(1+q)} f'\left(\frac{a+b}{2}\right)$$

$$\leqslant \frac{1}{b-a} \int_a^b f(x)_a d_q x - \frac{2}{b-a} \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)_a d_q x$$

$$\leqslant \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(qx + (1-q)a)_a d_q x\right]. \tag{3.16}$$

REMARK 3.6. If $q \to 1$, then (3.9), (3.13), and (3.15) reduce to

$$0 \leqslant \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx$$

$$\leqslant (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx\right],$$

which readily appeared in [18].

REMARK 3.7. If $q \to 1$, then (3.12), (3.14), and (3.16) reduces to

$$0 \leqslant \frac{1}{b-a} \int_a^b f(x)dx - \frac{2}{b-a} \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)dx \leqslant \frac{1}{2} \left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right],$$

which readily appeared in [18].

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