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A NEW PROOF OF THE ORLICZ–LORENTZ BUSEMANN–PETTY CENTROID INEQUALITY

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Abstract. The Orlicz-Lorentz Busemann-Petty centroid inequality was recently established by Nguyen [16]. In this paper, using shadow systems, a new proof of the Orlicz-Lorentz Busemann-Petty centroid inequality is given.

1. Introduction

The concept of centroid body is one of the central notions in Brunn-Minkowski theory. The classical affine isoperimetric inequality that relates the volume of a convex body with that of its centroid body was conjectured by Blaschke. This conjecture was first proved by Petty [17] which is now known as the *Busemann-Petty centroid inequality*. The L_p centroid body is a natural extension of the centroid body, which was introduced by Lutwak, Yang and Zhang [13]. Using Steiner symmetrization, the L_p Busemann-Petty centroid inequality was established in [13]. Based on the shadow system, an alternative proof of the L_p Busemann-Petty centroid inequality was given by Campi and Gronchi [2]. Using concepts introduced by Ludwig [11], Haberl and Schuster [6] were led to establish asymmetric versions of the L_p Busemann-Petty centroid inequality. In [15], Lutwak, Yang and Zhang introduced the Orlicz centroid body and established Orlicz Busemann-Petty centroid inequality for the convex body with the origin as an interior point. These works initiate an extension of the L_p Brunn-Minkowski theory to an Orlicz-Brunn-Minkowski theory. By extending the method of Lutwak, Yang and Zhang, an extension to star bodies was obtained by Zhu [29]. Another proof of the Orlicz Busemann-Petty centroid inequality was given by Li and Leng [10], who used shadow system. For more information on the L_p and Orlicz Brunn-Minkowski theory see references [1]-[10],[12]-[30].

Very recently, the *Orlicz-Lorentz centroid body* was introduced by Nguyen in [16], which includes Orlicz centroid body as a special case, and the Orlicz-Lorentz Busemann-Petty centroid inequality was also established.

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We consider a strictly convex function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ if $t > 0$, $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. The class of such ϕ will be denoted by $\mathscr{C}$. A function $\omega : I \to (0,\infty)$ is called a *weight function* if ω is nonincreasing function which is locally integrable with respect to the Lebesgue measure μ on $I = (0,|\mu|)$ such that $\int_I \omega(t)dt = \infty$ if $I = (0,\infty)$, here $|\mu|$ denotes the total measure of μ.

Let $K \subset \mathbb{R}^n$ be a convex body that contain the origin in its interior with volume $|K|$. In this paper, we consider the measure space $(K,\mathscr{B}_K,\mu^K)$. Using the Orlicz-Lorentz norm, the support function of Orlicz-Lorentz centroid body $\Gamma_{\phi,\omega}K$ is defined by [16]

$$h_{\Gamma_{\phi,\omega}K}(x) = \inf\left\{\lambda > 0 : \int_0^1 \phi\left(\frac{f^*_{x,K}(t)}{\lambda}\right)\omega(t)dt < 1\right\}, \tag{1.1}$$

where $f_{x,K}$ is defined as $f_{x,K}(y) = x\cdot y$ with $y \in K$, $x\cdot y$ denotes the standard inner product of vectors x and y in $\mathbb{R}^n$ and h_K denotes the support function of K, $f^*_{x,K}$ is the decreasing rearrangement of $f_{x,K}$ (See Section 2 for unexplained terminology and notation).

In particular, when $\omega \equiv 1$, the definition of Orlicz-Lorentz centroid body coincides with the definition of Orlicz centroid body given by Lutwak, Yang and Zhang for even convex function ϕ in $\mathbb{R}$. Note that Lutwak, Yang and Zhang defined the Orlicz centroid body for any convex function $\phi : \mathbb{R} \to [0,\infty)$ such that ϕ is decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$. Their definition is more general than ours in this case. Moreover, when $\phi(t) = t^p$ and $\omega \equiv 1$, we again obtain the defintion of the L_p centroid body given in [13].

In this paper, inspired by the works of Campi and Gronchi [2] and Nguyen [16], we will give another proof of the Orlicz-Lorentz Busemann-Petty centroid inequality.

Orlicz-Lorentz Busemann-Petty centroid inequality. *If $\phi \in \mathscr{C}$, ω is a weight function on (0, 1) and K is a convex body in $\mathbb{R}^n$ containing the origin in its interior, then the volume ratio*

$$\frac{|\Gamma_{\phi,\omega}K|}{|K|}$$

is minimized if and only if K is an origin-centered ellipsoid.

This paper is organized as follows. In Section 2, we collect some basic concepts and various facts of convex bodies. In Section 3, we prove some results which will be used. The proof of Orlicz-Lorentz Busemann-Petty centroid inequality will be given in Section 4.

2. Preliminaries

Good general references for the theory of convex bodies are provided by the book of Schneider [19]. Our setting will be Euclidean n-space $\mathbb{R}^n$. The set of all invertible $n\times n$ matrices will be denoted by $GL(n)$.

A convex body is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. For a convex body K, its support function h_K is defined by

$$h_K(x) = h(K,x) = \max\{x \cdot y : y \in K\} \quad \text{for} \quad x \in \mathbb{R}^n.$$

Let $\mathscr{K}^n$ denote the set of all convex bodies of $\mathbb{R}^n$ and let $\mathscr{K}_0^n$ denote the set of all convex bodies containing the origin in its interior of $\mathbb{R}^n$.

A subset $K \subset \mathbb{R}^n$ is a star-shaped about the origin if for any $x \in K$ then the segment $\{tx : t \in [0,1]\}$ is contained in K. For a star-shaped about the origin K, its radial function $\rho_K : \mathbb{R}\backslash\{0\} \to [0,\infty]$ is defined by

$$\rho_K(x) = \max \lambda > 0 : \lambda x \in K.$$

If ρ_K is strict positive and continuous, then we call K a star body. Let $\mathscr{S}_0^n$ denote the set of all star bodies with respect to the origin in $\mathbb{R}^n$.

Let (Ω, Σ, μ) be a measure space with an σ-finite, non atom measure μ. Given any measurable function $f : \Omega \to \mathbb{R}$, we define the distribution function of f by

$$\mu_f(t) = \mu(\{x : |f(x)| > t\}), \qquad \forall t > 0,$$

and the decreasing rearrangement of f by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leqslant t\}, \qquad t > 0.$$

We denote $I = (0, \mu(\Omega))$. For $\phi \in \mathscr{C}$ and a weight function ω, we define the Orilcz-Lorentz space $\Lambda_{\phi,\omega}$ on (Ω, Σ, μ) to be the set of all measurable functions f on Ω such that

$$\int_I \phi\left(\frac{f^*(t)}{\lambda}\right)\omega(t)dt < \infty,$$

for some $\lambda > 0$. If the function $f \in \Lambda_{\phi,\omega}$, its Orlicz-Lorentz norm is defined by

$$\|f\|_{\Lambda_{\phi,\omega}} = \inf\left\{\lambda > 0 : \int_I \phi\left(\frac{f^*(t)}{\lambda}\right)\omega(t)dt \leqslant 1\right\}. \tag{2.1}$$

Let K be a convex body in $\mathbb{R}^n$. We consider the measure space $(\Omega, \Sigma, \mu) = (K, \mathscr{B}_K, \mu_K)$ here and where $\mathscr{B}_K$ denotes σ-algebra of all Lebesgue measurable subset of K, and μ^K denotes the normalized measure on K whose density is $1_K(x)dx/|K|$ for any Lebesgue measurable $K \subset \mathbb{R}^n$ of positive measure.

A *shadow system* along the unit direction v is a family of convex hulls in $\mathbb{R}^n$,

$$K_t = conv\{z + \alpha(z)tv : \ z \in A \subset \mathbb{R}^n\},$$

where A is an arbitrary bounded set of points, α is a real bounded function on A, and the parameter t runs in an interval of the real axis.

A *parallel chord movement* along the unit direction v, a particular type of a shadow system, is a family of convex bodies K_t in $\mathbb{R}^n$ defined by

$$K_t = \{z + \beta(z|v^{\perp})tv : \ z \in K, \ 0 \leqslant t \leqslant 1\},$$

where K is a convex body in $\mathbb{R}^n$ and β is a continuous real function on $v^{\perp} = \{z \in \mathbb{R}^n : \langle v, z\rangle = 0\}$. Note that $|K_t|$ and the orthogonal projection $K_t|v^{\perp}$ of K_t are independent of t.

For a direction v, define a convex body by

$$K = \{x + sv : x \in K|v^{\perp}, s \in \mathbb{R}, f(x) \leqslant s \leqslant g(x)\}.$$

Then the parallel chord movement with speed function $\beta(x) = -(f(x)+g(x))$ and $t \in [0,1]$ is such that $K_0 = K$, $K_1 = K^v$, the reflection of K in the hyperplane $v^{\perp}$, and $K_{\frac{1}{2}}$ is the Steiner symmetral of K with respect to $v^{\perp}$.

THEOREM A. *Let $H_t, t \in [t_1,t_2]$, be a one-parameter family of convex bodies such that $H_t|v^{\perp}$ is independent of t. Assume the bodies H_t are defined by*

$$H_t = \left\{x + yv : x \in H_t|v^{\perp},\ y \in \mathbb{R},\ f_t(x) \leqslant y \leqslant g_t(x)\right\}, \quad \forall t \in [t_1,t_2],$$

for suitable functions f_t, g_t. Then $H_t, t \in [t_1,t_2]$ is a shadow system of convex sets along the direction v if and only if for every $x \in H_t|v^{\perp}$,

1. *$g_t(x)$ and $f_t(x)$ are convex functions of the parameter t in $[t_1,t_2]$,*
2. *$f_{\mu r+(1-\mu)s}(x) \leqslant \mu g_r(x) + (1-\mu) f_s(x) \leqslant g_{\mu r+(1-\mu)s}(x)$, for every $r,s \in [t_1,t_2]$, $\mu \in [0,1]$.*

3. Proofs of the main results

LEMMA 3.1. *If $\{K_t : 0 \leqslant t \leqslant 1\}$ is a parallel chord movement along the unit direction v, then the orthogonal projection of $\Gamma_{\phi,\omega}K_t$ onto $v^{\perp}$ is independent of t.*

Proof. If $x \in v^{\perp}$, then by the definition of distribution function and the fact $|K_t| = |K_0| = |K|$, we have

$$\begin{aligned}
\mu^{K_t}_{f_{x,K_t}}(s) &= \mu^{K_t}(\{y \in \mathbb{R}^n : |f_{x,K_t}(y)| \geqslant s\}) = \frac{1}{|K_t|}\int_{\{y \in K_t : |\langle x,y\rangle| \geqslant s\}} dy \\
&= \frac{1}{|K|}\int_{\{z \in K : |\langle x,z\rangle + \beta(z|v^{\perp})t\langle x,v\rangle| \geqslant s\}} dz = \frac{1}{|K|}\int_{\{z \in K : |\langle x,z\rangle| \geqslant s\}} dz = \mu^K_{f_{x,K}}(s),
\end{aligned}$$

this means $f^*_{x,K_t}(s) = f^*_{x,K}(s)$ for any $s > 0$. From the definition of Orilcz-Lorentz centroid body (1.1), then

$$h_{\Gamma_{\phi,\omega}K}(x) = \lambda \quad \Leftrightarrow \quad \int_0^1 \phi\left(\frac{f^*_{x,K}(t)}{\lambda}\right)\omega(t)dt = 1.$$

Hence for $x \in v^{\perp}$, then $h_{\Gamma_{\phi,\omega}K_t}(x) = h_{\Gamma_{\phi,\omega}K}(x)$. $\square$

The following lemma shows that $h_{\Gamma_{\phi,\omega}K_t}(x)$ is continuous with respect to t.

LEMMA 3.2. *The support function $h_{\Gamma_{\phi,\omega}K_t}(u)$ is a Lipschitz function of t, that is, for $t_1, t_2 \in [0,1]$ and $x \in \mathbb{R}^n \setminus \{0\}$*

$$|h_{\Gamma_{\phi,\omega}K_{t_1}}(x) - h_{\Gamma_{\phi,\omega}K_{t_2}}(x)| \leqslant |t_1 - t_2| \, \|\beta(\cdot|v^\perp)\langle x, v\rangle\|_{\phi,\omega}.$$

Proof. From the definition of $\|\cdot\|_{\phi,\omega}$, we have

$$\|f_1\|_{\phi,\omega} = \lambda_1 \quad \Longleftrightarrow \quad \int_I \phi\left(\frac{f_1^*(t)}{\lambda_1}\right)\omega(t)dt = 1,$$

and

$$\|f_2\|_{\phi,\omega} = \lambda_2 \quad \Longleftrightarrow \quad \int_I \phi\left(\frac{f_2^*(t)}{\lambda_2}\right)\omega(t)dt = 1.$$

By the fact that $\phi(g^*) = (\phi(|g|))^*$ for any measurable function g, we have

$$\phi\left(\frac{(f_1+f_2)^*(t)}{\lambda_1+\lambda_2}\right) = \left(\phi\left(\frac{|f_1+f_2|}{\lambda_1+\lambda_2}\right)\right)^*(t).$$

From the convexity of the function ϕ, we have

$$\phi\left(\frac{|f_1+f_2|}{\lambda_1+\lambda_2}\right) \leqslant \frac{\lambda_1}{\lambda_1+\lambda_2}\phi\left(\frac{|f_1|}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_1+\lambda_2}\phi\left(\frac{|f_2|}{\lambda_2}\right).$$

Therefore

$$\phi\left(\frac{(f_1+f_2)^*(t)}{\lambda_1+\lambda_2}\right) \leqslant \frac{\lambda_1}{\lambda_1+\lambda_2}\phi\left(\frac{f_1^*(t)}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_1+\lambda_2}\phi\left(\frac{f_2^*(t)}{\lambda_2}\right).$$

Hence

$$\|f_1+f_2\|_{\phi,\omega} \leqslant \lambda_1+\lambda_2 = \|f_1\|_{\phi,\omega} + \|f_2\|_{\phi,\omega}. \tag{3.1}$$

Thus

$$|\|f_1\|_{\phi,\omega} - \|f_2\|_{\phi,\omega}| \leqslant \|f_1 - f_2\|_{\phi,\omega}.$$

From the fact

$$h_{\Gamma_{\phi,\omega}K_t}(x) = \|\langle x,\cdot\rangle + \beta(\cdot|v^\perp)t\langle x,v\rangle\|_{\phi,\omega},$$

we have

$$|h_{\Gamma_{\phi,\omega}K_{t_1}}(x) - h_{\Gamma_{\phi,\omega}K_{t_2}}(x)| \leqslant |t_1 - t_2| \, \|\beta(\cdot|v^\perp)\langle x, v\rangle\|_{\phi,\omega}. \quad \square$$

The convex body $\Gamma_{\phi,\omega}K_t$ can be represented by

$$\Gamma_{\phi,\omega}K_t = \{x + lv : \; x \in \Gamma_{\phi,\omega}K_0|v^\perp, \; f_t(x) \leqslant l \leqslant g_t(x)\},$$

where f_t and $-g_t$ are convex functions defined on $\Gamma_{\phi,\omega}K_0|v^\perp$.

LEMMA 3.3. *If $\{K_t : 0 \leqslant t \leqslant 1\}$ is a parallel chord movement along the unit direction v, then for every $x \in \operatorname{relint} \Gamma_{\phi,\omega} K_0 | v^{\perp}$,*

$$g_t(x) = \inf_{u \in v^{\perp}} \{h_{\Gamma_{\phi,\omega} K_t}(u+v) - \langle x, u \rangle\}, \tag{3.2}$$

and

$$f_t(x) = \sup_{u \in v^{\perp}} \{\langle x, u \rangle - h_{\Gamma_{\phi,\omega} K_t}(u-v)\}. \tag{3.3}$$

Proof. Let $u \in v^{\perp}$. From the definition of the overgraph, it follows immediately that $x + g_t(x) \in \Gamma_{\phi,\omega} K_t$. The definition of the support function shows that

$$\langle x + g_t(x)v, u+v \rangle \leqslant h_{\Gamma_{\phi,\omega} K_t}(u+v).$$

Hence,

$$\langle x, u \rangle + g_t(x) \leqslant h_{\Gamma_{\phi,\omega} K_t}(u+v),$$

for $u \in v^{\perp}$.

Since $\Gamma_{\phi,\omega} K_t$ has support hyperplane at $x + g_t(x)v \in \partial(\Gamma_{\phi,\omega} K_t)$, for $x \in$ *relint* $\Gamma_{\phi,\omega} K_0 | v^{\perp}$, there exists a vector $u' + v$, with $u' \in v^{\perp}$, so that

$$\langle x + g_t(x)v, u' + v \rangle = h_{\Gamma_{\phi,\omega} K_t}(u' + v).$$

Therefore,

$$g_t(x) = \inf_{u \in v^{\perp}} \{h_{\Gamma_{\phi,\omega} K_t}(u+v) - \langle x, u \rangle\}.$$

Formula (3.3) can be shown in the same way. □

THEOREM 3.1. *If $\{K_t : 0 \leqslant t \leqslant 1\}$ is a parallel chord movement along the unit direction v, then $\Gamma_{\phi,\omega} K_t$ is a shadow system along the same direction v.*

Proof. In order to prove the family $\Gamma_{\phi,\omega} K_t$ is a shadow system, we need only prove the functions f_t and g_t satisfy the conditions (1) and (2) in Theorem A.

First, we prove $g_t(x)$ and $-f_t(x)$ are convex functions of t. We just prove the convexity of g_t, and the convexity of $-f_t$ can be proved in the same way. By Lemma

3.3, the triangular inequality of Orlicz-Lorentz norm (3.1), we have

$$\begin{aligned}
2g_{\frac{t_1+t_2}{2}}(x) &= \inf_{u\in v^\perp}\left\{h_{\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}}(2u+2v)-\langle x,2u\rangle\right\}\\
&= \inf_{u\in v^\perp}\left\{\|\langle 2u+2v,\cdot\rangle+\beta(\cdot|v^\perp)(t_1+t_2)\|_{\phi,\omega}-\langle x,2u\rangle\right\}\\
&= \inf_{u_1,u_2\in v^\perp}\left\{\|\langle u_1+u_2+2v,\cdot\rangle+\beta(\cdot|v^\perp)(t_1+t_2)\|_{\phi,\omega}-\langle x,u_1+u_2\rangle\right\}\\
&\leqslant \inf_{u_1,u_2\in v^\perp}\Big\{\|\langle u_1+v,\cdot\rangle+\beta(\cdot|v^\perp)t_1\|_{\phi,\omega}+\|\langle u_2+v,\cdot\rangle+\beta(\cdot|v^\perp)t_2\|_{\phi,\omega}\\
&\quad -x\cdot(u_1+u_2)\Big\}\\
&= \inf_{u_1\in v^\perp}\left\{\|\langle u_1+v,\cdot\rangle+\beta(\cdot|v^\perp)t_1\|_{\phi,\omega}-x\cdot u_1\right\}\\
&\quad +\inf_{u_2\in v^\perp}\left\{\|\langle u_2+v,\cdot\rangle+\beta(\cdot|v^\perp)t_2\|_{\phi,\omega}-x\cdot u_2\right\}\\
&= g_{t_1}(x)+g_{t_2}(x).
\end{aligned}$$

Second, we prove that f_t and g_t satisfy (2) of Theorem A. Let $u_1,u_2\in v^\perp$ and

$$h_{\Gamma_{\phi,\omega}K_{t_1}}(-\mu u_1+\mu v)=\lambda_1,\quad h_{\Gamma_{\phi,\omega}K_{\mu t_1+(1-\mu)t_2}}(u_2-v)=\lambda_2.$$

Then we have

$$\begin{aligned}
(1-\theta)f_{t_2}(x) &= \sup_{u\in v^\perp}\{\langle x,(1-\theta)u\rangle-h_{\Gamma_{\phi,\omega}K_{t_2}}((1-\theta)(u-v))\}\\
&= \sup_{-u_1,u_2\in v^\perp}\{\langle x,u_2-\theta u_1\rangle-h_{\Gamma_{\phi,\omega}K_{t_2}}(u_2-\theta u_1-(1-\theta)v)\}\\
&\geqslant \sup_{-u_1,u_2\in v^\perp}\{\langle x,u_2-\theta u_1\rangle-h_{\Gamma_{\phi,\omega}K_{t_1}}(-\theta u_1+\theta v)\\
&\quad -h_{\Gamma_{\phi,\omega}K_{\theta t_1+(1-\theta)t_2}}(u_2-v)\}\\
&= \sup_{-u_1\in v^\perp}\{\langle x,-\theta u_1\rangle-h_{\Gamma_{\phi,\omega}K_{t_1}}(-\theta u_1+\theta v)\}\\
&\quad +\sup_{u_2\in v^\perp}\{\langle x,u_2\rangle-h_{\Gamma_{\phi,\omega}K_{\theta t_1+(1-\theta)t_2}}(u_2-v)\}\\
&= -\theta g_{t_1}(x)+f_{\theta t_1+(1-\theta)t_2}(x).
\end{aligned}$$

This is the first inequality. The second inequality follows by interchanging t_1 with t_2 and x with $-x$.

Therefore, we deduce $\Gamma_{\phi,\omega}K_t$ is a shadow system along the same direction v. □

THEOREM 3.2. *If $\{K_t : 0\leqslant t\leqslant 1\}$ is a parallel chord movement with speed function β, then the volume of $\Gamma_{\phi,\omega}K_t$ is strictly convex function of t unless β is linear.*

Proof. By Fubini's theorem we have

$$|\Gamma_{\phi,\omega}K_t|=\int_{(\Gamma_{\phi,\omega}K_0)|v^\perp}[g_t(x)-f_t(x)]dx. \tag{3.4}$$

Hence the volume of $\Gamma_{\phi,\omega}K_t$ is a convex function of t follows from the convexity of $g_t(x)$ and $-f_t(x)$ with respect to t. If

$$|\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}| = \frac{1}{2}|\Gamma_{\phi,\omega}K_{t_1}| + \frac{1}{2}|\Gamma_{\phi,\omega}K_{t_2}|$$

for some $t_1,t_2 \in [0,1]$, then we deduce that

$$g_{\frac{t_1+t_2}{2}}(x) - f_{\frac{t_1+t_2}{2}}(x) = \frac{1}{2}(g_{t_1}(x)+g_{t_2}(x)) - \frac{1}{2}(f_{t_1}(x)+f_{t_2}(x)) \tag{3.5}$$

for almost every $x \in \Gamma_{\phi,\omega}K_0$. Take a point x from the interior of $(\Gamma_{\phi,\omega}K_0)|v^{\perp}$. Then there exist $u_1,u_2,u_3,u_4 \in v^{\perp}$ such that

$$\begin{aligned}
&(g_{t_1}(x)+g_{t_2}(x)) - (f_{t_1}(x)+f_{t_2}(x))\\
=&h_{\Gamma_{\phi,\omega}K_{t_1}}(u_1+v) + h_{\Gamma_{\phi,\omega}K_{t_2}}(u_2+v) + h_{\Gamma_{\phi,\omega}K_{t_1}}(u_3-v) + h_{\Gamma_{\phi,\omega}K_{t_2}}(u_4-v)\\
&-\langle x,u_1\rangle - \langle x,u_2\rangle - \langle x,u_3\rangle - \langle x,u_4\rangle\\
=&\|\langle u_1+v,\cdot\rangle + \beta(\cdot|v^{\perp})t_1\|_{\phi,\omega} + \|\langle u_2+v,\cdot\rangle + \beta(\cdot|v^{\perp})t_2\|_{\phi,\omega}\\
&+\|\langle u_3+v,\cdot\rangle - \beta(\cdot|v^{\perp})t_1\|_{\phi,\omega} + \|\langle u_4+v,\cdot\rangle - \beta(\cdot|v^{\perp})t_2\|_{\phi,\omega} - \langle x,u_1\rangle - \langle x,u_2\rangle - \langle x,u_3\rangle\\
&-\langle x,u_4\rangle.
\end{aligned}$$

By the triangular inequality of Orlicz-Lorentz norm we have

$$\begin{aligned}
&(g_{t_1}(x)+g_{t_2}(x)) - (f_{t_1}(x)+f_{t_2}(x))\\
\geqslant&2(\|\langle \frac{u_1+u_2}{2}+v,\cdot\rangle + \beta(\cdot|v^{\perp})\frac{t_1+t_2}{2}\|_{\phi,\omega} + \|\langle \frac{u_3+u_4}{2}+v,\cdot\rangle - \beta(\cdot|v^{\perp})\frac{t_1+t_2}{2}\|_{\phi,\omega}\\
&-\langle x,\frac{u_1+u_2}{2}\rangle - \langle x,\frac{u_3+u_4}{2}\rangle)\\
=&2(h_{\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}}(\frac{u_1+u_2}{2}+v) + h_{\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}}(\frac{u_3+u_4}{2}+v) - \langle x,\frac{u_1+u_2}{2}\rangle - \langle x,\frac{u_3+u_4}{2}\rangle)\\
\geqslant&2(g_{\frac{t_1+t_2}{2}}(x) - f_{\frac{t_1+t_2}{2}}(x)).
\end{aligned}$$

By (3.5) and the equality condition of the triangular inequality, there exists a constant c such that

$$\langle\frac{u_1+u_2}{2}+v,z\rangle + \beta(z|v^{\perp})\frac{t_1+t_2}{2} = c\langle\frac{u_3+u_4}{2}+v,z\rangle + c\beta(z|v^{\perp})\frac{t_3+t_4}{2}, \tag{3.6}$$

for every $z \in K_0$, owing to the continuity of β. Setting $z = z' + \lambda v$ in (3.6), where $z' \in K_0|v^{\perp}$. By differentiating with respect to the parameter λ, it turns out that $c = 1$. Then we conclude that β is a linear function. □

4. Proof of the Orlicz-Lorentz centroid inequality

In order to obatain the Orlicz-Lorentz Busemann-Petty centroid inequality, the following lemmas will be needed.

LEMMA 4.1. (Shephard [20]) *The volume of a shadow system is a convex function of the parameter* t.

LEMMA 4.2. (Nguyen [16]) *Let* $\phi \in \mathscr{C}$ *and* ω *is a weight function on* $(0,1)$. *If* $K \in \mathscr{S}_0^n$ *and* $A \in GL(n)$ *then* $\Gamma_{\phi,\omega}(AK) = A\Gamma_{\phi,\omega}K$.

LEMMA 4.3. (Nguyen [16]) *Let* $\phi \in \mathscr{C}$ *and* ω *is a weight function on* $(0,1)$. *If* $K_i, K \in \mathscr{S}_0^n$ *and* $K_i \to K$, *then* $\Gamma_{\phi,\omega}K_i \to \Gamma_{\phi,\omega}K$ *in* $\mathscr{K}_0^n$.

Proof of the Orlicz-Lorentz centroid inequality.

Proof. Theorem 3.1 and Lemma 4.1 imply that the volume of $\Gamma_{\phi,\omega}K_t$ is a convex function of t. From Lemma 4.2, we have $\Gamma_{\phi,\omega}(K^v) = (\Gamma_{\phi,\omega}K)^v$. Then

$$|\Gamma_{\phi,\omega}K_{\frac{1}{2}}| \leqslant \frac{1}{2}|\Gamma_{\phi,\omega}K_0| + \frac{1}{2}|\Gamma_{\phi,\omega}K_1| = |\Gamma_{\phi,\omega}K|,$$

that is, the volume of the Orlicz-Lorentz centroid body is not increased after a Steiner symmetrization. Lemma 4.3 implies that the ratio $|\Gamma_{\phi,\omega}K|/|K|$ is continuous in the Hausdorff metric. Therefore it attains its minimum value when K is a ball.

If the speed function β of the parallel chord movement is linear, then K_t is a linear image of K, for every t in the range of the movement. If K is not an origin symmetric ellipsoid, it is well known, see [18], that there exists a direction v such that the Steiner symmetral of K along the direction v is not an linear image of K. Therefore, by Theorem 3.2, $|\Gamma_{\phi,\omega}K|/|K|$ is minimized if and only if K is an ellipsoid centered at the origin. The Orlicz-Lorentz Busemann-Petty centroid inequality is proved. □

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