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UPPER AND LOWER BOUNDS FOR THE OPTIMAL CONSTANT IN THE EXTENDED SOBOLEV INEQUALITY. DERIVATION AND NUMERICAL RESULTS

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Abstract. We prove and give numerical results for two lower bounds and eleven upper bounds to the optimal constant $k_0 = k_0(n,\alpha)$ in the inequality

$$\|u\|_{2n/(n-2\alpha)} \leqslant k_0 \|\nabla u\|_2^{\alpha} \|u\|_2^{1-\alpha}, \qquad u \in H^1(\mathbb{R}^n),$$

for $n=1,\ 0<\alpha\leqslant 1/2$, and $n\geqslant 2,\ 0<\alpha<1$.

This constant k_0 is the reciprocal of the infimum $\lambda_{n,\alpha}$ for $u\in H^1(\mathbb{R}^n)$ of the functional

$$\Lambda_{n,\alpha} = \frac{\|\nabla u\|_2^{\alpha} \|u\|_2^{1-\alpha}}{\|u\|_{2n/(n-2\alpha)}}, \qquad u \in H^1(\mathbb{R}^n),$$

where for $n=1,\ 0<\alpha\leqslant 1/2$, and for $n\geqslant 2,\ 0<\alpha<1$.

The lowest point in the point spectrum of the Schrödinger operator $\tau = -\Delta + q$ on $\mathbb{R}^n$ with the real-valued potential q can be expressed in $\lambda_{n,\alpha}$ for all $q_- = \max(0,-q) \in L^p(\mathbb{R}^n)$, for $n=1,\ 1\leqslant p<\infty$, and $n\geqslant 2,\ n/2<p<\infty$, and the norm $\|q_-\|_p$.

1. Introduction

Here, we present the derivations and the results of some numerical evaluations for the optimal constant $k_0 = k_0(n,\alpha)$ in the estimate

$$\|u\|_{2n/(n-2\alpha)} \leqslant k_0 \|\nabla u\|_2^{\alpha} \|u\|_2^{1-\alpha}, \qquad u \in H^1(\mathbb{R}^n), \tag{1}$$

for $n=1,\ 0<\alpha\leqslant 1/2$, and $n\geqslant 2,\ 0<\alpha<1$.

For $n=1$, k_0 is known explicitly (see [1], [2], [3] and [4, Lemma 2.1, (2.4)])

$$k_0(1,\alpha) = 2^{\alpha}\alpha^{\alpha/2}(1-\alpha)^{-(1-\alpha)/2}(1-2\alpha)^{(1-2\alpha)/2} B\left(\frac{1}{2},\frac{1}{2\alpha}\right)^{-\alpha}, \tag{2}$$

for $0<\alpha<1/2$, and $k_0(1,1/2)=1$,

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where $B(p,q)$ is the Beta Function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad \Re p > 0, \quad \Re q > 0. \tag{3}$$

For $n \geqslant 2$, a number of authors has dealt with estimates for $k_0(n,\alpha)$ for some specific values or in a general sense: [5], [6], [7], [8], [9], [10], [11], [4], [12], [13], [14], [15].

The value k_0 equals the reciprocal value of the infimum $\lambda_{n,\alpha}$ of the functional $\Lambda_{n,\alpha}$:

$$\lambda_{n,\alpha} = \inf_{u\in H^1(\mathbb{R}^n)} \Lambda_{n,\alpha}, \quad \text{with} \tag{4}$$

$$\Lambda_{n,\alpha} = \frac{\|\nabla u\|_2^{\alpha}\, \|u\|_2^{1-\alpha}}{\|u\|_{2n/(n-2\alpha)}}, \quad u \in H^1(\mathbb{R}^n), \tag{5}$$

$$\text{where } 0 < \alpha \leqslant 1/2 \quad \text{if } n = 1, \quad \text{and } 0 < \alpha < 1 \quad \text{if } n \geqslant 2.$$

One of the motivations to study this functional comes from the fact that the lowest point in the point spectrum of the Schrödinger operator can be expressed by the infimum $\lambda_{n,\alpha}$ of this functional $\Lambda_{n,\alpha}$. So, for the Schrödinger operator $\tau = -\Delta + q$ on $\mathbb{R}^n$ with the real-valued potential q such that $q = q_+ - q_-$, where

$$q_+ = \max(0,q) \in L^2_{loc}(\mathbb{R}^n), \tag{6}$$

$$q_- = \max(0,-q) \in L^p(\mathbb{R}^n), \qquad \begin{aligned} n = 1: &\quad 1 \leqslant p < \infty, \\ n \geqslant 2: &\quad n/2 < p < \infty. \end{aligned} \tag{7}$$

the lowest point in the point spectrum for all such q expressed as

$$l(n,\alpha) = \inf_{q_- \in L^p(\mathbb{R}^n)} \inf_{u\in H^1(\mathbb{R}^n)} \frac{\|\nabla u\|_2^2 + \int_{\mathbb{R}^n} q\,|u|^2\,dx}{\|u\|_2^2} \|q_-\|_p^{-1/(1-\alpha)}, \tag{8}$$

$$\text{with } \alpha = n/(2p),$$

will be

$$l(n,\alpha) = -(1-\alpha)\alpha^{\alpha/(1-\alpha)}\lambda_{n,\alpha}^{-2/(1-\alpha)}, \begin{aligned} &0 < \alpha \leqslant 1/2 \text{ if } n = 1, \\ &0 < \alpha < 1 \text{ if } n \geqslant 2, \end{aligned} \tag{9}$$

see among others [10], [4].

The corresponding Euler equation belonging to the infimum $\lambda_{n,\alpha}$ of the functional $\Lambda_{n,\alpha}(u)$ reads

$$-\alpha\frac{\Delta u}{\|\nabla u\|_2^2} + (1-\alpha)\frac{u}{\|u\|_2^2} - \frac{u|u|^{\rho}}{\|u\|_{\rho+2}^{\rho+2}} = 0, \tag{10}$$

$$\text{with } \rho = \frac{4\alpha}{(n-2\alpha)}, \qquad \alpha = \frac{\rho n}{2(\rho+2)},$$

which can be scaled in the form (see [10], [4])

$$-\frac{d^2}{dr^2}u-\frac{(n-1)}{r}\frac{d}{dr}u-u|u|^{\rho}+u=0,\ r=|x|>0,$$
$$\frac{d}{dr}u(0)=0,\ \lim_{r\to\infty}u(r)=0. \tag{11}$$

We have used a scaling such that

$$\alpha\|u\|_2^2=(1-\alpha)\|\nabla u\|_2^2=\alpha(1-\alpha)\|u\|_{\rho+2}^{\rho+2}, \tag{12}$$

which is always possible by scaling the function and the argument. And the infimum $\lambda_{n,\alpha}$ will then be found as (with $\overline{u}_{n,\alpha}$ the unique positive (see [16]) solution of (11))

$$\frac{1}{k_0(n,\alpha)}=\lambda_{n,\alpha}=\alpha^{\alpha/2}(1-\alpha)^{(n(1-\alpha)-2\alpha)/(2n)}\left[\|\overline{u}_{n,\alpha}\|_2^2\right]^{\alpha/n}=\chi(\alpha)\left(\frac{\|\overline{u}_{n,\alpha}\|_2^2}{1-\alpha}\right)^{\alpha/n},$$
$$\text{for } 0<\alpha<1,\ n\geqslant 2, \tag{13}$$
$$\text{with } \chi(\alpha)=\sqrt{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}. \tag{14}$$

The values $k_0(n,\alpha)$ for $\alpha=1$ is covered by the special form of the Sobolev embedding

$$\|w\|_t\leqslant\frac{1}{C_T(n,s)}\|\nabla w\|_s,\ t=sn/(n-s),\ 1\leqslant s<n,\ w\in H^{1,s}(\mathbb{R}^n), \tag{15}$$

where $C_T(n,s)$ is the optimal constant and

$$H^{1,s}(\mathbb{R}^n)=\text{completion of }\{w\mid w\in C^1(\mathbb{R}^n),\|u\|_{1,s}^s=\|u\|_s^s+\|\nabla u\|_s^s<\infty\}$$
$$\text{with respect to the norm }\|\cdot\|_{1,s}. \tag{16}$$

If we take $\alpha=1$ and $s=2$ in (1), we have $k_0(n,1)=1/\lambda_{n,1}=1/C_T(n,2)$, $n\geqslant 3$. Since $H^1(\mathbb{R}^2)\not\hookrightarrow L^\infty(\mathbb{R}^2)$, it follows that $\lambda_{2,1}=C_T(2,2)=0$, and so $k_0(2,1)$ is not defined. The numbers $C_T(n,s)$ are known explicitly by the work of [17] and [18], see also [19]

$$C_T(n,s)=n^{1/s}\left(\frac{n-s}{s-1}\right)^{(s-1)/s}\left[\sigma_n B\left(\frac{n}{s},n+1-\frac{n}{s}\right)\right]^{1/n},\ 1<s<n, \tag{17}$$
$$C_T(n,1)=n\omega_n^{1/n},\ n\geqslant 2, \tag{18}$$

where σ_n the surface area of the unit ball in $\mathbb{R}^n$, ω_n the volume of the unit ball in $\mathbb{R}^n$

$$\omega_n=\pi^{n/2}/\Gamma(1+n/2), \tag{19}$$
$$\sigma_n=n\omega_n=2\pi^{n/2}/\Gamma(n/2), \tag{20}$$
$$B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b),\ a,b>0, \tag{21}$$

and there is equality in (15) for functions of the form

$$w_{n,s}(x_1,\ldots,x_n) = \left\{a + b|x|^{s/(s-1)}\right\}^{1-n/s}, \; a,b > 0, \; 1 < s < n. \tag{22}$$

From now on, we concentrate on the optimal constant $k_0(n,\alpha)$. Firstly, we list a number of estimates, two lower bounds and eleven different upper bounds for $k_0(n,\alpha)$ with references if published. Thereafter, we proof the estimates also for the published bounds.

2. Lower bounds

2.1. Lower bound 1

$$k_0 > \underline{k_0}(\alpha) = \left[\frac{\alpha^\alpha}{\pi^\alpha e^\alpha (1-\alpha)^\alpha \left[\ln\left(\frac{1}{1-\alpha}\right)\right]^\alpha}\right]^{1/2}, \qquad n=2, \quad 0<\alpha<1. \tag{23}$$

2.2. Lower bound 2

$$k_0 > \underline{\underline{k_0}}(n,\alpha) = \left[\frac{1}{n^n}\left(\frac{2}{\pi}\right)^{2\alpha}(n-2\alpha)^{n-2\alpha}\right]^{1/4}, \qquad n \geqslant 2, \quad 0<\alpha<1. \tag{24}$$

3. Upper bounds

3.1. Upper bound 1

$$k_0 < \overline{k_0}(n,\alpha) = \frac{1}{\chi(\alpha)}\left[\frac{\sigma_n}{2}B\left(\frac{n}{2},\frac{n(1-\alpha)}{2\alpha}\right)\right]^{\alpha/n} k_B\left(\frac{2n}{n+2\alpha}\right), \tag{25}$$

for $n \geqslant 2, \quad 0<\alpha<1,$

with $\chi(\alpha)$ defined in (14), σ_n defined in (20),
with $B(p,q)$ defined in (3),
and with

$$k_B(p) = \left[\left(\frac{p}{2\pi}\right)^{1/p}\left(\frac{p'}{2\pi}\right)^{-1/p'}\right]^{n/2}, \qquad \frac{1}{p}+\frac{1}{p'}=1. \tag{26}$$

See [10, Theorem 1], [12, Proposition 1] and [15, Theorem 1]. Remark that

$$n=2, \qquad B\left(1,\frac{1-\alpha}{\alpha}\right) = \frac{\alpha}{1-\alpha}.$$

3.2. Upper bound 2

$$k_0 < \overline{\overline{k_0}}(n,\alpha) = \frac{1}{\chi(\alpha)} \left[k_B \left(\frac{n}{n-2\alpha} \right) k_B^2 \left(\frac{2n}{n+2\alpha} \right) \|G(x)\|_{n/(n-2\alpha)} \right]^{1/2}, \quad (27)$$

for $n \geqslant 2,\ 0 < \alpha < 1$,

with $\chi(\alpha)$ defined in (14), $k_B(p)$ defined in (26),

and with $G(x) = \dfrac{K_{(n-2)/2}(|x|)}{|x|^{(n-2)/2}}$, $\quad K_\nu$ is the modified Bessel function. (28)

See [10, Theorem 2] and [15].

Remark that for $n = 2,\ \alpha = 1/2$

$$\|G(x)\|_2 = \left(2\pi \int_0^\infty K_0^2(r) r dr \right)^{1/2} = \pi^{1/2},$$

and for $n = 3$, and general α

$$\|G(x)\|_{3/(3-2\alpha)} = \sqrt{\frac{\pi}{2}} (4\pi)^{(3-2\alpha)/3} \left(\frac{3-2\alpha}{3} \right)^{2-2\alpha} \left[\Gamma\left(\frac{6-6\alpha}{3-2\alpha} \right) \right]^{(3-2\alpha)/3},$$

because $K_{1/2}(x) = \sqrt{\dfrac{\pi}{2x}} \exp(-x)$.

3.3. Upper bound 3

$$k_0 < \overline{\overline{\overline{k_0}}}(n,\alpha) = \frac{1}{\chi(\alpha)} \frac{1}{\sqrt{(1-\alpha)}} k_B \left(\frac{n}{n-\alpha} \right) k_B \left(\frac{2n}{n+2\alpha} \right) \times \|G(x)\|_{n/(n-\alpha)}, \qquad \text{for } n \geqslant 2,\ 0 < \alpha < 1, \quad (29)$$

with $\chi(\alpha)$ defined in (14), with $k_B(p)$ defined in (26), and with $G(x)$ defined in (28).

3.4. Upper bound 4

$$k_0 < \overline{k_{D,1}}(n,\alpha) = A(n,\alpha)^\gamma, \qquad n \geqslant 2, \quad 0 < \alpha < 1, \quad (30)$$

$$\text{with } A(n,\alpha) = \left[\frac{2\alpha(n-\alpha)}{\pi n (n-2\alpha)^2} \right]^{\theta/2} \left[1 - \frac{n\alpha}{2(n-\alpha)} \right]^{(n-2\alpha)/(2n)} \quad (31)$$

$$\times \left[\frac{\Gamma\left(\frac{n}{\alpha}-1\right)}{\Gamma\left(\frac{n}{\alpha}-1-\frac{n}{2}\right)} \right]^{\theta/n},$$

$$\text{and with } \theta = \frac{\alpha(n-2\alpha)}{2n-2\alpha-\alpha n}, \qquad \gamma = \frac{2n-2\alpha-\alpha n}{n-2\alpha}. \tag{32}$$

3.5. Upper bound 5

$$k_0 < \overline{k_{D,2}}(n,\alpha) = A(n,\alpha)^{\alpha}\overline{k_0}(n,\alpha)^{1-\theta}, \qquad n \geqslant 2, \quad 0 < \alpha < 1, \tag{33}$$

with $A(n,\alpha)$ defined in (31), $\overline{k_0}(n,\alpha)$ defined in (25),

and with $\theta = \dfrac{\alpha(n-2\alpha)}{2n-2\alpha-\alpha n}$, defined in (32).

Compare [4, Theorem 1.7 (1.30)].

3.6. Upper bound 6

$$k_0 < \overline{k_{D,3}}(n,\alpha) = A(n,\alpha)^{\alpha}\overline{\overline{k_0}}(n,\alpha)^{1-\theta}, \qquad n \geqslant 2, \quad 0 < \alpha < 1, \tag{34}$$

with $A(n,\alpha)$ defined in (31), θ defined in (32),

and with $\overline{\overline{k_0}}(n,\alpha)$ defined in (27).

Compare [4, Theorem 1.7 (1.30)].

3.7. Upper bound 7

$$k_0 < \overline{k_{I,1}}(n,\alpha) = 1/k_{V,1}(n,\alpha), \qquad n \geqslant 3, \quad 1/2 < \alpha < 1, \tag{35}$$

$$\text{with } k_{V,1}(n,\alpha) = \overline{k_0}\left(n, \frac{1}{2}\right)^{-\alpha_1} k_T(n)^{-(1-\alpha_1)}, \qquad \alpha_1 = 2(1-\alpha), \tag{36}$$

with $\overline{k_0}(n,\alpha)$ defined in (25),

$$\text{and with } k_T(n) = \frac{1}{C_T(n,2)} = \frac{1}{\sqrt{\pi n(n-2)}} \left[\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} \right]^{1/n}, \tag{37}$$

where $C_T(n,2)$ is defined in (17).

See [4, Theorem 1.7, (1.30), $\theta' = 1/2$, $\theta'' = 1$, with the restriction $n \geqslant 3$].

3.8. Upper bound 8

$$k_0 < \overline{k_{I,2}}(n,\alpha) = 1/k_{V,2}(n,\alpha), \qquad n \geqslant 3, \quad \alpha_V < \alpha < 1, \tag{38}$$

$$\text{with } k_{V,2}(n,\alpha) = \overline{k_0}(n,\alpha_V)^{-\alpha_2} k_T(n)^{-(1-\alpha_2)}, \tag{39}$$

with $\overline{k_0}(n,\alpha)$ defined in (25), $k_T(n)$ defined in (37),

$$\text{and with } \alpha_2 = \frac{1-\alpha}{1-\alpha_V}, \tag{40}$$

where α_V follows from

$$\alpha_V = \alpha_V(n) = \frac{n}{2p_V}, \text{ where } p_V \text{ is the solution of} \tag{41}$$

$$\ln\left(\frac{n-p}{p-1}\right) + \frac{n-p}{p(p-1)} + \psi(p) - \psi(n+1-p) = 0, \tag{42}$$

$$\psi(x) = \frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)}, \qquad x > 0, \qquad 1 < p < n, \quad n \geqslant 2.$$

See [4, Theorem 1.7 (1.30), $\theta' = \theta_N$ $(= \alpha_V)$, $\theta'' = 1$, with the restriction $n \geqslant 3$]. See Section 5.3 for numerical values of $\alpha_V(n)$, $n = 2, \cdots, 10$.

3.9. Upper bound 9

$$k_0 < \overline{k_{I,3}}(n,\alpha) = 1/k_{V,3}(n,\alpha), \qquad n \geqslant 3, \quad \alpha_V < \alpha < 1, \tag{43}$$

with α_V defined in (41),

$$\text{with } k_{V,3}(n,\alpha) = k_{L,V}(n,\alpha_V)^{\alpha_2} k_T(n)^{-(1-\alpha_2)}, \ \alpha_2 \text{ defined in (40)}, \tag{44}$$

$$\text{with } k_{L,V}(n,\alpha) = [\alpha C_T(n,2\alpha)]^{\alpha}, \tag{45}$$

with $C_T(n,s)$ defined in (17), that is

$$C_T(n,s) = n^{1/s}\left(\frac{n-s}{s-1}\right)^{(s-1)/s}\left[\sigma_n B\left(\frac{n}{s}, n+1-\frac{n}{s}\right)\right]^{1/n}, \qquad 1 < s < n,$$

and with $k_T(n)$ defined in (37), $k_T(n) = 1/C_T(n,2)$.

Compare [4, Theorem 1.7 (1.30) and (1.32), $\theta' = \theta_N$ $(= \alpha_V)$, $\theta'' = 1$, with the restriction $n \geqslant 3$].

3.10. Upper bound 10

$$k_0 < \overline{k_{L,V}}(n,\alpha) = [\alpha_V C_T(n,2\alpha_V)]^{-\alpha}, \tag{46}$$

$$\begin{aligned}&n \geqslant 2, \quad 0 < \alpha \leqslant \alpha_V,\\ &k_0 < \overline{k_{L,V}}(n,\alpha) = 1/k_{L,V}(n,\alpha) = [\alpha C_T(n,2\alpha)]^{-\alpha},\\ &n \geqslant 2, \quad \alpha_V \leqslant \alpha < 1,\end{aligned} \tag{47}$$

with α_V defined in (41), $C_T(n,s)$ defined in (17).

See [4, Theorem 1.7, (1.32)].

3.11. Upper bound 11

$$k_0 < \overline{k_B}(n,\alpha) = k_T(n)^{\alpha}, \qquad n \geqslant 3, \quad 0 < \alpha < 1, \tag{48}$$

with $k_T(n)$ defined in (37).

See [4, Theorem 1.7 (1.33), $\theta' = 0,\ \theta'' = 1$, with the restriction $n \geqslant 3$].

4. Proofs

4.1. Lower bounds

We take as trial function in (5) the function

$$u_{n,\alpha} = a\exp(-br^{\mu}), \qquad a,b,\mu > 0. \tag{49}$$

We need the following general integral (see [20, (5.9.1)])

$$\int_0^{\infty} \exp(-mr^{\mu})r^{\nu-1}dr = \frac{1}{\mu}\left(\frac{1}{m}\right)^{\nu/\mu}\Gamma\left(\frac{\nu}{\mu}\right). \tag{50}$$

For this trial function the following three integrals become ($\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$, the surface area of the unit ball in $\mathbb{R}^n$, see (20))

$$\int_{\mathbb{R}^n} u_{n,\alpha}^2(x)dx = \sigma_n \int_0^{\infty} a^2 e^{-2br^{\mu}} r^{n-1}dr = \sigma_n a^2 \frac{1}{\mu}\left(\frac{1}{2b}\right)^{n/\mu}\Gamma\left(\frac{n}{\mu}\right), \tag{51}$$

$$\begin{aligned}\int_{\mathbb{R}^n} (\nabla u_{n,\alpha}(x))^2 dx &= \sigma_n \int_0^{\infty} a^2 b^2 \mu^2 r^{2(\mu-1)} e^{-2br^{\mu}} r^{n-1}dr\\ &= \sigma_n a^2 \frac{\mu}{4}\left(\frac{1}{2b}\right)^{(n-2)/\mu}\Gamma\left(2+\frac{n-2}{\mu}\right),\end{aligned} \tag{52}$$

$$\begin{aligned}\int_{\mathbb{R}^n} u_{n,\alpha}^{\rho+2}(x)dx &= \sigma_n \int_0^{\infty} a^{\rho+2} e^{-(\rho+2)br^{\mu}} r^{n-1}dr\\ &= \sigma_n a^{\rho+2} \frac{1}{\mu}\left(\frac{1}{(\rho+2)b}\right)^{n/\mu}\Gamma\left(\frac{n}{\mu}\right).\end{aligned} \tag{53}$$

4.2. Lower bound 1

For $n=2$, and general μ the three integrals (51), (52) and (53) become

$$\int_{\mathbb{R}^2} u_{2,\alpha}^2(x)dx = 2\pi\int_0^\infty a^2 e^{-2br^\mu} r dr = \sigma_2 a^2 \frac{1}{\mu}\left(\frac{1}{2b}\right)^{2/\mu}\Gamma\left(\frac{2}{\mu}\right), \tag{54}$$

$$\int_{\mathbb{R}^2} (\nabla u_{2,\alpha}(x))^2 dx = 2\pi\int_0^\infty a^2 b^2 \mu^2 r^{2(\mu-1)} e^{-2br^\mu} r dr = \sigma_2 a^2 \frac{\mu}{4}\Gamma(2), \tag{55}$$

$$\int_{\mathbb{R}^2} u_{2,\alpha}^{\rho+2}(x)dx = 2\pi\int_0^\infty a^{\rho+2} e^{-(\rho+2)br^\mu} r dr \tag{56}$$

$$= \sigma_2 a^{\rho+2}\frac{1}{\mu}\left(\frac{1}{(\rho+2)b}\right)^{2/\mu}\Gamma\left(\frac{2}{\mu}\right).$$

Let a,b be variable and μ fixed, we use the two scaling relations (12)

$$\alpha\sigma_2 a^2 \frac{1}{\mu}\left(\frac{1}{2b}\right)^{2/\mu}\Gamma\left(\frac{2}{\mu}\right) = (1-\alpha)\sigma_2 a^2\frac{\mu}{4}\Gamma(2), \tag{57}$$

$$\sigma_2 a^2 \frac{1}{\mu}\left(\frac{1}{2b}\right)^{2/\mu}\Gamma\left(\frac{2}{\mu}\right) = (1-\alpha)\sigma_2 a^{\rho+2}\frac{1}{\mu}\left(\frac{1}{(\rho+2)b}\right)^{2/\mu}\Gamma\left(\frac{2}{\mu}\right). \tag{58}$$

This gives for the optimal values for $(a,b)=(a_0,b_0)$

$$a^\rho = a_0^\rho = \left(\frac{\rho+2}{2}\right)^{\frac{\mu+2}{\mu}},\ b^{2/\mu} = b_0^{2/\mu} = \frac{2\rho\Gamma\left(\frac{2}{\mu}\right)}{\mu^2 2^{2/\mu}}.$$

$$k_0(2,\alpha) = \frac{1}{\chi(\alpha)}\left(\frac{1-\alpha}{\|\overline{u}_{2,\alpha}\|_2^2}\right)^{\alpha/2} \tag{59}$$

$$> \underline{k_0}(\alpha) = \frac{1}{\chi(\alpha)}\left\{\frac{(1-\alpha)2\rho}{2\pi\left[\mu^{\rho/2}\left(\frac{\rho}{2}+1\right)^{1+2/\mu}\right]^{2/\rho}}\right\}^{\alpha/2}.$$

Consider now μ as variable to minimize $\underline{k_0}(\alpha)$ by maximizing the denominator

$$\max_{0<\mu<\infty}\left[\mu^{\rho/2}\left(\frac{\rho}{2}+1\right)^{1+2/\mu}\right] = \left[\frac{2e\ln(1+\rho/2)}{\rho/2}\right]^{\rho/2}(1+\rho/2),$$

for $\mu_0 = \dfrac{2\ln(1+\rho/2)}{\rho/2}$.

This gives for (59)

$$\underline{k_0}(\alpha) = \frac{1}{\chi(\alpha)}\left\{\frac{2(1-\alpha)(\rho/2)^2}{2\pi e\ln(1+\rho/2)(1+\rho/2)^{2/\rho}}\right\}^{\alpha/2}$$

$$= \left[\frac{\alpha^{\alpha}}{\pi^{\alpha} e^{\alpha} (1-\alpha)^{\alpha} \left[\ln\left(\frac{1}{1-\alpha}\right)\right]^{\alpha}} \right]^{1/2}, \tag{60}$$

which equals (23).

4.3. Lower bound 2

For general n and $\mu = 2$ the three integrals (51), (52) and (53) become

$$\int_{\mathbb{R}^n} u_{n,\alpha}^2(x)dx = \sigma_n \int_0^{\infty} a^2 \exp(-2br^2) r^{n-1} dr = \sigma_n a^2 \frac{1}{2} \left(\frac{1}{2b}\right)^{n/2} \Gamma\left(\frac{n}{2}\right), \tag{61}$$

$$\int_{\mathbb{R}^n} (\nabla u_{n,\alpha}(x))^2 dx = \sigma_n \int_0^{\infty} a^2 b^2 4r^2 \exp(-2br^2) r^{n-1} dr \tag{62}$$

$$= \sigma_n a^2 \frac{1}{2} \left(\frac{1}{2b}\right)^{(n-2)/2} \Gamma\left(1+\frac{n}{2}\right),$$

$$\int_{\mathbb{R}^n} u_{n,\alpha}^{\rho+2}(x)dx = \sigma_n \int_0^{\infty} a^2 \exp(-(\rho+2)r^2) r^{n-1} dr \tag{63}$$

$$= \sigma_n a^{\rho+2} \frac{1}{2} \left(\frac{1}{(\rho+2)b}\right)^{n/2} \Gamma\left(\frac{n}{2}\right).$$

Using the two scaling relations (12)

$$\alpha \sigma_n a^2 \frac{1}{2} \left(\frac{1}{2b}\right)^{n/2} \Gamma\left(\frac{n}{2}\right) = (1-\alpha) \sigma_n a^2 \frac{1}{2} \left(\frac{1}{2b}\right)^{(n-2)/2} \Gamma\left(1+\frac{n}{2}\right), \tag{64}$$

$$\sigma_n a^2 \frac{1}{2} \left(\frac{1}{2b}\right)^{n/2} \Gamma\left(\frac{n}{2}\right) = (1-\alpha) \sigma_n a^{\rho+2} \frac{1}{2} \left(\frac{1}{(\rho+2)b}\right)^{n/2} \Gamma\left(\frac{n}{2}\right), \tag{65}$$

we get $(a,b) = (a_0, b_0)$

$$a^{\rho} = a_0^{\rho} = \frac{1}{1-\alpha} \left(\frac{n}{n-2\alpha}\right)^{n/2}, \; b = b_0 = \frac{\alpha}{n(1-\alpha)},$$

where we use all the time the reation $\rho = \frac{4\alpha}{n-2\alpha}$. Using (61) and (13) we find lower bound 2 (24)

$$\underline{\underline{k_0}}(n,\alpha) = \left[\frac{1}{n^n} \left(\frac{2}{\pi}\right)^{2\alpha} (n-2\alpha)^{n-2\alpha} \right]^{1/4}, \qquad n \geqslant 2, \quad 0 < \alpha < 1. \tag{66}$$

4.4. Upper bounds

We introduce the standard notations

$$r = \frac{2n}{n-2\alpha}, \quad \rho = r - 2 = \frac{4\alpha}{n-2\alpha}, \tag{67}$$

and so

$$\alpha = \frac{\rho n}{2(\rho+2)} = \frac{n}{2}\left(\frac{r-2}{r}\right). \tag{68}$$

For the proof of upper bound 1 we need a less well-known inequality which we present here as Lemma.

LEMMA 1. *See* [21] *and* [13, Lemma 1]. *For* $u \in L^2(\mathbb{R}^n),\ |x|\,u \in L^2(\mathbb{R}^n),\ x \in \mathbb{R}^n$, $0 < \alpha < 1$,

$$\|u\|_{\frac{2n}{n+2\alpha}} \leqslant \frac{1}{\chi(\alpha)}\left[\frac{\sigma_n}{2}B\left(\frac{n}{2},\frac{n(1-\alpha)}{2\alpha}\right)\right]^{\alpha/n} \|\,|x|\,u\|_2^{\alpha}\|u\|_2^{1-\alpha}. \tag{69}$$

Equality will be reached for functions

$$u(x) = \frac{A}{\left(B+C|x|^2\right)^{\frac{n+2\alpha}{4\alpha}}}, \quad \textit{with } A,B,C \textit{ arbitrary.}$$

Proof. We start with the inequality

$$\int_{\mathbb{R}^n} f^s g^t\,dx \leqslant \left(\int_{\mathbb{R}^n} f dx\right)^s \left(\int_{\mathbb{R}^n} g dx\right)^t, \quad s+t=1, \tag{70}$$

and we make the choices

$$s = p/2, \quad t = 1-p/2. \quad f^s = \left(|u|^2\left(a+b|x|^2\right)\right)^{p/2}, \quad g^t = \left(a+b|x|^2\right)^{-p/2}.$$

This makes for (70)

$$\int_{\mathbb{R}^n} |u|^p\,dx \leqslant \left(\int_{\mathbb{R}^n}\left(|u|^2\left(a+b|x|^2\right)\right)dx\right)^{p/2}\left(\int_{\mathbb{R}^n}\left(a+b|x|^2\right)^{-\frac{p/2}{1-p/2}}dx\right)^{(1-p/2)},$$

or for $p = (\rho+2)/(\rho+1) = 2n/(n+2\alpha)$ and so $\rho = 4\alpha/(n-2\alpha)$

$$\int_{\mathbb{R}^n} |u|^p\,dx = \|u\|_{\frac{\rho+2}{\rho+1}}^{\frac{\rho+2}{\rho+1}} \leqslant \left(\int_{\mathbb{R}^n}\left(|u|^2\left(a+b|x|^2\right)\right)dx\right)^{\frac{\rho+2}{2(\rho+1)}} \times\left(\int_{\mathbb{R}^n}\left(a+b|x|^2\right)^{-\frac{\rho+2}{\rho}}dx\right)^{\frac{\rho}{2(\rho+1)}}. \tag{71}$$

We define

$$I_0 = \left(\int_{\mathbb{R}^n}\left(a+b|x|^2\right)^{-\frac{\rho+2}{\rho}}dx\right).$$

In a standard way this integral can be calculated as

$$I_0 = a^{-\frac{(4-(n-2)\rho)}{2\rho}} b^{-\frac{n}{2}}\left[\frac{\sigma_n}{2}B\left(\frac{n}{2},\frac{\rho+2}{\rho}-\frac{n}{2}\right)\right].$$

We make now the choice

$$b = \|u\|_2^2 / \| \, |x| \, u\|_2^2,$$

such that (71) transforms into

$$\|u\|^2_{\frac{\rho+2}{\rho+1}} \leqslant \left(\int_{\mathbb{R}^n} \left(|u|^2 \left(a + b\,|x|^2 \right) \right) dx \right) \times \left(a^{-\frac{(4-(n-2)\rho)}{2\rho}} b^{-\frac{n}{2}} \left[\frac{\sigma_n}{2} B\left(\frac{n}{2}, \frac{\rho+2}{\rho} - \frac{n}{2} \right) \right] \right)^{\frac{\rho}{(\rho+2)}},$$

or

$$\|u\|^2_{\frac{\rho+2}{\rho+1}} \leqslant (a+1)\, a^{-(1-\alpha)} \|u\|_2^{2-n\frac{2\alpha}{n}} \| \, |x| \, u\|_2^{2\left(-\frac{n}{2}\right)\frac{2\alpha}{n}} \left[\frac{\sigma_n}{2} B\left(\frac{n}{2}, \frac{n(1-\alpha)}{2\alpha} \right) \right]^{\frac{2\alpha}{n}}.$$

We still have the free parameter a. We minimalize the function $h(a) = (a+1)\, a^{-(1-\alpha)}$. By standard means this minimum will be found for $a_0 = (1-\alpha)/\alpha$ and $h(a_0) = \alpha^{-\alpha} (1-\alpha)^{-1+\alpha} = \chi^{-2}(\alpha)$, by (14). Finally, we arrive at

$$\|u\|_{\frac{\rho+2}{\rho+1}} = \|u\|_{\frac{2n}{n+2\alpha}} \leqslant \frac{1}{\chi(\alpha)} \left[\frac{\sigma_n}{2} B\left(\frac{n}{2}, \frac{n(1-\alpha)}{2\alpha} \right) \right]^{\frac{\alpha}{n}} \|u\|_2^{1-\alpha} \| \, |x| \, u\|_2^{\alpha}.$$

Equality in (70) will be reached if $f = Cg$, C arbitrary, so

$$\left(|u|^2 \left(a + b\,|x|^2 \right) \right) = C \left(a + b\,|x|^2 \right)^{-\frac{p/2}{1-p/2}}, \quad a, b \text{ arbitrary},$$

or

$$u(x) = C \left(a + b\,|x|^2 \right)^{-\frac{\rho+1}{\rho}} = \frac{C}{\left(A + B\,|x|^2 \right)^{\frac{n+2\alpha}{4\alpha}}}, \quad a, A, b, B \text{ arbitrary}. \quad \square$$

LEMMA 2. *See* [4, Theorem 1.7, Case *i*), formula (1.30)]. *For* $0 < \alpha < 1$, $n \geqslant 2$ *there holds the logconvexity of* $k_0(n, \alpha)$

$$k_0(n,\alpha) < (k_0(n,\alpha'))^{\theta} (k_0(n,\alpha''))^{1-\theta}, \quad 0 < \theta < 1, \qquad (72)$$

with $\alpha = \theta\alpha' + (1-\theta)\alpha''$, $\alpha' \neq \alpha''$.

Proof. By the Hölder inequality

$$\|v\|_r < \|v\|_{r'}^{\theta} \|v\|_{r''}^{1-\theta}, \quad 0 < \theta < 1, \quad 1/r = \theta/r' + (1-\theta)/r'', \; r' \neq r'', \qquad (73)$$

which inequality is strict, since $r' \neq r''$. For the choice $r = 2n/(n-2\alpha)$, the condition for application of (73) implies $\alpha = \theta\alpha' + (1-\theta)\alpha''$, and so

$$\Lambda_{N,\alpha}(v) = \frac{\|\nabla v\|_2^{\alpha} \|v\|_2^{1-\alpha}}{\|v\|_r} > \left(\frac{\|\nabla v\|_2^{\alpha'} \|v\|_2^{1-\alpha'}}{\|v\|_{r'}} \right)^{\theta} \left(\frac{\|\nabla v\|_2^{\alpha''} \|v\|_2^{1-\alpha''}}{\|v\|_{r''}} \right)^{1-\theta}$$

$$= \Lambda^{\theta}_{N,\alpha'}(v)\Lambda^{1-\theta}_{N,\alpha''}(v), \tag{74}$$

and this implies the assertion of Lemma 2, since (see (4))

$$\frac{1}{k_0(n,\alpha)} = \lambda_{n,\alpha} = \inf_{u\in H^1(\mathbb{R}^n)} \Lambda_{n,\alpha}. \quad \square$$

4.5. Upper bound 1

See the proof in [12, Proposition 1] or [15, Theorem 1]. For completeness we sketch the proof. We use the following sharp form of the Hausdorff-Young inequality due to Babenko (see [22, Section II. Babenko's inequality])

$$\|u\|_{\frac{2n}{n-2\alpha}} \leqslant k_b\left(\frac{2n}{n+2\alpha}\right)\|\widehat{u}\|_{\frac{2n}{n+2\alpha}}, \tag{75}$$

$$\text{with } \widehat{u} = \left(\frac{1}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n}\exp(-i(x,\xi))u(x)dx.$$

Application of Lemma 1 (69) for the Fourier Transform of u, the function $\widehat{u}$, gives (combined with (75))

$$\begin{aligned}\|u\|_{\frac{2n}{n-2\alpha}} &\leqslant k_b\left(\frac{2n}{n+2\alpha}\right)\|\widehat{u}\|_{\frac{2n}{n+2\alpha}}\\ &\leqslant k_b\left(\frac{2n}{n+2\alpha}\right)\frac{1}{\chi(\alpha)}\left[\frac{\sigma_n}{2}B\left(\frac{n}{2},\frac{n(1-\alpha)}{2\alpha}\right)\right]^{\alpha/n}\||\xi|\widehat{u}\|_2^{\alpha}\|\widehat{u}\|_2^{1-\alpha}.\end{aligned}$$

Due to the Parseval-Steklov relations for Fourier transforms $\|\widehat{u}\|_2 = \|u\|_2$ and $\||\xi|\widehat{u}\|_2 = \|\nabla u\|_2$, we arrive at formula (25), the first upper bound, so

$$\overline{k_0}(n,\alpha) = k_b\left(\frac{2n}{n+2\alpha}\right)\frac{1}{\chi(\alpha)}\left[\frac{\sigma_n}{2}B\left(\frac{n}{2},\frac{n(1-\alpha)}{2\alpha}\right)\right]^{\alpha/n}. \tag{76}$$

4.6. Upper bound 2

See the proof in [15, Theorem 1]. For completeness we sketch the proof. We apply the Beckner-Young's Inequality, see [22, Section III. Young's inequality], for $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$,

$$\|f*g\|_r \leqslant (A_pA_qA_{r'})^n\|f\|_p\|g\|_q, \qquad 1\leqslant p,q,r<\infty, \quad 1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}, \tag{77}$$

$$\text{where } A_p = \left[p^{1/p}/p'^{(1/p')}\right]^{1/2}, \qquad \text{with } \frac{1}{p}+\frac{1}{p'} = 1.$$

Note that $k_b(p) = (2\pi)^{(-1/p+1/p')n/2}A_p^n$.

We apply this inequality (77) for the solution of (11) $\overline{u}_{n,\alpha}(r)$ written as $\psi_0(x)$, $x \in \mathbb{R}^n$, in convolution form. ψ_0 satisfies

$$\Delta\psi_0 - \psi_0 = -\psi_0^{\rho+1}. \tag{78}$$

By application of the Fourier Transform on the equation

$$\Delta\psi_{0,\delta} - \psi_{0,\delta} = \delta, \qquad x \in \mathbb{R}^n,$$

with δ the Dirac delta function, we find for the Fourier Transform $\widehat{\psi_{0,\delta}}$

$$\widehat{\psi_{0,\delta}} = -\left(\frac{1}{2\pi}\right)^{n/2} \frac{1}{(1+\xi^2)}, \qquad \text{because } \widehat{\delta} = \left(\frac{1}{2\pi}\right)^{n/2},$$

which gives for $\psi_{0,\delta}$

$$\psi_{0,\delta} = -\left(\frac{1}{2\pi}\right)^{n/2} G(x), \qquad \text{with } G(x) = \frac{K_{(n-2)/2}(|x|)}{|x|^{\frac{n-2}{2}}},$$

see [23, Chapter 8, p. 289]. And so we find for ψ_0 the integral equation

$$\psi_0 = -\left(\frac{1}{2\pi}\right)^{n/2} G * \left(-\psi_0^{\rho+1}\right) = \left(\frac{1}{2\pi}\right)^{n/2} G * \psi_0^{\rho+1}. \tag{79}$$

Now, we apply (77) with $f = G$, $g = \psi_0^{\rho+1}$, $r = \rho+2$, $p = (\rho+2)/2$, $q = (\rho+2)/(\rho+1)$, so $r' = q$, and we have

$$\begin{aligned} \|\psi_0\|_{\rho+2} &= \left(\frac{1}{2\pi}\right)^{n/2} \left\|G * \psi_0^{\rho+1}\right\|_{\rho+2} \\ &\leqslant \left(\frac{1}{2\pi}\right)^{n/2} \left(A_{(\rho+2)/2} A^2_{(\rho+2)/(\rho+1)}\right)^n \|G\|_{(\rho+2)/2} \left\|\psi_0^{\rho+1}\right\|_{(\rho+2)/(\rho+1)} \\ &= k_b\left(\frac{\rho+2}{2}\right) k_b^2\left(\frac{\rho+2}{\rho+1}\right) \|G\|_{(\rho+2)/2} \|\psi_0\|^{\rho+1}_{\rho+2}. \end{aligned} \tag{80}$$

From (80) we get

$$\|\psi_0\|^{\rho+2}_{\rho+2} \geqslant \left[k_b\left(\frac{\rho+2}{2}\right) k_b^2\left(\frac{\rho+2}{\rho+1}\right) \|G\|_{(\rho+2)/2}\right]^{-\left(\frac{\rho+2}{\rho}\right)}. \tag{81}$$

By (12) this becomes

$$\|\psi_0\|_2^2 \geqslant (1-\alpha)\left[k_b\left(\frac{\rho+2}{2}\right) k_b^2\left(\frac{\rho+2}{\rho+1}\right) \|G\|_{(\rho+2)/2}\right]^{-\left(\frac{\rho+2}{\rho}\right)},$$

and by (13) we have

$$\chi(\alpha)\left(\frac{\|\overline{u}_{n,\alpha}\|_2^2}{1-\alpha}\right)^{\alpha/n} = \frac{1}{k_0(n,\alpha)}.$$

Since $\|\overline{u}_{n,\alpha}\|_2^2 = \|\psi_0\|_2^2$ (by definition) and $\alpha/n = \rho/(2(\rho+2))$

$$k_0(n,\alpha) \leqslant \frac{1}{\chi(\alpha)}\left[k_b\left(\frac{\rho+2}{2}\right)k_b^2\left(\frac{\rho+2}{\rho+1}\right)\|G\|_{(\rho+2)/2}\right]^{1/2}.$$

This equals the announced upper bound 2 (27), because $(\rho+2)/2 = n/(n-2\alpha)$ and $(\rho+2)/(\rho+1) = 2n/(n+2\alpha)$:

$$\begin{aligned} k_0(n,\alpha) &\leqslant \frac{1}{\chi(\alpha)}\left[k_B\left(\frac{n}{n-2\alpha}\right)k_B^2\left(\frac{2n}{n+2\alpha}\right)\|G(x)\|_{n/(n-2\alpha)}\right]^{1/2} \\ &= \overline{\overline{k_0}}(n,\alpha). \end{aligned} \tag{82}$$

4.7. Upper bound 3

We follow the same strategy as for the upper bound 2. We apply (77) with $f = G$, $g = \psi_0^{\rho+1}$, $p = 2(\rho+2)/(\rho+4)$, $q = (\rho+2)/(\rho+1)$, $r = 2$, so $r' = 2$, and we have

$$\begin{aligned} \|\psi_0\|_2 &= \left(\frac{1}{2\pi}\right)^{n/2}\left\|G*\psi_0^{\rho+1}\right\|_2 \\ &\leqslant \left(\frac{1}{2\pi}\right)^{n/2}\left(A_{2(\rho+2)/(\rho+4)}A_{(\rho+2)/(\rho+1)}\right)^n \times \|G\|_{2(\rho+2)/(\rho+4)}\left\|\psi_0^{\rho+1}\right\|_{(\rho+2)/(\rho+1)} \\ &= k_b\left(\frac{2(\rho+2)}{\rho+4}\right)k_b\left(\frac{\rho+2}{\rho+1}\right)\|G\|_{2(\rho+2)/(\rho+4)}\|\psi_0\|_{\rho+2}^{\rho+1}. \end{aligned} \tag{83}$$

By (12) this becomes

$$(1-\alpha)\|\psi_0\|_{\rho+2}^{\rho+2} \leqslant \left[k_b\left(\frac{2(\rho+2)}{\rho+4}\right)k_b\left(\frac{\rho+2}{\rho+1}\right)\|G\|_{2(\rho+2)/(\rho+4)}\right]^2\|\psi_0\|_{\rho+2}^{2(\rho+1)}.$$

This can be rewritten as

$$\|\psi_0\|_{\rho+2}^{\rho} \geqslant (1-\alpha)\left[k_b\left(\frac{2(\rho+2)}{\rho+4}\right)k_b\left(\frac{\rho+2}{\rho+1}\right)\|G\|_{2(\rho+2)/(\rho+4)}\right]^{-2}, \tag{84}$$

and by (13) we have

$$\chi(\alpha)\left(\frac{\|\overline{u}_{n,\alpha}\|_2^2}{1-\alpha}\right)^{\alpha/n} = \chi(\alpha)\left(\|\overline{u}_{n,\alpha}\|_{\rho+2}^{\rho+2}\right)^{\alpha/n} = \frac{1}{k_0(n,\alpha)}.$$

Since $\|\overline{u}_{n,\alpha}\|_{\rho+2}^{\rho+2} = \|\psi_0\|_{\rho+2}^{\rho+2}$ (by definition) and $\alpha/n = \rho/(2(\rho+2))$ there follows

$$k_0(n,\alpha) \leqslant \frac{1}{\chi(\alpha)}\frac{1}{\sqrt{(1-\alpha)}}\left[k_b\left(\frac{2(\rho+2)}{\rho+4}\right)k_b\left(\frac{\rho+2}{\rho+1}\right)\|G\|_{2(\rho+2)/(\rho+4)}\right].$$

This equals the announced upper bound 3 (29), because $2(\rho+2)/(\rho+4) = n/(n-\alpha)$ and $(\rho+2)/(\rho+1) = 2n/(n+2\alpha)$:

$$k_0(n,\alpha) \leqslant \frac{1}{\chi(\alpha)} \frac{1}{\sqrt{(1-\alpha)}} \left[k_b\left(\frac{n}{(n-\alpha)}\right) k_b\left(\frac{2n}{(n+2\alpha)}\right) \|G\|_{n/(n-\alpha)} \right] \tag{85}$$
$$= \overline{\overline{\overline{k_0}}}(n,\alpha).$$

4.8. Upper bound 4

We start with the inequality

$$\|u\|_{2p} \leqslant A \|\nabla u\|_2^{\theta} \|u\|_{p+1}^{1-\theta}, \quad u \in L^{p+1}(\mathbb{R}^n), \nabla u \in L^2(\mathbb{R}^n), |u|^{2p} \in L^1(\mathbb{R}^n), \tag{86}$$

for $n = 2,\ p > 1$, and for $n \geqslant 3,\ 1 < p \leqslant n/(n-2)$,

$$\theta = \frac{n(p-1)}{p(n+2-(n-2)p)}, \tag{87}$$

with the optimal constant

$$A = \left(\frac{y(p-1)^2}{2\pi n}\right)^{\frac{\theta}{2}} \left(\frac{2y-n}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma\left(y-\frac{n}{2}\right)}\right)^{\frac{\theta}{n}}, \qquad y = \frac{p+1}{p-1}, \tag{88}$$

see [24, Theorem 1].

Next, we apply the Cauch-Schwarz's Inequality in the form

$$\|u\|_{p+1} \leqslant \|u\|_{2p}^{\eta} \|u\|_2^{1-\eta}, \quad \text{for } \eta = \frac{p}{p+1}, \tag{89}$$

and insert this inequality in the right-hand side of (86) to obtain

$$\|u\|_{2p} \leqslant A \|\nabla u\|_2^{\theta} \|u\|_{2p}^{\eta(1-\theta)} \|u\|_2^{(1-\eta)(1-\theta)},$$

or

$$\|u\|_{2p}^{1-\eta(1-\theta)} \leqslant A \|\nabla u\|_2^{\theta} \|u\|_2^{(1-\eta)(1-\theta)},$$

or

$$\|u\|_{2p} \leqslant A^{\frac{1}{1-\eta(1-\theta)}} \|\nabla u\|_2^{\frac{\theta}{1-\eta(1-\theta)}} \|u\|_2^{\frac{(1-\eta)(1-\theta)}{1-\eta(1-\theta)}}. \tag{90}$$

For the choice of $p = n/(n-2\alpha)$ as in (1) we find after some calculations, using (87)

$$\theta = \frac{\alpha(n-2\alpha)}{2n-2\alpha-\alpha n}, \quad \frac{\theta}{1-\eta(1-\theta)} = \alpha, \tag{91}$$
$$\frac{(1-\eta)(1-\theta)}{1-\eta(1-\theta)} = 1-\alpha, \quad y = \frac{n-\alpha}{\alpha},$$

and

$$\frac{1}{1-\eta(1-\theta)} = \frac{2n-2\alpha-\alpha n}{n-2\alpha} \equiv \gamma. \tag{92}$$

Using the identities (91) and (92) we arrive at

$$\|u\|_{2n/(n-2\alpha)} \leqslant A^{\gamma} \|\nabla u\|_2^{\alpha} \|u\|_2^{1-\alpha}, \tag{93}$$

which is inequality (1) and where A^{γ} equals, using $y = n/\alpha - 1$, $p - 1 = 2\alpha/(n-2\alpha)$

$$A^{\gamma} = \left(\frac{2\alpha(n-\alpha)}{\pi n(n-2\alpha)^2}\right)^{\frac{\alpha}{2}} \left(1 - \frac{n\alpha}{2(n-\alpha)}\right)^{(2n-2\alpha-\alpha n)/(2n)} \times \left(\frac{\Gamma\left(\frac{n}{\alpha}-1\right)}{\Gamma\left(\frac{n}{\alpha}-1-\frac{n}{2}\right)}\right)^{\frac{\alpha}{n}}, \tag{94}$$

so we found the announced upper bound 4 (30)

$$\overline{k_{D,1}}(n,\alpha) = A^{\gamma}, \quad \text{with } A = A(n,\alpha) \text{ defined in (31).} \tag{95}$$

4.9. Upper bound 5

We observe that there holds trivially

$$k_0(n,\alpha) = k_0(n,\alpha)^{\theta} k_0(n,\alpha)^{1-\theta}. \tag{96}$$

Make now the choice $\theta = \alpha(n-2\alpha)/(2n-2\alpha-\alpha n)$ see (32), then

$$k_0(n,\alpha)^{\theta} < \overline{k_{D,1}}(n,\alpha)^{\theta} = (A(n,\alpha)^{\gamma})^{\theta} = A(n,\alpha)^{\alpha}, \tag{97}$$

since $\gamma\theta = \alpha$ (see (92)) and further

$$k_0(n,\alpha)^{1-\theta} < \overline{k_0}(n,\alpha)^{1-\theta}. \tag{98}$$

Insertation of (97) and (98) into (96) gives upper bound 5:

$$k_0 < \overline{k_{D,2}}(n,\alpha) = A(n,\alpha)^{\alpha} \overline{k_0}(n,\alpha)^{1-\theta}, \qquad n \geqslant 2, \quad 0 < \alpha < 1. \tag{99}$$

4.10. Upper bound 6

There holds trivially

$$k_0(n,\alpha) = k_0(n,\alpha)^{\theta} k_0(n,\alpha)^{1-\theta}. \tag{100}$$

Make now the choice $\theta = \alpha(n-2\alpha)/(2n-2\alpha-\alpha n)$ see (32), then

$$k_0(n,\alpha)^{\theta} < \overline{k_{D,1}}(n,\alpha)^{\theta} = (A(n,\alpha)^{\gamma})^{\theta} = A(n,\alpha)^{\alpha}, \tag{101}$$

since $\gamma\theta = \alpha$ (see (92)) and further

$$k_0(n,\alpha)^{1-\theta} < \overline{\overline{k_0}}(n,\alpha)^{1-\theta}. \tag{102}$$

Insertation of (101) and (102) into (100) gives upper bound 6:

$$k_0 < \overline{k_{D,3}}(n,\alpha) = A(n,\alpha)^{\alpha} \overline{\overline{k_0}}(n,\alpha)^{1-\theta}, \qquad n \geqslant 2, \quad 0 < \alpha < 1. \tag{103}$$

By the way, it is clear that in this way more upper bounds can be constructed.

4.11. Upper bound 7

This inequality is an application of [4, Theorem 1.7, (1.30), $\theta' = 1/2$, $\theta'' = 1$, with the restriction $n \geqslant 3$], as follows. Apply Lemma 2 with the choices $\alpha' = 1/2$, $\alpha'' = 1$ and $\theta = 2(1-\alpha)$. See the results for the case $\alpha = 1$ in the Introduction, equation (15). Application of (72) for $n \geqslant 3$:

$$k_0(n,\alpha) < \overline{k_0}\left(n,\frac{1}{2}\right)^{2(1-\alpha)} k_0(n,1)^{2\alpha-1} = \overline{k_0}\left(n,\frac{1}{2}\right)^{2(1-\alpha)} (C_T(n,2))^{-2\alpha+1}$$
$$= \overline{k_0}\left(n,\frac{1}{2}\right)^{2(1-\alpha)} (k_T(n))^{2\alpha-1}, \quad n \geqslant 3, \quad 1/2 < \alpha < 1. \tag{104}$$

The last restriction comes from the requirement that $\theta < 1$. We made the choice to bound $k_0\left(n,\frac{1}{2}\right)$ by $\overline{k_0}\left(n,\frac{1}{2}\right)$. Equation (104) represents the announced upper bound 7

$$\overline{k_{I,1}}(n,\alpha) = \overline{k_0}\left(n,\frac{1}{2}\right)^{\alpha_1} k_T(n)^{(1-\alpha_1)}, \alpha_1 = 2(1-\alpha), n \geqslant 3, 1/2 < \alpha < 1. \tag{105}$$

4.12. Upper bound 8

This inequality is an application of [4, Theorem 1.7, (1.30), $\theta' = \theta_N (= \alpha_V)$, $\theta'' = 1$, with the restriction $n \geqslant 3$], as follows. Apply Lemma 2 with the choices $\alpha' = \alpha_V$, $\alpha'' = 1$ and $\theta = \alpha_2 = (1-\alpha)/(1-\alpha_V)$. See the results for the case $\alpha = 1$ in the Introduction, equation (15). Application of (72) for $n \geqslant 3$ and for $\alpha_V < \alpha < 1$:

$$k_0(n,\alpha) < \overline{k_0}(n,\alpha_V)^{\alpha_2} k_0(n,1)^{1-\alpha_2} = \overline{k_0}(n,\alpha_V)^{\alpha_2} (C_T(n,2))^{-(1-\alpha_2)}$$
$$= \overline{k_0}(n,\alpha_V)^{\alpha_2} (k_T(n))^{(1-\alpha_2)}, \quad n \geqslant 3, \quad \alpha_V < \alpha < 1. \tag{106}$$

We again made the choice to bound $k_0(n,\alpha_V)$ by $\overline{k_0}(n,\alpha_V)$. The value α_V can be chosen freely and has been chosen here as the argument value for the optimum of the expression $\alpha C_T(n,2\alpha)$, see further at the proof for upper bound 10. Equation (106) represents the announced upper bound 8

$$\overline{k_{I,2}}(n,\alpha) = \overline{k_0}(n,\alpha_V)^{\alpha_2} k_T(n)^{(1-\alpha_2)}, \alpha_2 = (1-\alpha)/(1-\alpha_V), n \geqslant 3, \alpha_V < \alpha < 1. \tag{107}$$

4.13. Upper bound 9

This inequality is an application of [4, Theorem 1.7, (1.30), $\theta' = \theta_N (= \alpha_V)$, $\theta'' = 1$, with the restriction $n \geqslant 3$], as follows. Apply Lemma 2 with the choices $\alpha' = \alpha_V$, $\alpha'' = 1$ and $\theta = \alpha_2 = (1-\alpha)/(1-\alpha_V)$. See the results for the case $\alpha = 1$ in the Introduction, equation (15). Application of (72) for $n \geqslant 3$ and for $\alpha_V < \alpha < 1$:

$$k_0(n,\alpha) < \overline{k_{L,V}}(n,\alpha_V)^{\alpha_2} k_0(n,1)^{1-\alpha_2} = (\alpha_V C_T(n,2\alpha_V))^{-\alpha_V\alpha_2} (C_T(n,2))^{-(1-\alpha_2)}$$
$$= (\alpha_V C_T(n,2\alpha_V))^{-\alpha_V\alpha_2} (k_T(n))^{(1-\alpha_2)}, \quad n \geqslant 3, \quad \alpha_V < \alpha < 1. \tag{108}$$

Here, we bounded $k_0(n,\alpha_V)$ by $\overline{k_{L,V}}(n,\alpha_V)$, i.e. the upper bound 10 (46). The value α_V can be chosen freely and has been chosen here as the argument value for the optimum of the expression $\alpha C_T(n,2\alpha)$, see further at the proof for upper bound 10. Equation (108) represents the announced upper bound 9

$$\overline{k_{I,3}}(n,\alpha) = (\alpha_V C_T(n,2\alpha_V))^{-\alpha_V\alpha_2} k_T(n)^{(1-\alpha_2)}, \qquad (109)$$
$$\alpha_2 = (1-\alpha)/(1-\alpha_V), \qquad n \geqslant 3, \quad \alpha_V < \alpha < 1.$$

4.14. Upper bound 10

Firstly, we prove

$$k_0(n,\alpha) < (\alpha C_T(n,2\alpha))^{-\alpha}, \quad n \geqslant 2, \quad 1/2 < \alpha < 1. \qquad (110)$$

This result has been given in [4, Theorem 1.7, (1.31)] and was inspired by [6, (1.5)], by making the transformation $w = u^{1/\alpha}$ for $v > 0$ in (15) as follows

$$\begin{aligned}
C_T(n,s) &\leqslant \frac{\|\nabla w\|_s}{\|w\|_t} = \frac{\|\nabla u^{1/\alpha}\|_s}{\|u^{1/\alpha}\|_t} = \frac{1/\alpha \|u^{(1-\alpha)/\alpha}\nabla u\|_s}{\|u^{1/\alpha}\|_t} \quad [t = sn/(n-s)] \\
&= \frac{1}{\alpha}\frac{\left(\int (\nabla u)^s u^{s(1-\alpha)/\alpha} dx\right)^{1/s}}{\left(\int u^{t/\alpha} dx\right)^{1/t}} \quad \text{[apply Hölder inequality, } 1/P + 1/Q = 1] \\
&\leqslant \frac{1}{\alpha}\frac{\left(\int (\nabla u)^{sP} dx\right)^{1/(sP)} \left(\int u^{Qs(1-\alpha)/\alpha} dx\right)^{1/(sQ)}}{\left(\int u^{t/\alpha} dx\right)^{1/t}} \quad \text{[take } P = 2/s, \ Q = 2/(2-s)] \\
&= \frac{1}{\alpha}\frac{\left(\int (\nabla u)^{2} dx\right)^{1/2} \left(\int u^{Qs(1-\alpha)/\alpha} dx\right)^{(2-s)/(2s)}}{\left(\int u^{t/\alpha} dx\right)^{1/t}} \quad \text{[take } s = 2\alpha \text{, and } r = t/\alpha = 2n/(n-2\alpha)] \\
&= \frac{1}{\alpha}\frac{\|\nabla u\|_2 \|u\|_2^{(1-\alpha)/\alpha}}{\|u\|_r^{1/\alpha}} = \frac{1}{\alpha}(\Lambda_{n,\alpha}(u))^{1/\alpha}, \qquad (111)
\end{aligned}$$

for the choice $s = 2\alpha$. We have to restrict α to the interval $1/2 \leqslant \alpha \leqslant 1$ to give $C_T(n,2\alpha)$ a meaning. Again, the inequality is strict since $w = \overline{u}^{\alpha}_{n,\alpha}$ does not equal a function $w_{n,s}$ (see (22)), with $s = 2\alpha$. So (111) implies

$$\lambda_{n,\alpha} = \inf_{u\in H^1(\mathbb{R}^n)} \Lambda_{n,\alpha}(u) > (\alpha C_T(n,2\alpha))^{\alpha},$$

and this equivalent with

$$k_0(n,\alpha) = 1/\lambda_{n,\alpha} < (\alpha C_T(n,2\alpha))^{-\alpha}, \quad n \geqslant 2, \quad 1/2 < \alpha < 1.$$

Application of Lemma 2 with $\alpha'' = 0$, $\theta = \alpha/\alpha'$, and $k_0(n,0) = 1$ gives

$$k_0(n,\alpha) < \left(\left(\alpha' C_T(n,2\alpha')\right)^{-\alpha'}\right)^{\alpha/\alpha'} = \left(\alpha' C_T(n,2\alpha')\right)^{-\alpha}.$$

Since α' can still be chosen freely, we can improve this inequality by maximizing the $(\alpha' C_T(n, 2\alpha'))$. In a standard way we find that there is a unique value $\alpha_V \in (1/2, 1)$ which optimizes this expression, see [4, Proof Theorem 1.7, (1.32)] for details. Finally we find the announced upper bound 10

$$k_0 < \overline{k_{L,V}}(n,\alpha) = [\alpha_V C_T(n, 2\alpha_V)]^{-\alpha}, \qquad n \geqslant 2,\ 0 < \alpha \leqslant \alpha_V, \tag{112}$$
$$k_0 < \overline{k_{L,V}}(n,\alpha) = 1/k_{L,V}(n,\alpha) = [\alpha C_T(n, 2\alpha)]^{-\alpha},\ n \geqslant 2,\ \alpha_V \leqslant \alpha < 1, \tag{113}$$

where the value for α_V follows from

$$\alpha_V = \alpha_V(n) = \frac{n}{2p_V}, \text{ where } p_V \text{ is the solution of} \tag{114}$$
$$\ln\left(\frac{n-p}{p-1}\right) + \frac{n-p}{p(p-1)} + \psi(p) - \psi(n+1-p) = 0, \tag{115}$$
$$\psi(x) = \frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)}, \qquad x > 0, \qquad 1 < p < n, \quad n \geqslant 2.$$

In both expressions (112) and (113) the second argument in C_T is larger than 1, as required. The value α_V has also been used in the upper bounds 8 and 9.

4.15. Upper bound 11

This inequality is a combination of the Hölder inequality (73)

$$\|u\|_r < \|u\|_{r'}^{\theta} \|u\|_{r''}^{1-\theta}, \quad 0 < \theta < 1, \quad 1/r = \theta/r' + (1-\theta)/r'',\ r' \neq r'', \tag{116}$$

and the Sobolev embedding (15)

$$\|u\|_t \leqslant \frac{1}{C_T(n,2)} \|\nabla u\|_2,\ t = 2n/(n-2),\ n \geqslant 3. \tag{117}$$

For the choice $r = 2n/(n-2\alpha)$, $\theta = \alpha$, $r'' = 2$ in (116), we find $r' = 2n/(n-2)$, which is just the value applicable for the Sobolev embedding (117). These two estimates combined gives

$$\|u\|_{2n/(n-2\alpha)} < \left(\frac{1}{C_T(n,2)}\right)^{\alpha} \|\nabla u\|_2^{\alpha} \|u\|_2^{1-\alpha} = k_T(n)^{\alpha} \|\nabla u\|_2^{\alpha} \|u\|_2^{1-\alpha}, n \geqslant 3. \tag{118}$$

So, we found the announced upper bound 11

$$k_0 < \overline{k_B}(n,\alpha) = k_T(n)^{\alpha}, \qquad n \geqslant 3, \quad 0 < \alpha < 1. \tag{119}$$

5. Numerical evaluations lower and upper bounds

In order to assess the quality of the estimates we have calculated the numbers $\lambda_{n,\alpha}$ for $n = 2, 3, 4, 5, 10$ and $\alpha = 0.05 + (i-1)0.005$, $i = 1, 2, 3, \cdots, 176$ up till $\theta = 0.925$. The method is the same as used in the paper [4]. This method to find $\lambda_{n,\alpha}$ consists of a shooting technique to find that value $\overline{u}(0) = u_0$ such that $\overline{u}(r)$ is a positive solution of (11) with $\lim_{r\to\infty}\overline{u}(r) = 0$. Therefore, we transformed the interval $r \in (0,\infty)$ into $s = r/(1+r) \in (0,1)$. The transformed differential equation becomes, with $w(s) = u(r)$, $0 < s < 1$,

$$(1-s)^4 \frac{d^2}{ds^2} w + \left\{ \left(\frac{(n-1)}{s} - 2 \right) (1-s)^3 \right\} \frac{d}{ds} w - w|w|^{(n+2\alpha)/(n-2\alpha)-1} - w = 0,$$
$$w(0) = v_0, \qquad \frac{d}{ds} w(0) = 0. \tag{120}$$

The aim now is to find a value v_0 such that for $w(0) = v_0$, $\frac{d}{ds}w(0) = 0$, we find $w(1) = 0$. We solved the transformed differential equation (120) by means of a numerical integration method (Runge-Kutta of the fourth order) with a self-adapting step-size routine such that a prescribed maximal relative error (ε_{rel}) in each component $(w(s), \frac{d}{ds}w(s))$ has been satisfied. We made the choice $\varepsilon_{rel} = 10^{-15}$. For every value of v_0 the numerical integrator will find some point $s = s(v_0) \in (0,1)$ where either $w(s) < 0$, or $\frac{d}{ds}w(s) > 0$. At that point s the integration will be stopped. This integrator is coupled to a numerical zero-finding routine (see ([25])), which can also be applied for finding a discontinuity. The function f for which such a discontinuity has to been found is specified by if $w(s(v_0)) < 0$, $f(v_0) = -(1-s(v_0))$ else (that means thus $\frac{d}{ds}w(s(v_0)) > 0$) $f(v_0) = (1-s(v_0))$. The sought value v_0 has been found if this numerical routine has come up with two values v_0 and v_0^1 such that $\left|v_0 - v_0^1\right| < r_p|v_0| + a_p$, (with $r_p = a_p = 10^{-15}$ relative and absolute precisions, respectively) and $|f(v_0)| \leqslant |f(v_0^1)|$, while $sign(f(v_0) = -sign(f(v_0^1))$. During the integration processes the norms in (12) will be calculated. As a check upon this procedure the following expressions

$$\|\overline{u}_{n,\alpha}\|_2^2/(1-\alpha), \quad \|\nabla\overline{u}_{n,\alpha}\|_2^2/\alpha, \quad \|\overline{u}_{n,\alpha}\|_{2n/(n-2\alpha)}^{2n/(n-2\alpha)}, \tag{121}$$

are compared. They should be all equal, see (12). The eigenvalue $\lambda_{n,\alpha}$ is found then by (13).

5.1. Some numerical results for values for $\alpha = 1/3,\ 2/3$ and $n = 2$

Here, we give for $n = 2$ and for particular values of α ($\alpha = 1/3$ and $2/3$) the upper and lower bounds which are applicable. Compare these with [10, $\alpha = 1/3$] and [6, $\alpha = 2/3$].

α	k_0	$\underline{k_0}$	$\underline{\underline{k_0}}$
$n = 2$			
1/3	7.2493833e-001	7.2431703e-001	7.2184608e-001
2/3	6.0129905e-001	5.9737503e-001	5.6854280e-001

Table 1: Functional, $n = 2$, Lower bounds 1 - 2.

α	k_0	$\overline{k_0}$	$\overline{\overline{k_0}}$	$\overline{\overline{\overline{k_0}}}$
$n=2$				
1/3	7.2493833e-001	7.2978972e-001	7.3987840e-001	7.8567080e-001
2/3	6.0129905e-001	6.4335375e-001	6.1742806e-001	7.2152108e-001

Table 2: Functional, $n=2$, Upper bounds 1 - 3.

α	$\overline{k_{D,1}}$	$\overline{k_{D,2}}$	$\overline{k_{D,3}}$	$\overline{k_{L,V}}$
$n=2$				
1/3	7.3907188e-001	7.3132861e-001	7.3974392e-001	7.7547470e-001
2/3	6.8278406e-001	6.5623746e-001	6.3848696e-001	6.1088706e-001

Table 3: $n=2$, Upper bounds 4 - 6 and 10.

5.2. Numerical results for $\alpha = 0.05, \cdots, 0.925$ $(\Delta = 0.005)$ and $n = 2,3,4,5,10$

In the Supplementary Material to this paper we present tables which give the results of the numerical calculations of the functional $k_0(n,\alpha)$ and the lower and upper bounds, based on the technique described above (see also [4]).

Values ”0.0000000e+000” has to be interpreted as ”Not Applicable”. The lower and upper bounds have been calculated using the software package Matlab™.

5.3. Results for the zeros p_V and $\alpha_V = n/(2p_V)$

The zeros p_V as defined in (42) are given below in the Table 4; $\alpha_V(n) = n/(2p_V)$. The asymptotic expressions are

$$p_V(n) = 2n/3 + 5/18 + O(1/n), \qquad n \to \infty, \tag{122}$$
$$\alpha_V(n) = 3/4 - 5/(16n) + O(1/n^2), \qquad n \to \infty, \tag{123}$$

n	p_V	$p_{V,asymp}$	$p_V - p_{V,asymp}$
		$= 2n/3 + 5/18$	
2	1.6474176e+000	1.6111111e+000	3.6306497e-002
3	2.3044430e+000	2.2777778e+000	2.6665194e-002
4	2.9654018e+000	2.9444444e+000	2.0957401e-002
5	3.6283253e+000	3.6111111e+000	1.7214200e-002
6	4.2923606e+000	4.2777778e+000	1.4582787e-002
7	4.9570820e+000	4.9444444e+000	1.2637555e-002
8	5.6222549e+000	5.6111111e+000	1.1143822e-002
9	6.2877400e+000	6.2777778e+000	9.9621751e-003
10	6.9534493e+000	6.9444444e+000	9.0048448e-003

Table 4: The zeros p_V for $n = 2, \cdots, 10$ and their asymptotic approximations.

n	α_V	$\alpha_{V,asymp}$	$\alpha_V - \alpha_{V,asymp}$
		$= 3/4 - 5/(16n)$	
2	6.0701063e-001	5.9375000e-001	1.3260630e-002
3	6.5091652e-001	6.4583333e-001	5.0831867e-003
4	6.7444485e-001	6.7187500e-001	2.5698490e-003
5	6.8902311e-001	6.8750000e-001	1.5231128e-003
6	6.9891612e-001	6.9791667e-001	9.9945530e-004
7	7.0606054e-001	7.0535714e-001	7.0339854e-004
8	7.1145831e-001	7.1093750e-001	5.2081118e-004
9	7.1567845e-001	7.1527778e-001	4.0067485e-004
10	7.1906759e-001	7.1875000e-001	3.1758674e-004

Table 5: The zeros $\alpha_V = n/(2p_V)$ for $n = 2,\cdots,10$ and their asymptotic approximations.

6. Discussion

With respect to the lower bounds it is clear based on the numerical results in the Supplementary Material to this paper (Tables 4-8 and Fig. 3 in ”Comparison Functional with Lower bounds for Functional” therein) that the lower bound for $n = 2$, $\underline{k_0}(\alpha)$, is superior to the lower bound $\underline{\underline{k_0}}(2,\alpha)$.

With respect to the upper bounds the situation is more complicated. For the range of n $(n = 2,3,4,5$ and $n = 10)$ and α $(0.05 \leqslant \alpha \leqslant 0.925$ with steps $\Delta\alpha = 0.005)$ we have examined there are just four upper bounds which are superior, see the Table 6 and the Figures 1, 2, 3, 4 and 5.

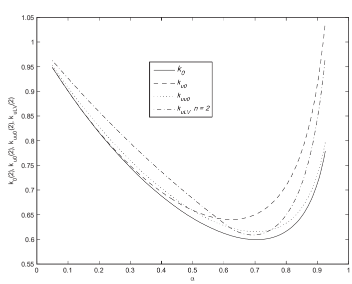

Figure 1: Best bounds for $n = 2$.

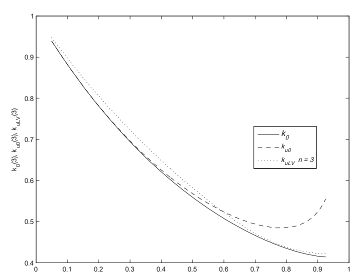

Figure 2: Best bounds for $n = 3$.

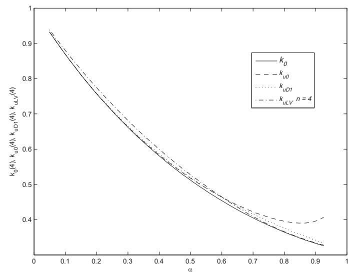

Figure 3: Best bounds for $n = 4$.

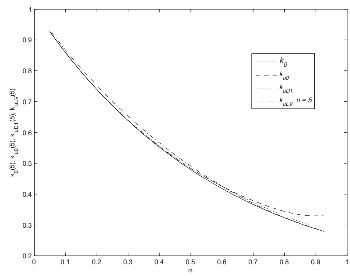

Figure 4: Best bounds for $n = 5$.

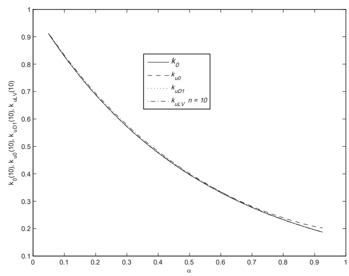

Figure 5: Best bounds for $n = 10$.

n	Range α	Upper bound #	Expression Upper bound
2	$(0.050, 0.495)$	1	$\overline{k_0}(2,\alpha)$
2	0.500	$1 = 2$	$\overline{k_0}(2,1/2) = \overline{\overline{k_0}}(2,1/2)$
2	$[0.505, 0.615)$	2	$\overline{\overline{k_0}}(2,\alpha)$
2	$(0.620, 0.745)$	10	$\overline{k_{L,V}}(2,\alpha)$
2	$(0.750, 0.925)$	2	$\overline{\overline{k_0}}(2,\alpha)$
3	$(0.050, 0.590)$	1	$\overline{k_0}(3,\alpha)$
3	$(0.595, 0.925)$	10	$\overline{k_{L,V}}(3,\alpha)$
4	$(0.050, 0.590)$	1	$\overline{k_0}(4,\alpha)$
4	$(0.595, 0.605)$	4	$\overline{k_{D,1}}(4,\alpha)$
4	$(0.610, 0.925)$	10	$\overline{k_{L,V}}(4,\alpha)$
5	$(0.050, 0.565)$	1	$\overline{k_0}(5,\alpha)$
5	$(0.570, 0.630)$	4	$\overline{k_{D,1}}(5,\alpha)$
5	$(0.635, 0.925)$	10	$\overline{k_{L,V}}(5,\alpha)$
10	$(0.050, 0.535)$	1	$\overline{k_0}(10,\alpha)$
10	$(0.540, 0.675)$	4	$\overline{k_{D,1}}(10,\alpha)$
10	$(0.680, 0.925)$	10	$\overline{k_{L,V}}(10,\alpha)$

Table 6: Optimal upper bounds for $n = 2, 3, 4, 5, 10$.

We remark that $\overline{k_0}(2,1/2) = \overline{\overline{k_0}}(2,1/2) = 2^1 3^{-3/4}\pi^{-1/4}$, and $\overline{k_0}(3,3/4) = \overline{\overline{k_0}}(3,3/4) = 2^{7/4}3^{-3/2}\pi^{-1/4}$ see [15, equation (12) and (17)].

As can been seen from the figures in the Supplementary Material to this paper, for larger values of n almost all bounds come close to the actual value for $k_0(n,\alpha)$; see the Figures 7, 12, 28, 32, 37, 42, 46 and 51 therein, for $n = 10$.

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