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ON THE NEGATIVE SOLUTIONS OF L_p–BUSEMANN–PETTY PROBLEM

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Abstract. Intersection bodies led to the solutions of Busemann-Petty problem by Lutwak. Associated with Haberl and Ludwig's L_p-intersection bodies, Yuan and Cheung researched related the L_p-Busemann-Petty problem. In this paper, we sequentially study L_p-Busemann-Petty problem of L_p-intersection bodies and give its two negative forms.

1. Introduction

If K is a compact star shaped (about the origin) in n-dimensional Euclidean space $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \backslash \{0\} \to [0,\infty)$, is defined by (see [3])

$$\rho(K,x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. The set of star bodies (about the origin) in $\mathbb{R}^n$ is denoted by $\mathscr{S}_o^n$, for the set of all origin-symmetric star bodies, we write $\mathscr{S}_{os}^n$.

The well-known Busemann-Petty problem is one of essential questions in Brunn-Minkowski theory, it may be stated as follows:

PROBLEM 1.1 (BUSEMANN-PETTY PROBLEM). Let K and L be origin-symmetric convex bodies. For all $u \in S^{n-1}$, is there the implication

$$V_{n-1}(K \cap u^\perp) \subseteq V_{n-1}(L \cap u^\perp) \Rightarrow V(K) \leqslant V(L)?$$

Here S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, $u^\perp$ is the $(n-1)$-dimensional hyperplane orthogonal to u, V_{n-1} and $V(K)$ respectively denote the $(n-1)$- and n-dimensional volume of body K.

Intersection bodies led to the solutions of Busemann-Petty problem by Lutwak. In 1988, Lutwak ([10]) introduced the intersection bodies as follows: For $K \in \mathscr{S}_o^n$, the intersection body, IK, of K is a star body whose radial function is defined by

$$\rho(IK,u) = V_{n-1}(K \cap u^\perp)$$

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for all $u \in S^{n-1}$.

Further, Lutwak ([10]) showed the following general Busemann-Petty problem for star bodies by intersection bodies:

PROBLEM 1.2 (GENERAL BUSEMANN-PETTY PROBLEM). For $K, L \in \mathscr{S}_o^n$, is there the implication

$$IK \subseteq IL \Rightarrow V(K) \leqslant V(L)?$$

Obviously, if K and L both are origin-symmetric convex bodies, then problem 1.2 is problem 1.1. For problem 1.1, Gardner ([1]), Zhang ([24]) showed that it has an affirmative answer for $n \leqslant 4$ and a negative answer for $n \geqslant 5$. For problem 1.2, Lutwak ([10]) gave its an affirmative answer if K is restricted to the class of intersection bodies and two negative answers if K is not origin-symmetric star body or L is not intersection body.

THEOREM 1.A. *For $K, L \in \mathscr{S}_o^n$, if K is an intersection body, then*

$$IK \subseteq IL \Rightarrow V(K) \leqslant V(L).$$

And $V(K) = V(L)$ if and only if $K = L$.

THEOREM 1.B. *For $K \in \mathscr{S}_o^n$, if $K \notin \mathscr{S}_{os}^n$, then there exists $L \in \mathscr{S}_{os}^n$, such that*

$$IK \subset IL.$$

But

$$V(K) > V(L).$$

THEOREM 1.C. *Suppose $L \in \mathscr{S}_o^n$ is sufficiently smooth. If L is not intersection body, then there exists $K \in \mathscr{S}_{os}^n$, such that*

$$IK \subset IL.$$

But

$$V(K) > V(L).$$

During the past nearly three decades, the investigation of Busemann-Petty problem have received considerable attention (see [1, 2, 3, 4, 7, 8, 9, 10, 12, 14, 21, 22, 23, 24]).

In 2006, Haberl and Ludwig ([6]) introduced L_p-intersection bodies as follows: For $K \in \mathscr{S}_o^n$, nonzero $p < 1$, the L_p-intersection body, I_pK, of K is an origin-symmetric star body whose radial function is given by (see [6])

$$\rho_{I_pK}^p(z) = \int_K |z \cdot x|^{-p} dx = \frac{1}{n-p} \int_{S^{n-1}} |z \cdot v|^{-p} \rho_K(v)^{n-p} dv \tag{1.1}$$

for all $z \in \mathbb{R}^n$. Here dv is the element with respect to spherical Lebesgue measure on S^{n-1}. Meanwhile, Haberl and Ludwig ([6]) pointed out that the intersection body of K is obtained can as a limit of L_p-intersection body of K, i.e., for $K \in \mathscr{S}_o^n$ and any $z \in \mathbb{R}^n$,

$$\rho_{IK}(z) = \lim_{p \to 1^-} \frac{1-p}{2} \rho_{I_pK}^p(z).$$

In addition, Haberl and Ludwig ([6]) also introduced the notion of asymmetric L_p-intersection bodies.

Associated with Haberl and Ludwig's L_p-intersection bodies, Yuan and Cheung ([20]) considered the following L_p-Busemann-Petty problem.

PROBLEM 1.3 (L_p-BUSEMANN-PETTY PROBLEM). For $K,L \in \mathscr{S}_o^n$, nonzero $p<1$, is there the implication

$$I_pK \subseteq I_pL \Rightarrow V(K) \leqslant V(L)?$$

For problem 1.3, Yuan and Cheung ([20]) gave an affirmative answer and a negative answer as follows:

THEOREM 1.D. *For $K,L \in \mathscr{S}_o^n$ and nonzero $p<1$. If K is an L_p-intersection body, then for $0<p<1$,*

$$I_pK \subseteq I_pL \Rightarrow V(K) \leqslant V(L);$$

for $p<0$,

$$I_pK \subseteq I_pL \Rightarrow V(K) \geqslant V(L).$$

And $V(K)=V(L)$ if and only if $K=L$.

THEOREM 1.E. *For $K \in \mathscr{S}_o^n$, $0<p<1$. If $K \notin \mathscr{S}_{os}^n$, then there exists $L \in \mathscr{S}_{os}^n$, such that*

$$I_pK \subset I_pL.$$

But

$$V(K) > V(L).$$

Obviously, Theorem 1.D, Theorem 1.E is the L_p-version of Theorem 1.A, Theorem 1.B, respectively.

Meanwhile, associated with asymmetric L_p-intersection bodies, Haberl ([5]) researched corresponding L_p-Busemann-Petty problem and obtained its affirmative and negative forms. Recently, according to general L_p-intersection bodies, Wang and Li ([18]) considered general L_p-Busemann-Petty problem. For the studies of L_p-Busemann-Petty problem, also see [11, 15, 17].

The main goal of this paper is to study the negative forms of L_p-Busemann-Petty problem. Our works belong to the field of L_p-dual Brunn-Minkowski theory. We first extend the scope of negative solutions from $\mathscr{S}_{os}^n$ to $\mathscr{S}_o^n$ in Theorem 1.E.

THEOREM 1.1. *For $K \in \mathscr{S}_o^n$, $0<p<1$. If $K \notin \mathscr{S}_{os}^n$, then there exists $L \in \mathscr{S}_o^n$, such that*

$$I_pK \subset I_pL.$$

But

$$V(K) > V(L).$$

Next, combining with the L_p-intersection bodies, we give the L_p-analogues of Theorem 1.C.

THEOREM 1.2. *Suppose $L \in \mathscr{S}_{os}^n$ is sufficiently smooth. If L is not L_p-intersection body, then there exists $K \in \mathscr{S}_{os}^n$, such that for $0 < p < 1$,*

$$I_pK \subset I_pL.$$

But

$$V(K) > V(L).$$

The proofs of theorems 1.1-1.2 are completed in Section 3.

2. Background materials

2.1. L_p-radial Blaschke combinations, general L_p-radial Blaschke bodies

In 2015, Wang and Wang ([16]) defined the L_p-radial Blaschke combinations (also called the L_p-dual Blaschke combinations) of star bodies as follows: For $K, L \in \mathscr{S}_o^n$, $p \neq n$ and $\lambda, \mu \geqslant 0$ (not both 0), the L_p-radial Blaschke combination, $\lambda \cdot K \widehat{+}_p \mu \cdot L \in \mathscr{S}_o^n$, of K and L is defined by

$$\rho(\lambda \cdot K \widehat{+}_p \mu \cdot L, \cdot)^{n-p} = \lambda \rho(K, \cdot)^{n-p} + \mu \rho(L, \cdot)^{n-p}. \tag{2.1}$$

Here $\lambda \cdot K = \lambda^{1/(n-p)}K$. If $p = 1$, then $\lambda \cdot K \widehat{+}_p \mu \cdot L$ is the radial Blaschke combination $\lambda \cdot K \widehat{+} \mu \cdot L$.

Now, in order to prove our results, we will give the general L_p-radial Blaschke bodies (also called the general L_p-dual Blaschke bodies) as follows: Let

$$\lambda = f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \quad \mu = f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)} \tag{2.2}$$

with $\tau \in [-1, 1]$ and $L = -K$ in (2.1), and write

$$\widetilde{\nabla}_p^\tau K = f_1(\tau) \cdot K \widehat{+}_p f_2(\tau) \cdot (-K). \tag{2.3}$$

We call $\widetilde{\nabla}_p^\tau K$ the general L_p-radial Blaschke body of K. From (2.2) and (2.3), we easily see that $\widetilde{\nabla}_p^1 K = K$, $\widetilde{\nabla}_p^{-1} K = -K$ and

$$\widetilde{\nabla}_p^0 K = \frac{1}{2} \cdot K \widehat{+}_p \frac{1}{2} \cdot (-K). \tag{2.4}$$

Here $\widetilde{\nabla}_p^0 K$ is the L_p-radial Blaschke body $\widetilde{\nabla}_p K$ whose definition was given by Haberl (see [5]).

For the general L_p-radial Blaschke bodies, by (2.2) we know

$$f_1(\tau) + f_2(\tau) = 1. \tag{2.5}$$

Hence, if $K \in \mathscr{S}_{os}^n$ then $\widetilde{\nabla}_p^\tau K \in \mathscr{S}_{os}^n$. If $K \notin \mathscr{S}_{os}^n$, we have the following conclusion.

PROPOSITION 2.1. *For $K,L \in \mathscr{S}_o^n$ and $p \neq n$. If $K \notin \mathscr{S}_{os}^n$, then for $\tau \in [-1,1]$,*

$$\widetilde{\nabla}_p^\tau K \in \mathscr{S}_{os}^n \Leftrightarrow \tau = 0. \tag{2.6}$$

Proof. If $\tau = 0$, by (2.4) we immediately get $\widetilde{\nabla}_p^\tau K \in \mathscr{S}_{os}^n$.

Conversely, notice that for any $M \in \mathscr{S}_o^n$ and $u \in S^{n-1}$, $\rho_M(-u) = \rho_{-M}(u)$, thus if $\widetilde{\nabla}_p^\tau K \in \mathscr{S}_{os}^n$, then $\widetilde{\nabla}_p^\tau K = -\widetilde{\nabla}_p^\tau K$, i.e., for all $u \in S^{n-1}$,

$$\rho_{\widetilde{\nabla}_p^\tau K}^{n-p}(u) = \rho_{-\widetilde{\nabla}_p^\tau K}^{n-p}(u) = \rho_{\widetilde{\nabla}_p^\tau K}^{n-p}(-u),$$

by (2.3) we have

$$\rho_{f_1(\tau)\cdot K \widehat{+}_p f_2(\tau)\cdot(-K)}^{n-p}(u) = \rho_{f_1(\tau)\cdot K \widehat{+}_p f_2(\tau)\cdot(-K)}^{n-p}(-u).$$

This together with (2.1) yields

$$f_1(\tau)\rho_K^{n-p}(u) + f_2(\tau)\rho_{-K}^{n-p}(u) = f_1(\tau)\rho_K^{n-p}(-u) + f_2(\tau)\rho_{-K}^{n-p}(-u),$$

hence

$$f_1(\tau)\rho_K^{n-p}(u) + f_2(\tau)\rho_{-K}^{n-p}(u) = f_1(\tau)\rho_{-K}^{n-p}(u) + f_2(\tau)\rho_K^{n-p}(u),$$

i.e.,

$$[f_1(\tau) - f_2(\tau)][\rho_K^{n-p}(u) - \rho_{-K}^{n-p}(u)] = 0.$$

Since $K \notin \mathscr{S}_{os}^n$ implies $\rho_K^{n-p}(u) - \rho_{-K}^{n-p}(u) \neq 0$, thus we obtain

$$f_1(\tau) - f_2(\tau) = 0.$$

This and (2.2) give $\tau = 0$. $\square$

2.2. L_p dual mixed volumes

For $K,L \in \mathscr{S}_o^n$, real $p \neq 0$ and $\lambda, \mu \geqslant 0$ (not both 0), the L_p-radial Minkowski combination, $\lambda \circ K \widetilde{+}_p \mu \circ L \in \mathscr{S}_o^n$, of K and L is defined by (see [5, 13])

$$\rho(\lambda \circ K \widetilde{+}_p \mu \circ L, \cdot)^p = \lambda \rho(K,\cdot)^p + \mu \rho(L,\cdot)^p.$$

Here $\lambda \circ K = \lambda^{1/p} K$. The case $p = 1$ yields the radial Minkowski combination $\lambda \circ K \widetilde{+} \mu \circ L$.

Based on the L_p-radial Minkowski combinations of star bodies, Haberl ([5]) showed a class of L_p-dual mixed volumes: For $M,N \in \mathscr{S}_o^n$, $p > 0$ and $\varepsilon > 0$, the L_p-dual mixed volume, $\widetilde{V}_p(M,N)$, of M and N is defined by (for $p \geqslant 1$ see [19])

$$\frac{n}{p}\widetilde{V}_p(M,N) = \lim_{\varepsilon \to 0^+} \frac{V(M \widetilde{+}_p \varepsilon \circ N) - V(M)}{\varepsilon}.$$

From above definition, Haberl ([5]) gave the following integral representation of L_p-dual mixed volume:

$$\widetilde{V}_p(M,N) = \frac{1}{n}\int_{S^{n-1}} \rho_M^{n-p}(u)\rho_N^p(u)du. \tag{2.7}$$

Here du is the element with respect to spherical Lebesgue measure on S^{n-1}.

Taking $M = N$ in (2.7), then we get

$$\widetilde{V}_p(M,M) = V(M) = \frac{1}{n}\int_{S^{n-1}} \rho_M^n(u)du. \tag{2.8}$$

For the L_p-dual mixed volumes, the following corresponding Minkowski inequality was obtained by Haberl ([5]):

THEOREM 2.A. *If $M,N \in \mathscr{S}_o^n$, $n \neq p > 0$, then for $p < n$,*

$$\widetilde{V}_p(M,N) \leqslant V(M)^{\frac{n-p}{n}} V(N)^{\frac{p}{n}}, \tag{2.9}$$

for $p > n$,

$$\widetilde{V}_p(M,N) \geqslant V(M)^{\frac{n-p}{n}} V(N)^{\frac{p}{n}}.$$

In every inequality, equality holds if and only if M and N are dilatates.

2.3. L_{-p}-cosine transformations

In 2008, Haberl ([5]) introduced the L_{-p}-cosine transformations as follows: For nonzero $p < 1$ and function $f \in C(S^{n-1})$, the L_{-p}- cosine transformation is defined by

$$C_{-p}f(u) = \int_{S^{n-1}} |u\cdot v|^{-p} f(v)dv, \quad u \in S^{n-1}. \tag{2.10}$$

Here $C(S^{n-1})$ denotes the set of all continuous functions on S^{n-1}.

From (2.10) and (1.1), we easily see that for all $u \in S^{n-1}$,

$$\rho_{I_pK}^p(u) = \frac{1}{n-p} C_{-p}\rho_K^{n-p}(u). \tag{2.11}$$

If $F,G \in C(S^{n-1})$, write

$$(F,G) = \frac{1}{n}\int_{S^{n-1}} F(u)G(u)du,$$

then by (2.10) we have

$$(C_{-p}f,g) = (f,C_{-p}g) = \frac{1}{n}\int_{S^{n-1}}\int_{S^{n-1}} |u\cdot v|^{-p} f(u)g(u)dudv. \tag{2.12}$$

For the L_{-p}-cosine transformation C_{-p}, Haberl ([5]) proved the following fact.

THEOREM 2.B. *If nonzero $p < 1$ is not an integer, then $C_{-p} : C_e(S^{n-1}) \to C_e(S^{n-1})$ is injective.*

Here $C_e(S^{n-1})$ denotes the set of all even continuous functions on S^{n-1}.

3. Proofs of theorems 1.1-1.2

Theorems 1.1-1.2 show two negative forms of L_p-Busemann-Petty problem. In this section, we will prove theorems 1.1-1.2. The proof of Theorem 1.1 needs the following lemmas.

LEMMA 3.1. *If $K \in \mathscr{S}_o^n$, nonzero $p<1$ and $\tau \in [-1,1]$, then*

$$V(\widetilde{\nabla}_p^{\tau} K) \leqslant V(K). \tag{3.1}$$

Equality holds for $\tau \in (-1,1)$ if and only if K is origin-symmetric; for $\tau = \pm 1$, (3.1) becomes an equality.

Proof. According to (2.1) and (2.7), we have for any $Q \in \mathscr{S}_o^n$,

$$\begin{aligned}\widetilde{V}_p(\lambda \cdot K \widehat{+}_p \mu \cdot L, Q) &= \frac{1}{n}\int_{S^{n-1}} \rho^{n-p}_{\lambda\cdot K\widehat{+}_p\mu\cdot L}(u)\rho_Q^p(u)du \\ &= \frac{\lambda}{n}\int_{S^{n-1}} \rho_K^{n-p}(u)\rho_Q^p(u)du + \frac{\mu}{n}\int_{S^{n-1}} \rho_L^{n-p}(u)\rho_Q^p(u)du \\ &= \lambda \widetilde{V}_p(K,Q) + \mu \widetilde{V}_p(L,Q).\end{aligned}$$

For nonzero $p<1$, by inequality (2.9) we obtain

$$\widetilde{V}_p(\lambda \cdot K \widehat{+}_p \mu \cdot L, Q) \leqslant [\lambda V(K)^{\frac{n-p}{n}} + \mu V(L)^{\frac{n-p}{n}}]V(Q)^{\frac{p}{n}}.$$

Let $Q = \lambda \cdot K \widehat{+}_p \mu \cdot L$ in above inequality, then

$$V(\lambda \cdot K \widehat{+}_p \mu \cdot L)^{\frac{n-p}{n}} \leqslant \lambda V(K)^{\frac{n-p}{n}} + \mu V(L)^{\frac{n-p}{n}}. \tag{3.2}$$

And the equality condition of inequality (2.9) implies that equality holds in (3.2) for $\lambda, \mu > 0$ if and only if K and L are dilatates (if $\lambda = 0$ or $\mu = 0$, then (3.2) becomes an equality).

From (3.2), (2.3), (2.2) and (2.5), and notice that $V(K) = V(-K)$, we get

$$V(\widetilde{\nabla}_p^{\tau} K)^{\frac{n-p}{n}} \leqslant V(K)^{\frac{n-p}{n}},$$

this together with nonzero $p<1$ gives inequality (3.1).

Because of $f_1(\tau), f_2(\tau) > 0$ when $\tau \in (-1,1)$. Therefore, according to the equality condition of (3.2), we know that equality holds in (3.1) for $\tau \in (-1,1)$ if and only if K and $-K$ are dilatates, that is K is origin-symmetric.

If $\tau = \pm 1$, then by $\widetilde{\nabla}_p^{\pm 1} K = \pm K$ we see that (3.1) becomes an equality. □

LEMMA 3.2. *For $K \in \mathscr{S}_o^n$, nonzero $p<1$ and $\tau \in [-1,1]$, then*

$$I_p(\widetilde{\nabla}_p^{\tau} K) = I_p K. \tag{3.3}$$

Proof. From (1.1), (2.1), (2.3) and (2.5), and notice that $I_pK = I_p(-K)$, we have that for all $u \in S^{n-1}$,

$$\begin{aligned}\rho(I_p(\widetilde{\nabla}_p^\tau K),u)^p &= \frac{1}{n-p}\int_{S^{n-1}} |u\cdot v|^{-p}\rho^{n-p}_{\widetilde{\nabla}_p^\tau K}(v)dv \\ &= \frac{1}{n-p}\int_{S^{n-1}} |u\cdot v|^{-p}[f_1(\tau)\rho_K^{n-p}(v)+f_2(\tau)\rho_{-K}^{n-p}(v)]dv \\ &= f_1(\tau)\rho^p_{I_pK}(u)+f_2(\tau)\rho^p_{I_p(-K)}(u)=\rho^p_{I_pK}(u).\end{aligned}$$

This gives (3.3). $\square$

Proof of Theorem 1.1. Since $K \notin \mathscr{S}^n_{os}$, thus by inequality (3.1) we know that for $0<p<1$ and $\tau\in(-1,1)$,

$$V(\widetilde{\nabla}_p^\tau K) < V(K).$$

Choose $\varepsilon>0$ such that

$$V((1+\varepsilon)\widetilde{\nabla}_p^\tau K) < V(K).$$

From this, let $L=(1+\varepsilon)\widetilde{\nabla}_p^\tau K$, then $L\in\mathscr{S}^n_o$ (Proposition 2.1 gives that for $\tau=0$, $L\in\mathscr{S}^n_{os}$; for $\tau\in(-1,1)$ and $\tau\neq 0$, $L\in\mathscr{S}^n_o\backslash\mathscr{S}^n_{os}$) and satisfies $V(L)<V(K)$.

But by (3.3) and notice that $I_p(cK)=c^{\frac{n-p}{p}}I_pK$ for $c>0$, we obtain that for $0<p<1$,

$$I_pL = I_p((1+\varepsilon)\widetilde{\nabla}_p^\tau K) = (1+\varepsilon)^{\frac{n-p}{p}}I_p(\widetilde{\nabla}_p^\tau K) = (1+\varepsilon)^{\frac{n-p}{p}}I_pK\supset I_pK. \quad \square$$

Proof of Theorem 1.2. Let $C_e^\infty(S^{n-1})$ denotes the set of all even and infinite smooth functions on S^{n-1}. Because of $L\in\mathscr{S}^n_{os}$ is infinite smooth, thus $\rho_L\in C_e^\infty(S^{n-1})$. By Theorem 2.B we know that there exists $\varphi\in C_e^\infty(S^{n-1})$, such that for nonzero $p<1$, $\rho_L^p = C_{-p}\varphi$. Since L is not L_p-intersection body, hence function φ must be negative. Otherwise, if $\varphi\geqslant 0$ and notice $\varphi\in C_e^\infty(S^{n-1})$, then there exists infinite smooth $Q\in\mathscr{S}^n_{os}$ such that $\frac{1}{n-p}\rho_Q^{n-p}=\varphi$. From this, we know that $C_{-p}\varphi=\frac{1}{n-p}C_{-p}\rho_Q^{n-p}$, this together with (2.11) yields $\rho_L^p=\rho^p_{I_pQ}$, i.e., L is an L_p-intersection body. This leads to contradiction.

Therefore, choose $F\in C_e^\infty(S^{n-1})$ and F is not identically zero, such that $F>0$ when $\varphi<0$; $F=0$ when $\varphi\geqslant 0$. From this, we have

$$(F,\varphi)=\frac{1}{n}\int_{S^{n-1}}F(v)\varphi(v)dv<0. \tag{3.4}$$

And according to $F\in C_e^\infty(S^{n-1})$, by Theorem 2.B we know that there exists $g\in C_e^\infty(S^{n-1})$, such that $F=C_{-p}g$. Because of $\rho_L>0$ ($L\in\mathscr{S}^n_{os}$), thus there exists $\varepsilon>0$, such that $\rho_L^{n-p}-\varepsilon g>0$. Notice that $\rho_L^{n-p}-\varepsilon g\in C_e^\infty(S^{n-1})$, then there exists $K\in\mathscr{S}^n_{os}$ is infinite smooth, such that $\rho_K^{n-p}=\rho_L^{n-p}-\varepsilon g$. This yields

$$C_{-p}\rho_K^{n-p}=C_{-p}\rho_L^{n-p}-\varepsilon C_{-p}g,$$

i.e.,

$$\rho_{I_pK}^p = \rho_{I_pL}^p - \varepsilon F < \rho_{I_pL}^p.$$

This and $0 < p < 1$ give $I_pK \subset I_pL$.

But by (2.7), (2.11) and (3.4) we have

$$\begin{aligned} V(L) - \widetilde{V}_p(K,L) &= \widetilde{V}_p(L,L) - \widetilde{V}_p(K,L) = (\rho_L^{n-p}, \rho_L^p) - (\rho_K^{n-p}, \rho_L^p) = (\rho_L^{n-p} - \rho_K^{n-p}, \rho_L^p) \\ &= (\rho_L^{n-p} - \rho_K^{n-p}, C_{-p}\varphi) = (C_{-p}\rho_L^{n-p} - C_{-p}\rho_K^{n-p}, \varphi) = (\rho_{I_pL}^p - \rho_{I_pK}^p, \varphi) \\ &= (\varepsilon F, \varphi) = \varepsilon(F, \varphi) < 0. \end{aligned}$$

Using inequality (2.9) we obtain

$$V(L) < \widetilde{V}_p(K,L) \leqslant V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}},$$

i.e., $V(K) > V(L)$. $\square$

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