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REMARK ON THE PAPER OF ZHENG JIE SUN AND LING ZHU

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Dedicated to Professor Josip Pečarić

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Abstract. In this short review note we show that the new proof of Theorem 1.1 given by Zheng Jie Sun and Ling Zhu in the paper *Simple proofs of the Cusa-Huygens-type and Becker-Stark-type inequalities* is logically incorrect and present another simple proof of the same.

1. Remarks

The sharp circular inequality [1, 5]

$$\frac{\sin x}{x} < \frac{2+\cos x}{3};\ x \in (0, \pi/2) \tag{1}$$

is known as Cusa-Huygens inequality. C.-P. Chen, W.-S. Cheung [2] and József Sándor [6] extended and sharpened inequality (1) independently. Their common result is as stated below:

$$\left(\frac{2+\cos x}{3}\right)^{\theta} < \frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)^{\vartheta};\ x \in (0, \pi/2) \tag{2}$$

with the best positive constants $\theta \approx 1.1137399$ and $\vartheta = 1$.

In 2013, Zheng Jie Sun and Ling Zhu [7, Theorem 1.1] presented new proof of inequalities in (2). The authors of this paper [7] obtained that

$$g(x) > \left(\frac{x\cos x - \sin x}{x\sin x} + \frac{\sin x}{2+\cos x}\right)\ln\left(\frac{\sin x}{x}\right) \tag{3}$$

where

$$g(x) = \frac{x\cos x - \sin x}{x\sin x}\ln\left(\frac{2+\cos x}{3}\right) + \frac{\sin x}{2+\cos x}\ln\left(\frac{\sin x}{x}\right).$$

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Using (3) they proved (2). In what follows, we explain how intermediate result (3) is logically incorrect.

By virtue of (1) we have

$$\ln\left(\frac{\sin x}{x}\right) < \ln\left(\frac{2+\cos x}{3}\right)$$

which gives

$$\frac{x\cos x - \sin x}{x\sin x}\ln\left(\frac{2+\cos x}{3}\right) < \frac{x\cos x - \sin x}{x\sin x}\ln\left(\frac{\sin x}{x}\right)$$

since $x\cos x - \sin x < 0$ as $\cos x < \frac{\sin x}{x}$ [3].
This in turn results in to

$$g(x) < \left(\frac{x\cos x - \sin x}{x\sin x} + \frac{\sin x}{2+\cos x}\right)\ln\left(\frac{\sin x}{x}\right).$$

Thus it is clear that, the result in (3) is logically incorrect. The authors of [7] still proved their main result (2)[7, Theorem 1.1] using this incorrect result (3), which is a mathematical mistake. So their proof as they claimed cannot be considered as a new proof of inequalities in (2). However, they gave new and simple proof of another theorem [7, Theorem 1.2].

2. Main result

We give simple proof of (2) by using following lemma.

LEMMA 1. *(l'Hôpital's Rule* [4] *of monotonicity): Let f,g be two real valued functions which are continuous on $[a,b]$ and derivable on (a,b) and $g' \neq 0$. Then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are increasing(or decreasing) on (a,b) if f'/g' is increasing(or decreasing) on (a,b). The monotonicity in the conclusion is strict if f'/g' is strictly monotone.*

Simple Proof of Double Inequality (2):
Consider,

$$f(x) = \frac{\ln(\sin x/x)}{\ln\left(\frac{2+\cos x}{3}\right)} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = \ln(\sin x/x)$ and $f_2(x) = \ln\left(\frac{2+\cos x}{3}\right)$ with $f_1(0+) = 0 = f_2(0)$. By differentiation

$$\frac{f_1'(x)}{f_2'(x)} = \frac{(\sin x - x\cos x)(2+\cos x)}{x\sin^2 x} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = (\sin x - x\cos x)(2+\cos x)$ and $f_4(x) = x\sin^2 x$ with $f_3(0) = 0 = f_4(0)$. Again differentiating we get

$$\frac{f_3'(x)}{f_4'(x)} = \frac{2x\cos x + 2x - \sin x}{2x\cos x + \sin x} = 1 + \frac{2x - 2\sin x}{2x\cos x + \sin x} = 1 + \frac{2 - 2\sin x/x}{2\cos x + \sin x/x}$$
$$= 1 + g(x)h(x)$$

such that $g(x) = 2 - 2\frac{\sin x}{x}$ and $h(x) = \frac{1}{2\cos x + \frac{\sin x}{x}}$.
Now $\cos x$ and $\sin x/x$ are clearly positive decreasing functions and $\sin x/x < 1$, we have that $g(x)$ and $h(x)$ are both positive increasing functions which are differentiable on $(0,\pi/2)$. Therefore $h(x), h'(x) > 0$ and $g(x), g'(x) > 0$. Hence, $[g(x)h(x)]' > 0$, which shows that $f_3'(x)/f_4'(x)$ is strictly increasing in $(0,\pi/2)$. By Lemma 1, $f(x)$ is also strictly increasing in $(0,\pi/2)$. Therefore

$$f(0+) < f(x) < f(\pi/2); 0 < x < \pi/2.$$

Consequently, $\theta = f(\pi/2) = \frac{\ln(2/\pi)}{\ln(2/3)} \approx 1.1137399$ and $\vartheta = f(0+) = 1$ by l'Hôpital's rule. □

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