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REVERSE HILBERT TYPE INEQUALITIES

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Abstract. In this paper, some reverse Hilbert-Pachpatte's type inequalities involving series of nonnegative terms are established, which provide new estimates on inequality of this type.

1. Introduction

The well-known Hilbert's inequality can be stated below ([1, p.226])

THEOREM A. *Let $a_m, b_n \geqslant 0$, $0 < \sum_1^\infty a_m^p \leqslant \infty$ and $0 < \sum_1^\infty b_n^q \leqslant \infty$. If $p > 1$ and $q = p/(p-1)$, then*

$$\sum_1^\infty \sum_1^\infty \frac{a_m b_n}{m+n} \leqslant \frac{\pi}{sin(\pi/p)} \left(\sum_1^\infty a_m^p\right)^{1/p} \left(\sum_1^\infty b_n^q\right)^{1/q}. \tag{1.1}$$

Hilbert's inequality and its integral form were studied extensively and numerous variants, generalizations, and extensions appeared in the literature [2-11] and the references cited therein. Some researches on reverse Hilbert inequalities were published in [12-13] and et al. The latest research on this type of inequality can be found in the literature [14]. In 1998, Pachpatte [15] proved new inequalities similar to Hilbert's inequality (1.1) and the main results are the following theorems.

THEOREM B. *Let $p \geqslant 1, q \geqslant 1$ and $\{a_m\}$ and $\{b_n\}$ be two non-negative sequences of real numbers defined for $m = 1, \ldots, k$ and $n = 1, \ldots, r$, where k, r are natural numbers. Let $A_m = \sum_{s=1}^m a_s$ and $B_n = \sum_{t=1}^n b_t$. Then*

$$\sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{m+n} \leqslant C(p,q,k,r) \left(\sum_{m=1}^k (k-m+1)\left(a_m A_m^{p-1}\right)^2\right)^{1/2} \times \left(\sum_{n=1}^r (r-n+1)\left(b_n B_n^{q-1}\right)^2\right)^{1/2} \tag{1.2}$$

where

$$C(p,q,k,r) = \frac{1}{2} pq(kr)^{1/2}. \tag{1.3}$$

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THEOREM C. *Let $\{a_m\}$, $\{b_n\}$, A_m, B_n be as defined in Theorem A. Let $\{p_m\}$ and $\{q_n\}$ be positive sequences for $m = 1,\dots,k$ and $n = 1,\dots,r$, where k,r are natural numbers. Define $P_m = \sum_{s=1}^{m} p_s$ and $Q_n = \sum_{t=1}^{n} q_t$. Let ϕ and ψ be nonnegative, convex, sub-multiplicative functions defined on $\mathbb{R}_+ = [0,+\infty)$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{m+n} \leqslant M(k,r)\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^2\right)^{1/2} \times\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^2\right)^{1/2}, \tag{1.4}$$

where

$$M(k,r) = \frac{1}{2}\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)^{1/2}\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)^{1/2}. \tag{1.5}$$

The main purpose of this paper is to establish reverse Hilbert-Pachpatte's type inequalities of (1.1) and (1.4), these new inequalities provide new estimates on inequalities of these type.

2. Lemmas

LEMMA 2.1. *[1, p.39] If x and y are positive real numbers and $0 \leqslant \alpha \leqslant 1$, then*

$$\alpha x^{\alpha-1}(x-y) \leqslant x^\alpha - y^\alpha \leqslant \alpha y^{\alpha-1}(x-y). \tag{2.1}$$

LEMMA 2.2. *[16] If a,b are positive real numbers and $\frac{1}{p}+\frac{1}{q} = 1$ and $p > 1$, then*

$$S\left(\frac{a}{b}\right)a^{1/p}b^{1/q} \geqslant \frac{a}{p}+\frac{b}{q}, \tag{2.2}$$

where

$$S(h) = \frac{h^{1/(h-1)}}{e\log h^{1/(h-1)}},\ 0 < h \neq 1. \tag{2.3}$$

LEMMA 2.3. *If $\{a_i\}$ and $\{b_i\}$ $(i = 1,\dots,n)$ are positive real sequences, then*

$$\sum_{i=1}^{n} S\left(\frac{Ya_i^p}{Xb_i^q}\right)a_ib_i \geqslant \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} b_i^q\right)^{1/q}, \tag{2.4}$$

where $S\left(\frac{Ya_i^p}{Xb_i^q}\right)$ is as in (2.3), and

$$X = \sum_{i=1}^{n} a_i^p, \quad Y = \sum_{i=1}^{n} b_i^q.$$

Proof. Let

$$a=\frac{\left(S\left(\frac{Ya_i^p}{Xb_i^q}\right)\right)^{-1}\cdot a_i^p}{X},\quad b=\frac{\left(S\left(\frac{Ya_i^p}{Xb_i^q}\right)\right)^{-1}\cdot b_i^q}{Y},$$

and by using Lemma 2.2, we have

$$\begin{aligned}&S\left(\frac{Ya_i^p}{Xb_i^q}\right)\cdot\frac{a_ib_i}{X^{1/p}Y^{1/q}}\left(S\left(\frac{Ya_i^p}{Xb_i^q}\right)\right)^{-1/p}\left(S\left(\frac{Ya_i^p}{Xb_i^q}\right)\right)^{-1/q}\\ \geqslant&\frac{1}{p}\frac{a_i^p}{X}\left(S\left(\frac{Ya_i^p}{Xb_i^q}\right)\right)^{-1}+\frac{1}{q}\frac{b_i^q}{Y}\left(S\left(\frac{Ya_i^p}{Xb_i^q}\right)\right)^{-1}.\end{aligned}$$

Hence

$$S\left(\frac{Ya_i^p}{Xb_i^q}\right)\frac{a_ib_i}{X^{1/p}Y^{1/q}}\geqslant\frac{1}{p}\frac{a_i^p}{X}+\frac{1}{q}\frac{b_i^q}{Y}.$$

Taking the sum over i from 1 to n, we obtain

$$\frac{1}{X^{1/p}Y^{1/q}}\sum_{i=1}^{n}S\left(\frac{Ya_i^p}{Xb_i^q}\right)a_ib_i\geqslant\frac{1}{pX}\sum_{i=1}^{n}a_i+\frac{1}{qY}\sum_{i=1}^{n}b_i=1.$$

Hence

$$\sum_{i=1}^{n}S\left(\frac{Ya_i^p}{Xb_i^q}\right)a_ib_i\geqslant\left(\sum_{i=1}^{n}a_i^p\right)^{1/p}\left(\sum_{i=1}^{n}b_i^q\right)^{1/q}.\quad\square$$

LEMMA 2.4. *[2] (Jensen's inequality) If $f(x)$ is continuous and convex function and p_i $(i=1,2,\dots,n)$ are nonnegative real numbers (not all 0), then*

$$f\left(\frac{\sum_{i=1}^{n}p_ix_i}{\sum_{i=1}^{n}p_i}\right)\leqslant\frac{\sum_{i=1}^{n}p_if(x_i)}{\sum_{i=1}^{n}p_i},\tag{2.5}$$

with equality if and only if $x_1=\cdots=x_n$.

This inequality is reversed if $f(x)$ is concave function.

3. Main results

First, we establish a reverse Hilbert type inequality of (1.1). Our main result is given in the following theorem.

THEOREM 3.1. *Let $0\leqslant p\leqslant 1, 0\leqslant q\leqslant 1$ and $\{a_m\}$ and $\{b_n\}$ be two non-negative and decreasing sequences of real numbers defined for $m=1,2,\dots,k$ and $n=1,2,\dots,r$ where k,r are the natural numbers and define $A_m=\sum_{s=1}^{m}a_s$, $B_n=\sum_{t=1}^{n}b_t$ and $A_0=B_0=0$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{S_{p,q,k,r,m,n}A_m^pB_n^q}{(mn)^{1/2}}$$

$$\geqslant 2C(p,q,k,r)\left(\sum_{m=1}^{k}(k-m+1)(a_mA_m^{p-1})^2\right)^{1/2}\left((r-n+1)\sum_{n=1}^{r}(b_nB_n^{q-1})^2\right)^{1/2}, \tag{3.1}$$

where $C(p,q,k,r)$ *is as in* (1.3)*, and*

$$S_{p,q,k,r,m,n}=S\left(\frac{r\sum_{t=1}^{n}(b_tB_t^{q-1})^2}{\sum_{t=1}^{r}(b_tB_t^{q-1})^2(r-t+1)}\right)S\left(\frac{k\sum_{s=1}^{m}(a_sA_s^{p-1})^2}{\sum_{s=1}^{k}(a_sA_s^{p-1})^2(k-s+1)}\right)$$
$$\times S\left(\frac{m(a_uA_u^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)S\left(\frac{n(b_vB_v^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right),$$

where

$$S\left(\frac{m(a_uA_u^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)=\max\left\{S\left(\frac{m(a_1A_1^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right);S\left(\frac{m(a_mA_m^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)\right\},$$

and

$$S\left(\frac{n(b_vB_v^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right)=\max\left\{S\left(\frac{n(b_1B_1^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right);S\left(\frac{n(b_nB_n^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right)\right\},$$

and S *is as in* (2.3).

Proof. From Lemma 2.1, we obtain

$$\sum_{m=0}^{k-1}pA_{m+1}^{p-1}(A_{m+1}-A_m)\leqslant\sum_{m=0}^{k-1}(A_{m+1}^p-A_m^p).$$

Namely

$$A_k^p\geqslant p\sum_{m=0}^{k-1}a_{m+1}A_{m+1}^{p-1}. \tag{3.2}$$

By replacing m with s first, and then replacing k with m in (3.2), we have

$$A_m^p\geqslant p\sum_{s=1}^{m}a_sA_s^{p-1}\quad m=1,2,\ldots,k.$$

Hence

$$S\left(\frac{m(a_uA_u^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)A_m^p\geqslant p\sum_{s=1}^{m}\left(S\left(\frac{m(a_uA_u^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)\cdot a_sA_s^{p-1}\right).$$

Because of

$$\frac{m(a_1A_1^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\geqslant\cdots\geqslant 1\geqslant\cdots\geqslant\frac{m(a_mA_m^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}.$$

On the other hand, S function is decreasing on $(0,1)$ and increasing on $(1,\infty)$. So, we know that one of the $S\left(\frac{m(a_1A_1^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)a_sA_s^{p-1}$ and $S\left(\frac{m(a_mA_m^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)a_sA_s^{p-1}$ is maximum. Therefore

$$S\left(\frac{m(a_uA_u^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)A_m^p \geqslant p\sum_{s=1}^{m}\left(S\left(\frac{m(a_sA_s^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)\cdot a_sA_s^{p-1}\right). \tag{3.3}$$

Similarly

$$S\left(\frac{n(b_vB_v^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right)B_n^q \geqslant q\sum_{t=1}^{n}\left(S\left(\frac{n(b_tB_t^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right)\cdot b_tB_t^{q-1}\right), \tag{3.4}$$

where $S\left(\frac{m(a_uA_u^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)$ and $S\left(\frac{n(b_vB_v^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right)$ are defined in Theorem 3.1. From (3.3) and (3.4), and in view of Lemma 2.3, we have

$$\begin{aligned}
&S\left(\frac{m(a_uA_u^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)S\left(\frac{n(b_vB_v^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right)A_m^pB_n^q\\
&\geqslant pq\sum_{s=1}^{m}\left(S\left(\frac{m(a_sA_s^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)\cdot a_sA_s^{p-1}\times 1\right)\sum_{t=1}^{n}\left(S\left(\frac{n(b_tB_t^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right)\cdot b_tB_t^{q-1}\times 1\right)\\
&\geqslant pq(mn)^{1/2}\left(\sum_{s=1}^{m}\left(a_sA_s^{p-1}\right)^2\right)^{1/2}\cdot\left(\sum_{t=1}^{n}\left(b_tB_t^{q-1}\right)^2\right)^{1/2}.
\end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by

$$S\left(\frac{k\sum_{s=1}^{m}(a_sA_s^{p-1})^2}{\sum_{s=1}^{k}(a_sA_s^{p-1})^2(k-s+1)}\right)S\left(\frac{r\sum_{t=1}^{n}(b_tB_t^{q-1})^2}{\sum_{t=1}^{r}(b_tB_t^{q-1})^2(r-t+1)}\right),$$

we have

$$\begin{aligned}
&S\left(\frac{r\sum_{t=1}^{n}(b_tB_t^{q-1})^2}{\sum_{t=1}^{r}(b_tB_t^{q-1})^2(r-t+1)}\right)S\left(\frac{k\sum_{s=1}^{m}(a_sA_s^{p-1})^2}{\sum_{s=1}^{k}(a_sA_s^{p-1})^2(k-s+1)}\right)\\
&\times S\left(\frac{m(a_uA_u^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)S\left(\frac{n(b_vB_v^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right)A_m^pB_n^q\\
&\geqslant pq(mn)^{1/2}S\left(\frac{k\sum_{s=1}^{m}(a_sA_s^{p-1})^2}{\sum_{s=1}^{k}(a_sA_s^{p-1})^2(k-s+1)}\right)\left(\sum_{s=1}^{m}\left(a_sA_s^{p-1}\right)^2\right)^{1/2}\\
&\times S\left(\frac{r\sum_{t=1}^{n}(b_tB_t^{q-1})^2}{\sum_{t=1}^{r}(b_tB_t^{q-1})^2(r-t+1)}\right)\left(\sum_{t=1}^{n}\left(b_tB_t^{q-1}\right)^2\right)^{1/2}.
\end{aligned} \tag{3.6}$$

Dividing both sides of (3.6) by $(mn)^{1/2}$ first and then taking the sum over n from 1 to r and the sum over m from 1 to k, and by using Lemma 2.3, we obtain

$$\begin{aligned}
&\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{S_{p,q,k,r,m,n}A_m^pB_n^q}{(mn)^{1/2}}\\
&\geqslant pq\sum_{m=1}^{k}S\left(\frac{k\sum_{s=1}^{m}(a_sA_s^{p-1})^2}{\sum_{s=1}^{k}(a_sA_s^{p-1})^2(k-s+1)}\right)\left(\sum_{s=1}^{m}\left(a_sA_s^{p-1}\right)^2\right)^{1/2}\\
&\quad\times\sum_{n=1}^{r}S\left(\frac{r\sum_{t=1}^{n}(b_tB_t^{q-1})^2}{\sum_{t=1}^{r}(b_tB_t^{q-1})^2(r-t+1)}\right)\left(\sum_{t=1}^{n}\left(b_tB_t^{q-1}\right)^2\right)^{1/2}\\
&=pq\sum_{m=1}^{k}S\left(\frac{k\sum_{s=1}^{m}(a_sA_s^{p-1})^2}{\sum_{m=1}^{k}\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)\left(\left(\sum_{s=1}^{m}\left(a_sA_s^{p-1}\right)^2\right)^{1/2}\times 1\right)\\
&\quad\times\sum_{n=1}^{r}S\left(\frac{r\sum_{t=1}^{n}(b_tB_t^{q-1})^2}{\sum_{n=1}^{r}\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right)\left(\left(\sum_{t=1}^{n}\left(b_tB_t^{q-1}\right)^2\right)^{1/2}\times 1\right)\\
&\geqslant pq(kr)^{1/2}\left(\sum_{m=1}^{k}\sum_{s=1}^{m}(a_sA_s^{p-1})^2\right)^{1/2}\left(\sum_{n=1}^{r}\sum_{t=1}^{n}(b_tB_t^{q-1})^2\right)^{1/2}\\
&=2C(p,q,k,r)\left(\sum_{s=1}^{k}(a_sA_s^{p-1})^2(k-s+1)\right)^{1/2}\left(\sum_{t=1}^{r}(b_tB_t^{q-1})^2(r-t+1)\right)^{1/2}\\
&=2C(p,q,k,r)\left(\sum_{m=1}^{k}(a_mA_m^{p-1})^2(k-m+1)\right)^{1/2}\left(\sum_{n=1}^{r}(b_nB_n^{q-1})^2(r-n+1)\right)^{1/2},
\end{aligned}$$

where $C(p,q,k,r)$ is as in (1.3), and

$$\begin{aligned}
S_{p,q,k,r,m,n}=&S\left(\frac{n\sum_{t=1}^{n}(b_tB_t^{q-1})^2}{\sum_{t=1}^{r}(b_tB_t^{q-1})^2(r-t+1)}\right)S\left(\frac{m\sum_{s=1}^{m}(a_sA_s^{p-1})^2}{\sum_{s=1}^{k}(a_sA_s^{p-1})^2(k-s+1)}\right)\\
&\times S\left(\frac{m(a_uA_u^{p-1})^2}{\sum_{s=1}^{m}(a_sA_s^{p-1})^2}\right)S\left(\frac{n(b_vB_v^{q-1})^2}{\sum_{t=1}^{n}(b_tB_t^{q-1})^2}\right).
\end{aligned}$$

This completes the proof. □

REMARK 3.1. From (1.2) in Theorem B, we may estimate the following product and can get a lower bound.

$$\left(\sum_{m=1}^{k}(k-m+1)\left(a_mA_m^{p-1}\right)^2\right)^{1/2}\left(\sum_{n=1}^{r}(r-n+1)\left(b_nB_n^{q-1}\right)^2\right)^{1/2} \tag{3.7}$$

Namely

$$\left(\sum_{m=1}^{k}(k-m+1)\left(a_mA_m^{p-1}\right)^2\right)^{1/2}\left(\sum_{n=1}^{r}(r-n+1)\left(b_nB_n^{q-1}\right)^2\right)^{1/2}$$

$$\geqslant (C(p,q,k,r))^{-1}\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^pB_n^q}{m+n}.$$

On the other hand, from (3.1) in Theorem 3.1, we have

$$\left(\sum_{m=1}^{k}(k-m+1)(a_mA_m^{p-1})^2\right)^{1/2}\left(\sum_{n=1}^{r}(r-n+1)(b_nB_n^{q-1})^2\right)^{1/2}$$
$$\leqslant \frac{1}{2}(C(p,q,k,r))^{-1}\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{S_{p,q,k,r,m,n}A_m^pB_n^q}{mn}.$$

This is just an upper bound of the product (3.7).

Next, we establish a reverse Hilbert type inequality of (1.2). Our main result is given in the following theorem.

THEOREM 3.2. *Let $\{a_m\},\{b_n\}$ be two non-negative sequences of real numbers defined for $m=1,2,\ldots,k$ and $n=1,2,\ldots,r$ where k,r are the natural numbers. Let $\{p_m\}$ and $\{q_n\}$ be two positive sequences. Let ϕ and ψ be two nonnegative, concave and super-multiplicative functions defined on $\mathbb{R}_+$, and define*

$$\bar{A}_m=\sum_{s=1}^{m}S\left(\frac{m\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}{\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)a_s,\quad \bar{B}_n=\sum_{t=1}^{n}S\left(\frac{n\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}{\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}\right)b_t,$$

and

$$P_m=\sum_{s=1}^{m}S\left(\frac{m\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}{\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)p_s,\quad Q_n=\sum_{t=1}^{n}S\left(\frac{n\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}{\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}\right)q_t.$$

Then

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{S_{k,r,m,n}\phi(\bar{A}_m)\psi(\bar{B}_n)}{(mn)^{1/2}}$$
$$\geqslant 2M(k,r)\left(\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2(k-s+1)\right)^{1/2}\left(\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2(r-t+1)\right)^{1/2}, \tag{3.8}$$

where $M(k,r)$ is as in (1.5), and

$$S_{k,r,m,n}=S\left(\frac{\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2(k-s+1)\left(\frac{\phi(P_m)}{P_m}\right)^2}{\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)$$

$$\times S\left(\frac{\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2(r-t+1)\left(\frac{\psi(Q_n)}{Q_n}\right)^2}{\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}\right),$$

and $S(\cdot)$ *is as in* (2.3).

Proof. From Lemmas 2.3 and 2.4, and in view of ϕ is a super-multiplicative function, we obtain

$$\begin{aligned}
\phi(\bar{A}_m) &= \phi\left(\frac{P_m\sum_{s=1}^{m}S\left(\frac{m\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}{\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)p_sa_s/p_s}{\sum_{s=1}^{m}S\left(\frac{m\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}{\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)p_s}\right)\\
&\geqslant \phi(P_m)\phi\left(\frac{\sum_{s=1}^{m}S\left(\frac{m\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}{\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)p_sa_s/p_s}{\sum_{s=1}^{m}S\left(\frac{m\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}{\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)p_s}\right)\\
&\geqslant \frac{\phi(P_m)}{P_m}\sum_{s=1}^{m}S\left(\frac{m\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}{\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)p_s\phi\left(\frac{a_s}{p_s}\right)\\
&\geqslant \frac{\phi(P_m)}{P_m}m^{1/2}\left(\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2\right)^{1/2}. \qquad (3.9)
\end{aligned}$$

Similarly

$$\psi(\bar{B}_n) \geqslant \frac{\psi(Q_n)}{Q_n}n^{1/2}\left(\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2\right)^{1/2}. \qquad (3.10)$$

Multiplying both sides of (3.9) and (3.10), respectively, by

$$S\left(\frac{\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2(k-s+1)\cdot\left(\frac{\phi(P_m)}{P_m}\right)^2}{\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)\cdot\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right),$$

and

$$S\left(\frac{\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2(r-t+1)\cdot\left(\frac{\psi(Q_n)}{Q_n}\right)^2}{\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)\cdot\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}\right),$$

and then multiplying these two inequalities, we have

$$
\begin{aligned}
&S\left(\frac{\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2(k-s+1)\cdot\left(\frac{\phi(P_m)}{P_m}\right)^2}{\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)\cdot\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)\phi(\bar{A}_m)\\
&\quad\times S\left(\frac{\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2(r-t+1)\cdot\left(\frac{\psi(Q_n)}{Q_n}\right)^2}{\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)\cdot\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}\right)\psi(\bar{B}_n)\\
&\geqslant S\left(\frac{\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2(k-s+1)\cdot\left(\frac{\phi(P_m)}{P_m}\right)^2}{\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)\cdot\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)\frac{\phi(P_m)}{P_m}m^{1/2}\left(\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2\right)^{1/2}\\
&\quad\times S\left(\frac{\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2(r-t+1)\cdot\left(\frac{\psi(Q_n)}{Q_n}\right)^2}{\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)\cdot\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}\right)\frac{\psi(Q_n)}{Q_n}n^{1/2}\left(\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2\right)^{1/2}.
\end{aligned}
\tag{3.11}
$$

Dividing both sides of (3.11) by $(mn)^{1/2}$ first and then taking the sum over n from 1 to r first and then the sum over m from 1 to k, and by using the inequality in Lemma 2.3, we obtain

$$
\begin{aligned}
&\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{S_{k,r,m,n}\phi(\bar{A}_m)\psi(\bar{B}_n)}{(mn)^{1/2}}\\
&\geqslant\sum_{m=1}^{k}S\left(\frac{\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2(k-s+1)\cdot\left(\frac{\phi(P_m)}{P_m}\right)^2}{\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)\cdot\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)\times\frac{\phi(P_m)}{P_m}\left(\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2\right)^{1/2}\\
&\quad\times\sum_{n=1}^{r}S\left(\frac{\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2(r-t+1)\cdot\left(\frac{\psi(Q_n)}{Q_n}\right)^2}{\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)\cdot\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}\right)\\
&\quad\times\frac{\psi(Q_n)}{Q_n}\left(\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2\right)^{1/2}\\
&=\sum_{m=1}^{k}S\left(\frac{\left(\sum_{m=1}^{k}\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2\right)\cdot\left(\frac{\phi(P_m)}{P_m}\right)^2}{\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)\cdot\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)\frac{\phi(P_m)}{P_m}\left(\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2\right)^{1/2}
\end{aligned}
$$

$$\times\sum_{n=1}^{r}S\left(\frac{\left(\sum_{n=1}^{r}\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2\right)\cdot\left(\frac{\psi(Q_n)}{Q_n}\right)^2}{\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)\cdot\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}\right)\frac{\psi(Q_n)}{Q_n}\left(\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2\right)^{1/2}$$

$$\geqslant\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)^{1/2}\left(\sum_{m=1}^{k}\left(\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2\right)\right)^{1/2}$$

$$\times\sum_{n=1}^{r}\left(\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)^{1/2}\left(\sum_{n=1}^{r}\left(\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2\right)\right)^{1/2}$$

$$\geqslant 2M(k,r)\left(\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2(k-s+1)\right)^{1/2}\left(\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2(r-t+1)\right)^{1/2},$$

where $M(k,r)$ is as in (1.5), and

$$S_{k,r,m,n}=S\left(\frac{\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2(k-s+1)\left(\frac{\phi(P_m)}{P_m}\right)^2}{\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^2}\right)$$

$$\times S\left(\frac{\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2(r-t+1)\left(\frac{\psi(Q_n)}{Q_n}\right)^2}{\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^2}\right).\quad\square$$

REMARK 3.2. From (1.4) in Theorem C, we may estimate the following product and can get the lower bound.

$$\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^2\right)^{1/2}\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^2\right)^{1/2}. \tag{3.12}$$

Namely

$$\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^2\right)^{1/2}\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^2\right)^{1/2}$$

$$\geqslant(M(k,r))^{-1}\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{m+n}.$$

On the other hand, from (3.1) in Theorem 3.2, we have

$$\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^2\right)^{1/2}\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^2\right)^{1/2} \tag{3.13}$$

$$\leqslant \frac{1}{2}(M(k,r))^{-1}\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{S_{k,r,m,n}\phi(\bar{A}_m)\psi(\bar{B}_n)}{(mn)^{1/2}}. \tag{3.14}$$

This is just an upper bound of the product (3.12).

THEOREM 3.3. *Let $\{a_m\},\{b_n\}$ be two non-negative sequences of real numbers defined for $m=1,2,\ldots,k$ and $n=1,2,\ldots,r$ where k,r are the natural numbers. Define*

$$A'_m=\sum_{s=1}^{m}S\left(\frac{ma_s^2}{\sum_{s=1}^{m}a_s^2}\right)a_s,\quad B'_n=\sum_{t=1}^{n}S\left(\frac{nb_t^2}{\sum_{t=1}^{n}b_t^2}\right)b_t.$$

Then

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{S_{k,r,m,n}A'_mB'_n}{(mn)^{1/2}}\geqslant (kr)^{1/2}\left(\sum_{m=1}^{k}a_m^2(k-m+1)\right)^{1/2}\left(\sum_{n=1}^{r}b_n^2(r-n+1)\right)^{1/2},$$

where

$$S_{k,r,m,n}=S\left(\frac{\sum_{s=1}^{k}a_s^2(k-s+1)}{k\sum_{s=1}^{m}a_s^2}\right)S\left(\frac{\sum_{t=1}^{r}b_t^2(r-t+1)}{r\sum_{t=1}^{n}b_t^2}\right).$$

and $S(\cdot)$ is as in (2.3).

Proof. This follows immediately from Theorem 3.2 with $\phi(x)=x$ and $\psi(y)=y$. □

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