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GEOMETRIC PROPERTIES OF MATHIEU–TYPE POWER SERIES INSIDE UNIT DISK

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Abstract. In the present investigation we study normalized Mathieu-type power series and find sufficient conditions, so that the normalized Mathieu-type power series have certain geometric properties like close-to-convexity and starlikeness inside the unit disc.

1. Introduction

The following infinite series is named after Émile Leonard Mathieu (1835-1890) who investigated it in his 1890 monograph [10] on elasticity of solid bodies:

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2}\ (r>0). \tag{1}$$

Closed integral representation of the series $S(r)$ is given by (see [8])

$$S(r)=\frac{1}{r}\int_0^{\infty}\frac{t\sin(rt)}{e^t-1}dt. \tag{2}$$

The Mathieu-type power series is defined by (see [17])

$$S(r;z)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2}z^n\ \ (r>0,\ |z|<1). \tag{3}$$

Originally it is defined for function of real variable but we are defining it for function of complex variable. H. Alzer, J. L. Brenner and O. G. Ruehr in [1] obtained

$$\frac{1}{r^2+\frac{1}{2\zeta(3)}}<S(r)<\frac{1}{r^2+1/6} \tag{4}$$

where ζ denotes the zeta function. There has been a rich literature on the study of Mathieu's series, its generalization and its inequalities, one can refer [1, 3, 4, 5, 7, 11, 14]. In the present paper, our aim is to study geometric properties of Matheiu-type power series. For this we need the following well known definitions from geometric

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function theory.
Let $\mathcal{H}$ denote the class of analytic functions inside the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the class of analytic functions inside the unit disk $\mathbb{D}$, having the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in \mathbb{D}. \tag{5}$$

We denote by $\mathcal{S}$, the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{D}$ i. e.

$$\mathcal{S} = \{f \in \mathcal{A} \mid f \text{ is one-to-one in } \mathbb{D}\}.$$

A set G in the complex plane is called starlike with respect to origin if for any point z in G the line segment joining origin to z lies interior of G. A function $f \in \mathcal{A}$ that maps unit disk $\mathbb{D}$ onto a starlike domain is called starlike function and class of such functions is denoted by $\mathcal{S}^*$. Analytically a function $f \in \mathcal{A}$ is called starlike function if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

A set G in the plane is called convex if for every pair of points z_1 and z_2 interior of G, the line segment joining z_1 and z_2 is also in the interior of G. A function $f \in \mathcal{A}$ that maps $\mathbb{D}$ onto a convex domain is called convex function and class of such functions is denoted by $\mathcal{K}$. Analytically a function $f \in \mathcal{A}$ is called convex function if

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

An analytic function f in $\mathbb{D}$ is said to be close-to-convex with respect to a fixed convex function g (need not be normalized), denoted by $\mathcal{C}_g$, if $\mathfrak{Re}\{f'(z)/g'(z)\} > 0, z \in \mathbb{D}$. Every starlike function is close-to-convex. However, the converse is not true. The Noshiro-Warschawski theorem implies that close-to-convex functions are univalent in $\mathbb{D}$, but not necessarily the converse. It is easy to verify that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S}$. Geometrically an analytic function f is called close-to-convex in $\mathbb{D}$, if complement of $f(\mathbb{D})$ can be written as the union of non-intersecting half-lines. For more details see [6].
It is obvious to see that $S(r;z) \notin \mathcal{A}$, so using the following normalization, we have

$$\mathbb{S}(r;z) = \frac{(r^2+1)^2}{2}\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2}z^n = z + \sum_{n=2}^{\infty}\frac{n(r^2+1)^2}{(n^2+r^2)^2}z^n. \tag{6}$$

The aim of the present work is to find sufficient condition such that normalized Mathieu-type power series is close-to-convex and starlike in $\mathbb{D}$, for some related results we refer to [2, 12, 16]. For this we need the following lemmas:

LEMMA 1.1. (Ozaki [13]) *Let* $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$. *Suppose*

$$1 \geqslant 2a_2 \geqslant \cdots \geqslant (n+1)a_{n+1} \geqslant \cdots \geqslant 0 \tag{7}$$

or

$$1 \leqslant 2a_2 \leqslant \cdots \leqslant (n+1)a_{n+1} \leqslant \cdots \leqslant 2. \tag{8}$$

Then f *is close-to-convex with respect to convex function* $-log(1-z)$.

LEMMA 1.2. (Féjer [9]) *If* $a_n \geqslant 0$, $\{na_n\}$ *and* $\{na_n-(n+1)a_{n+1}\}$ *both are non-increasing, then the function* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ *is in* $\mathscr{S}^*$.

LEMMA 1.3. (Féjer [9]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that* $a_1=1$, *and that for* $n \geqslant 2$ *the sequence* $\{a_n\}$ *is a convex decreasing, i.e.*

$$a_1-a_2 \geqslant \cdots \geqslant a_k-a_{k+1} \geqslant \cdots \geqslant 0.$$

Then

$$\mathfrak{Re}\left(\sum_{n=1}^{\infty} a_n z^{n-1}\right) > 1/2\ (z \in \mathbb{D}). \tag{9}$$

Note that each convex decreasing sequence generates also a convex null sequence. Recall that the sequence $a_0,\ a_1,\ldots$ of nonnegative numbers is called a convex null sequence if

$$\lim_{k\to\infty} a_k = 0 \quad \text{and} \quad a_0-a_1 \geqslant a_1-a_2 \geqslant \cdots \geqslant a_k-a_{k+1} \geqslant \cdots \geqslant 0.$$

For a convex null sequence $a_0,\ a_1,\ldots,\ a_0>0$, we have instead of (9) the following inequality

$$\mathfrak{Re}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n z^n\right) > 0\ (z \in \mathbb{D}).$$

2. Close-to-convexity and starlikeness of Mathieu-type power series

Main results are contained in following theorems:

THEOREM 2.1. *For* $r>0$ *and* $z \in \mathbb{D}$

$$|\mathbb{S}(r;z)| \leqslant \frac{(r^2+1)^2}{2(r^2+1/6)} \tag{10}$$

Proof. Using (4) and triangle inequality in $\mathbb{D}$ we get the result. □

THEOREM 2.2. *If* $0<r\leqslant\sqrt{2}$ *then* $\mathbb{S}(r;z)$ *is close-to-convex with respect to* $-log(1-z)$ *in* $\mathbb{D}$.

Proof. Using (6) and (5), we have

$$na_n-(n+1)a_{n+1}=\frac{n^2(r^2+1)^2}{(n^2+r^2)^2}-\frac{(n+1)^2(r^2+1)^2}{((n+1)^2+r^2)^2}=\frac{(r^2+1)^2}{(n^2+r^2)^2((n+1)^2+r^2)^2}X(n),$$

where

$$X(n)=n^2((n+1)^2+r^2)^2-(n+1)^2(n^2+r^2)^2.$$

To show $X(n) \geqslant 0$, it is sufficient to show that

$$
\begin{aligned}
n^2((n+1)^2+r^2)^2 &\geqslant (n+1)^2(n^2+r^2)^2 \\
i.e.\ n((n+1)^2+r^2) &\geqslant (n+1)(n^2+r^2) \\
i.e.\ n^2+n-r^2 &\geqslant 0,
\end{aligned}
$$

which is true for all $n \geqslant 1$ and $0 < r \leqslant \sqrt{2}$ and hence by using Lemma 1.1 we get the required result. □

THEOREM 2.3. *If* $0 < r \leqslant \sqrt{4-\sqrt{13}}$ *then* $\mathbb{S}(r;z)$ *is starlike in* $\mathbb{D}$.

Proof. It is already proved in Theorem 2.2 that $\{na_n\}$ is nonincreasing sequence for all $0 < r \leqslant \sqrt{2}$. To show $\mathbb{S}(r;z)$ is starlike in $\mathbb{D}$, using Lemma 1.2, it is sufficient to show that the sequence $\{na_n - (n+1)a_{n+1}\}$ is also nonincreasing. That is

$$
\begin{aligned}
& na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} \geqslant 0 \\
\Longleftrightarrow\ & (r^2+1)^2\left[\frac{n^2}{(n^2+r^2)^2} - 2\frac{(n+1)^2}{((n+1)^2+r^2)^2} + \frac{(n+2)^2}{((n+2)^2+r^2)^2}\right] \geqslant 0 \\
\Longleftrightarrow\ & [f(n) - 2f(n+1) + f(n+2)] \geqslant 0,
\end{aligned}
$$

where

$$
f(x) = \frac{x^2}{(x^2+r^2)^2}, \quad x \geqslant 1.
$$

To show $[f(n) - 2f(n+1) + f(n+2)] \geqslant 0$, $n = 1,2,3,4,\ldots$, it is sufficient to prove that $f(x)$ is a convex function in the real sense or that $f''(x) \geqslant 0, x \geqslant 1$. Differentiating twice, we have

$$
f''(x) = \frac{2(3x^4 - 8x^2r^2 + r^4)}{(x^2+r^2)^4}, \ x \geqslant 1. \tag{11}
$$

Denominator is already positive for all $x \geqslant 1$ and $r > 0$. Let $\phi(x) = 3x^4 - 8x^2r^2 + r^4$. Obviously $\phi'(x) = 12x^3 - 16r^2x \geqslant 0$ for all $x \geqslant 1$ and $0 < r \leqslant \sqrt{3}/2$. Thus $f''(x) \geqslant 0$ provided $\phi(1) \geqslant 0$, which in turns gives $0 < r \leqslant \sqrt{4-\sqrt{13}}$. This completes the proof. □

THEOREM 2.4. *For* $0 < r \leqslant 1$,

$$
\mathfrak{Re}\left\{\frac{\mathbb{S}(r;z)}{z}\right\} > \frac{1}{2}\ (z \in \mathbb{D}). \tag{12}
$$

Proof. First we prove that

$$
\{a_n\}_{n=1}^{\infty} = \left\{\frac{n(r^2+1)^2}{(n^2+r^2)^2}\right\}_{n=1}^{\infty}
$$

is a decreasing sequence, for this we show

$$a_n - a_{n+1} \geqslant 0 \; \forall n \in \mathbb{N}.$$

Now

$$\begin{aligned} a_n - a_{n+1} \geqslant 0 \iff & (r^2+1)^2 \left[\frac{n}{(n^2+r^2)^2} - \frac{n+1}{((n+1)^2+r^2)^2} \right] \geqslant 0 \\ \iff & (r^2+1)^2 \left[f(n) - f(n+1) \right] \geqslant 0, \end{aligned}$$

where

$$f(x) = \frac{x}{(x^2+r^2)^2} \; (x \geqslant 1). \tag{13}$$

To show $f(n) - f(n+1) \geqslant 0$, $n = 1,2,3,\dots$, it is sufficient to prove that $f(x)$ is a decreasing function in the real sense or that $f'(x) < 0$, $x \geqslant 1$. We have

$$f'(x) = \frac{r^2 - 3x^2}{(x^2+r^2)^3} \leqslant 0 \; (x \geqslant 1 \text{ and } 0 < r \leqslant \sqrt{3}).$$

Next we prove that $\{a_n\}_{n=1}^{\infty}$ is a convex decreasing sequence. For this we show

$$a_{n+2} - a_{n+1} \geqslant a_{n+1} - a_n \;\; \forall \, n \in \mathbb{N}.$$

Now

$$\begin{aligned} & a_n - 2a_{n+1} + a_{n+2} \geqslant 0 \\ \iff & (r^2+1)^2 \left[\frac{n}{(n^2+r^2)^2} - 2\frac{n+1}{((n+1)^2+r^2)^2} + \frac{n+2}{((n+2)^2+r^2)^2} \right] \geqslant 0 \\ \iff & (r^2+1)^2 \left[f(n) - 2f(n+1) + f(n+2) \right] \geqslant 0, \end{aligned}$$

where $f(x)$ is given by (13). To show $[f(n)+f(n+2)-2f(n+1)] \geqslant 0, n=1,2,3,4,\dots,$ it suffices to prove that $f(x)$ is a convex function in the real sense or that $f''(x) \geqslant 0$, $x \geqslant 1$. We have

$$f''(x) = \frac{12x(x^2-r^2)}{(x^2+r^2)^4} \geqslant 0 \;\; (x \geqslant 1 \text{ and } 0 < r \leqslant 1).$$

Thus $\{a_n\}_{n=1}^{\infty}$ is a convex decreasing sequence. Now applying Lemma 1.3 on $\{a_n\}_{n=1}^{\infty}$, we have

$$\mathfrak{Re} \left\{ \sum_{n=1}^{\infty} a_n z^{n-1} \right\} > 1/2, \quad z \in \mathbb{D}.$$

which is equivalent to

$$\mathfrak{Re} \left\{ \frac{\mathbb{S}(r;z)}{z} \right\} > 1/2, \quad z \in \mathbb{D}. \quad \square$$

THEOREM 2.5. *For* $0 < r \leqslant \sqrt{4-\sqrt{13}}$,

$$\mathfrak{Re}\left\{\mathbb{S}'(r;z)\right\} > \frac{1}{2}\ (z \in \mathbb{D}). \tag{14}$$

Proof. From (6),

$$\mathbb{S}'(r;z) = 1 + \sum_{n=2}^{\infty} \frac{n^2(r^2+1)^2}{(n^2+r^2)^2} z^{n-1}. \tag{15}$$

So taking

$$a_n = \frac{n^2(r^2+1)^2}{(n^2+r^2)^2}$$

and proceeding similarly as in Theorem 2.4, we get the proof. □

THEOREM 2.6. *If*

$$r \in \left[\sqrt{3-\sqrt{6}}, \sqrt{3+\sqrt{6}}\right] \tag{16}$$

then

$$\int_0^z \frac{\mathbb{S}(r;t)}{t} \mathrm{d}t \tag{17}$$

is in the class $\mathscr{S}^*$ *of starlike functions.*

Proof. We have

$$\int_0^z \frac{\mathbb{S}(r;t)}{t} \mathrm{d}t = z + \frac{(r^2+1)^2}{2} \sum_{n=2}^{\infty} \frac{2z^n}{(n^2+r^2)^2}. \tag{18}$$

It is known that $\sum_{n=2}^{\infty} n|a_n| < 1$ implies $z + \sum_{n=2}^{\infty} a_n z^n$ is starlike. Hence, from (18), to show that (17) is a starlike function it suffices to prove that

$$\frac{(r^2+1)^2}{2} \sum_{n=2}^{\infty} \frac{2n}{(n^2+r^2)^2} < 1. \tag{19}$$

From (4) we have

$$\frac{(r^2+1)^2}{2} \sum_{n=2}^{\infty} \frac{2n}{(n^2+r^2)^2} < \frac{(r^2+1)^2}{2} \left\{ \frac{1}{r^2+1/6} - \frac{2}{(1+r^2)^2} \right\}.$$

After some simple calculations we can see that for nonnegative r (16) is equivalent to

$$\frac{(r^2+1)^2}{2} \left\{ \frac{1}{r^2+1/6} - \frac{2}{(1+r^2)^2} \right\} \leqslant 1.$$

Simplifying, we get $3r^4 - 6r^2 + 1 \leqslant 0$, which in turns gives $\sqrt{3-\sqrt{6}} \leqslant r \leqslant \sqrt{3+\sqrt{6}}$. This proves the starlikeness of (17). □

We have

$$z+\sum_{n=2}^{\infty}\frac{n(r^2+1)^2}{(n^2+r^2)^2}z^n=\frac{z}{(1-z)^2}*h_2(z)*h_2(z),$$

where

$$h_2(z)=z+\sum_{n=2}^{\infty}\frac{r^2+1}{r^2+n^2}z^n.$$

It is known that

$$h_1(z)=z+\sum_{n=2}^{\infty}\frac{r^2+1}{r^2+n}z^n$$

is convex univalent, so it is possible that $h_2(z)$ is convex univalent too. If we will be able to prove that $h_2(z)$ is convex univalent, then by [15], $\mathbb{S}(r;t)=\frac{z}{(1-z)^2}*h_2(z)*h_2(z)$ is starlike for all nonnegative r.

CONJECTURE. *For all r, $\mathfrak{Re}\{r\}>0$, the function $h_2(z)$ is convex univalent and so $\mathbb{S}(r;t)$ is starlike.*

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