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FURTHER INEQUALITIES FOR THE NUMERICAL RADIUS OF HILBERT SPACE OPERATORS

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Abstract. In this article, we present some new inequalities for numerical radius of Hilbert space operators via convex functions. Our results generalize and improve earlier results by El-Haddad and Kittaneh. Among several results, we show that if $A \in \mathbb{B}(\mathscr{H})$ and $r \geqslant 2$, then

$$w^r(A) \leqslant \|A\|^r - \inf_{\|x\|=1} \left\| |A| - w(A) |^{\frac{r}{2}} x \right\|^2$$

where $w(\cdot)$ and $\|\cdot\|$ denote the numerical radius and usual operator norm, respectively.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a Hilbert space $\mathscr{H}$. As customary, we reserve m, M for scalars. An operator A on $\mathscr{H}$ is said to be positive (in symbol: $A \geqslant 0$) if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. We write $A > 0$ if A is positive and invertible. For self-adjoint operators A and B, we write $A \geqslant B$ if $A - B$ is positive, i.e., $\langle Ax, x\rangle \geqslant \langle Bx, x\rangle$ for all $x \in \mathscr{H}$. In particular, for some scalars m and M, we write $m \leqslant A \leqslant M$ if $m\langle x,x\rangle \leqslant \langle Ax,x\rangle \leqslant M\langle x,x\rangle$ for all $x \in \mathscr{H}$. Here $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A.

If $A \in \mathbb{B}(\mathscr{H})$, the usual operator norm and the numerical radius of A are defined, respectively, by $\|A\| = \sup_{\|x\|=1} \|Ax\|$ and $w(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|$. The numerical radius satisfies

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|, \tag{1.1}$$

which show that $w(A)$ is a norm equivalent to $\|A\|$. We also remark that if $R(A) \perp R(A^*)$, then $w(A) = \frac{1}{2}\|A\|$ (see, e.g., [11, Theorem 1.3.4]).

An improvement of the second inequality in (1.1) has been given in [13, Theorem 1]. It says that for $A \in \mathbb{B}(\mathscr{H})$,

$$w(A) \leqslant \frac{1}{2}\||A| + |A^*|\| \leqslant \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{\frac{1}{2}}\right). \tag{1.2}$$

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Paper JMI-13-68

Consequently, if $A^2 = 0$, then $w(A) = \frac{\|A\|}{2}$. The first inequality of (1.2) was extended in [7] in the following form:

$$w^r(A) \leqslant \frac{1}{2}\left\| |A|^{2rv} + |A^*|^{2r(1-v)} \right\|, \qquad r \geqslant 1,\ 0 < v < 1. \tag{1.3}$$

Also, in the same paper, it was shown that

$$\|A+B\|^2 \leqslant \left\| |A|^2 + |B|^2 \right\| + \left\| |A^*|^2 + |B^*|^2 \right\|. \tag{1.4}$$

The following result concerning the product of two operators was proved in [5]:

$$w^r(B^*A) \leqslant \frac{1}{2}\left\| |A|^{2r} + |B|^{2r} \right\|, \qquad r \geqslant 1. \tag{1.5}$$

A general numerical radius inequality has been proved by Shebrawi and Albadawi [16], it has been shown that if $A, X, B \in \mathbb{B}(\mathscr{H})$, then

$$w^r(A^*XB) \leqslant \frac{1}{2}\left\| \left(A^*|X^*|^{2v}A\right)^r + \left(B^*|X|^{2(1-v)}B\right)^r \right\|, \qquad r \geqslant 1,\ 0 < v < 1. \tag{1.6}$$

Some interesting numerical radius inequalities improving inequalities (1.1) have been obtained by several mathematicians (see [2, 18], and references therein). For a comprehensive overview of the connections among these and other known inequalities in the literature, we refer to [4].

The purpose of this work is to establish some new inequalities for the numerical radius of bounded linear operators in Hilbert spaces. We provide a new estimate for the sum of two operators. After that, we generalize and improve the inequality (1.6). An improvement of inequality $w(A) \leqslant \|A\|$ is also given in the end of Section 2. Section 3 devoted to studying numerical radius inequalities involving f-connection of operators.

2. Inequalities for sums and products of operators

We start this section by an operator norm inequality related to (1.4). In fact we give another upper bound for $\|A+B\|^2$.

THEOREM 2.1. *Let $A, B \in \mathbb{B}(\mathscr{H})$, then*

$$\|A+B\|^2 \leqslant \frac{1}{2}\left[\left\| |A^*|^2 + |B^*|^2 \right\| + \left\| |A^*|^2 - |B^*|^2 \right\| \right] + w(BA^*) + 2\|A\|\,\|B\|. \tag{2.1}$$

Proof. We use the following inequality which is shown in the proof of Theorem 3 in [6]:

$$|\langle z, x\rangle|^2 + |\langle z, y\rangle|^2 \leqslant \|z\|^2 \max\left(\|x\|^2, \|y\|^2\right) + |\langle x, y\rangle|,$$

where $x, y, z \in \mathscr{H}$. Taking $x = A^*y$, $y = B^*y$, and $z = x$ with $\|x\| = \|y\| = 1$, we get

$$|\langle x, A^*y\rangle|^2 + |\langle x, B^*y\rangle|^2 \leqslant \max\left(\|A^*y\|^2, \|B^*y\|^2\right) + |\langle A^*y, B^*y\rangle|.$$

The above inequality is equivalent to

$$|\langle Ax,y\rangle|^2+|\langle Bx,y\rangle|^2\leqslant\frac{1}{2}\left[\langle AA^*+BB^*y,y\rangle+|\langle AA^*-BB^*y,y\rangle|\right]+|\langle BA^*y,y\rangle|$$

thanks to $\max\{a,b\}=\frac{1}{2}(a+b+|a-b|),(a,b\in\mathbb{R})$.

Now, it follows from the tringle inequality that

$$\begin{aligned}|\langle A+Bx,y\rangle|^2\leqslant&|\langle Ax,y\rangle|^2+|\langle Bx,y\rangle|^2+2|\langle Ax,y\rangle||\langle Bx,y\rangle|\\ \leqslant&\frac{1}{2}\left[\langle AA^*+BB^*y,y\rangle+|\langle AA^*-BB^*y,y\rangle|\right]+|\langle BA^*y,y\rangle|+2|\langle Ax,y\rangle||\langle Bx,y\rangle|.\end{aligned}$$

By taking the supremum over $x,y\in\mathscr{H}$ with $\|x\|=\|y\|=1$, we deduce the desired result. □

The following examples show that there is no ordering between our inequality (2.1) and Kittaneh inequality (1.4) in general.

EXAMPLE 2.1. Let $A=\begin{pmatrix}1&0\\-3&1\end{pmatrix}$, $B=\begin{pmatrix}-1&2\\0&1\end{pmatrix}$. After brief computation,

$$\|A+B\|^2\approx 14.52,$$

$$\frac{1}{2}\left[\left\||A^*|^2+|B^*|^2\right\|+\left\||A^*|^2-|B^*|^2\right\|\right]+w(BA^*)+2\|A\|\|B\|\approx 29.58,$$

and

$$\left\||A|^2+|B|^2\right\|+\left\||A^*|^2+|B^*|^2\right\|\approx 25.28.$$

Thus,

$$\begin{aligned}\|A+B\|^2\lneqq&\left\||A|^2+|B|^2\right\|+\left\||A^*|^2+|B^*|^2\right\|\\ \lneqq&\frac{1}{2}\left[\left\||A^*|^2+|B^*|^2\right\|+\left\||A^*|^2-|B^*|^2\right\|\right]+w(BA^*)+2\|A\|\|B\|.\end{aligned}$$

EXAMPLE 2.2. Let $A=\begin{pmatrix}2&0\\3&1\end{pmatrix}$, $B=\begin{pmatrix}0&1\\0&1\end{pmatrix}$. A simple computation shows that

$$\|A+B\|^2\approx 17.94,$$

$$\frac{1}{2}\left[\left\||A^*|^2+|B^*|^2\right\|+\left\||A^*|^2-|B^*|^2\right\|\right]+w(BA^*)+2\|A\|\|B\|\approx 25.4,$$

and

$$\left\||A|^2+|B|^2\right\|+\left\||A^*|^2+|B^*|^2\right\|\approx 29.44.$$

Thus,

$$\begin{aligned}\|A+B\|^2\lneqq&\frac{1}{2}\left[\left\||A^*|^2+|B^*|^2\right\|+\left\||A^*|^2-|B^*|^2\right\|\right]+w(BA^*)+2\|A\|\|B\|\\ \lneqq&\left\||A|^2+|B|^2\right\|+\left\||A^*|^2+|B^*|^2\right\|.\end{aligned}$$

REMARK 2.1. It follows from Theorem 2.1 that

$$\|A+B\|^2 \leqslant \frac{1}{2}\left[\left\||A|^2+|B|^2\right\|+\left\||A|^2-|B|^2\right\|\right]+w\left(BA^*\right)+2\|A\|\|B\|,$$

whenever A and B are two normal operators.

Letting $x=y$ in the proof of Theorem 2.1, we find that:

COROLLARY 2.1. *Let $A,B\in\mathbb{B}(\mathscr{H})$, then*

$$w^2(A+B) \leqslant \frac{1}{2}\left[\left\||A^*|^2+|B^*|^2\right\|+\left\||A^*|^2-|B^*|^2\right\|\right]+w\left(BA^*\right)+2w(A)\,w(B).$$

The following lemmas are useful for generalizing and improving inequality (1.6). The first lemma is known as the generalized mixed Schwarz inequality (see, e.g., [14, Theorem 1]).

LEMMA 2.1. *Let $A\in\mathbb{B}(\mathscr{H})$ and $x,y\in\mathscr{H}$ be any vectors. If f,g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)\,g(t)=t,(t\geqslant 0)$, then*

$$|\langle Ax,y\rangle| \leqslant \|f(|A|)\,x\|\,\|g(|A^*|)\,y\|.$$

The second lemma is well known in the literature as the Mond–Pečarić inequality [15].

LEMMA 2.2. *If f is a convex function on a real interval J containing the spectrum of the self-adjoint operator A, then for any unit vector $x\in\mathscr{H}$,*

$$f(\langle Ax,x\rangle) \leqslant \langle f(A)x,x\rangle \tag{2.2}$$

and the reverse inequality holds if f is concave.

The third lemma is a direct consequence of [3, Theorem 2.3].

LEMMA 2.3. *Let f be a non-negative non-decreasing convex function on $[0,\infty)$ and let $A,B\in\mathbb{B}(\mathscr{H})$ be positive operators. Then for any $0<v<1$,*

$$\|f((1-v)A+vB)\| \leqslant \|(1-v)\,f(A)+v f(B)\|.$$

The above three lemmas admit the following more general result.

PROPOSITION 2.1. *Let $A,B,X\in\mathbb{B}(\mathscr{H})$, and let f and g be non-negative functions on $[0,\infty)$ which are continuous and satisfy the relation $f(t)\,g(t)=t$ for all $t\in[0,\infty)$. If h is a non-negative increasing convex function on $[0,\infty)$, then for any $0<v<1$*

$$h\left(w^2\left(A^*XB\right)\right) \leqslant \left\|(1-v)\,h\left(\left(B^*f^2(|X|)\,B\right)^{\frac{1}{1-v}}\right)+vh\left(\left(A^*g^2(|X^*|)\,A\right)^{\frac{1}{v}}\right)\right\|. \tag{2.3}$$

In particular,

$$w^{2r}\left(A^*XB\right) \leqslant \frac{1}{2}\left\|\left(B^* f^2\left(|X|\right)B\right)^{2r} + \left(A^* g^2\left(|X^*|\right)A\right)^{2r}\right\|, \tag{2.4}$$

for all $r \geqslant 1$.

Proof. For any unit vector $x \in \mathscr{H}$, we have

$$\begin{aligned}
|\langle A^*XBx,x\rangle|^2 &= |\langle XBx,Ax\rangle|^2 \\
&\leqslant \left\langle B^* f^2\left(|X|\right)Bx,x\right\rangle \left\langle A^* g^2\left(|X^*|\right)Ax,x\right\rangle \qquad (2.5)\\
&= \left\langle \left(\left(B^* f^2\left(|X|\right)B\right)^{\frac{1}{1-v}}\right)^{1-v} x,x\right\rangle \left\langle \left(\left(A^* g^2\left(|X^*|\right)A\right)^{\frac{1}{v}}\right)^{v} x,x\right\rangle \\
&\leqslant \left\langle \left(B^* f^2\left(|X|\right)B\right)^{\frac{1}{1-v}} x,x\right\rangle^{1-v} \left\langle \left(A^* g^2\left(|X^*|\right)A\right)^{\frac{1}{v}} x,x\right\rangle^{v} \qquad (2.6)\\
&\leqslant (1-v)\left\langle \left(B^* f^2\left(|X|\right)B\right)^{\frac{1}{1-v}} x,x\right\rangle + v\left\langle \left(A^* g^2\left(|X^*|\right)A\right)^{\frac{1}{v}} x,x\right\rangle \qquad (2.7)\\
&= \left\langle (1-v)\left(B^* f^2\left(|X|\right)B\right)^{\frac{1}{1-v}} + v\left(A^* g^2\left(|X^*|\right)A\right)^{\frac{1}{v}} x,x\right\rangle,
\end{aligned}$$

where (2.5) follows from Lemma 2.1, (2.6) follows from Mond–Pečarić inequality for concave function $f(t) = t^v\,(0<v<1)$, and the weighted arithmetic-geometric mean inequality implies (2.7).

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, we infer that

$$w^2\left(A^*XB\right) \leqslant \left\|(1-v)\left(B^* f^2\left(|X|\right)B\right)^{\frac{1}{1-v}} + v\left(A^* g^2\left(|X^*|\right)A\right)^{\frac{1}{v}}\right\|.$$

On account of assumptions on h, we can write

$$\begin{aligned}
h\left(w^2\left(A^*XB\right)\right) &\leqslant h\left(\left\|(1-v)\left(B^* f^2\left(|X|\right)B\right)^{\frac{1}{1-v}} + v\left(A^* g^2\left(|X^*|\right)A\right)^{\frac{1}{v}}\right\|\right) \\
&= \left\|h\left((1-v)\left(B^* f^2\left(|X|\right)B\right)^{\frac{1}{1-v}} + v\left(A^* g^2\left(|X^*|\right)A\right)^{\frac{1}{v}}\right)\right\| \\
&\leqslant \left\|(1-v)h\left(\left(B^* f^2\left(|X|\right)B\right)^{\frac{1}{1-v}}\right) + vh\left(\left(A^* g^2\left(|X^*|\right)A\right)^{\frac{1}{v}}\right)\right\|, \qquad (2.8)
\end{aligned}$$

where (2.8) follows from Lemma 2.3.

The inequality (2.4) follows directly from (2.3) by taking $h(t) = t^r\,(r \geqslant 1)$ and $v = \frac{1}{2}$. □

Our aim in the next result is to improve (1.6) under some mild conditions. To do this end, we need the following refinement of arithmetic-geometric mean inequality [9, 10].

LEMMA 2.4. *Suppose that $a,b>0$ and positive real numbers m, M satisfy $\min\{a,b\}\leqslant m<M\leqslant \max\{a,b\}$. Then*

$$\frac{M+m}{2\sqrt{Mm}}\sqrt{ab}\leqslant \frac{a+b}{2}.$$

Proof. Consider $f(x)=\frac{2\sqrt{x}}{1+x}$ on $(1\leqslant)\,\frac{M}{m}\leqslant x$. Since $f'(x)=\frac{1-x}{\sqrt{x}(x+1)^2}\leqslant 0, (x\geqslant 1)$ we get $f(x)\leqslant f\left(\frac{M}{m}\right)$, which implies the result by a simple calculation. □

THEOREM 2.2. *Let $A,B,X\in\mathbb{B}(\mathscr{H})$, f and g be non-negative functions on $[0,\infty)$ which are continuous and satisfy the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$, and let h be a non-negative increasing convex function on $[0,\infty)$. If*

$$0<B^*f^2(|X|)B\leqslant m<M\leqslant A^*g^2(|X^*|)A$$

or

$$0<A^*g^2(|X^*|)A\leqslant m<M\leqslant B^*f^2(|X|)B,$$

then

$$h(w(A^*XB))\leqslant \frac{\sqrt{Mm}}{M+m}\left\|h\left(B^*f^2(|X|)B\right)+h\left(A^*g^2(|X^*|)A\right)\right\|. \tag{2.9}$$

Proof. It follows from Lemma 2.1 that

$$|\langle A^*XBx,x\rangle|\leqslant \sqrt{\langle B^*f^2(|X|)Bx,x\rangle\langle A^*g^2(|X^*|)Ax,x\rangle}. \tag{2.10}$$

Lemma 2.4 ensures that

$$\begin{aligned}\sqrt{\langle B^*f^2(|X|)Bx,x\rangle\langle A^*g^2(|X^*|)Ax,x\rangle}&\leqslant \frac{\sqrt{Mm}}{M+m}\left(\left\langle B^*f^2(|X|)Bx,x\right\rangle+\left\langle A^*g^2(|X^*|)Ax,x\right\rangle\right)\\&=\frac{\sqrt{Mm}}{M+m}\left\langle B^*f^2(|X|)B+A^*g^2(|X^*|)Ax,x\right\rangle.\end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11), we get

$$|\langle A^*XBx,x\rangle|\leqslant \frac{\sqrt{Mm}}{M+m}\left\langle B^*f^2(|X|)B+A^*g^2(|X^*|)Ax,x\right\rangle.$$

Taking the supremum over $x\in\mathscr{H}$ with $\|x\|=1$, we infer that

$$w(A^*XB)\leqslant \frac{\sqrt{Mm}}{M+m}\left\|B^*f^2(|X|)B+A^*g^2(|X^*|)A\right\|.$$

Now, since h is a non-negative increasing convex function, we have

$$h(w(A^*XB))\leqslant h\left(\frac{2\sqrt{Mm}}{M+m}\left\|\frac{B^*f^2(|X|)B+A^*g^2(|X^*|)A}{2}\right\|\right)$$

$$\leqslant \frac{2\sqrt{Mm}}{M+m} h\left(\left\|\frac{B^* f^2(|X|) B + A^* g^2(|X^*|) A}{2}\right\|\right) \tag{2.12}$$
$$= \frac{2\sqrt{Mm}}{M+m}\left\| h\left(\frac{B^* f^2(|X|) B + A^* g^2(|X^*|) A}{2}\right)\right\|$$
$$\leqslant \frac{\sqrt{Mm}}{M+m}\left\| h\left(B^* f^2(|X|) B\right) + h\left(A^* g^2(|X^*|) A\right)\right\|, \tag{2.13}$$

where the inequality (2.12) follows from the fact if f is non-negative convex function and $\alpha \leqslant 1$, then $f(\alpha t) \leqslant \alpha f(t)$ (of course, $\frac{2\sqrt{Mm}}{M+m} \leqslant 1$), and the inequality (2.13) is due to Lemma 2.3. $\square$

REMARK 2.2. Following (2.9), we list here some particular inequalities of interest.

- If $r \geqslant 1$ and $0 \leqslant v \leqslant 1$, then

$$w^r(A^*XB) \leqslant \frac{\sqrt{Mm}}{M+m}\left\|\left(B^*|X|^{2(1-v)}B\right)^r + \left(A^*|X^*|^{2v}A\right)^r\right\|,$$

whenever $0 < B^*|X|^{2(1-v)}B \leqslant m < M \leqslant A^*|X^*|^{2v}A$ or $0 < A^*|X^*|^{2v}A \leqslant m < M \leqslant B^*|X|^{2(1-v)}B$. The above inequality improves (1.6).

- If $r \geqslant 1$ and $0 \leqslant v \leqslant 1$, then

$$w^r(X) \leqslant \frac{\sqrt{Mm}}{M+m}\left\||X|^{2r(1-v)} + |X^*|^{2rv}\right\|,$$

whenever $0 < |X|^{2(1-v)} \leqslant m < M \leqslant |X^*|^{2v}$ or $0 < |X^*|^{2v} \leqslant m < M \leqslant |X|^{2(1-v)}$. The above inequality improves (1.3).

- If $r \geqslant 1$, then

$$w^r(A^*B) \leqslant \frac{\sqrt{Mm}}{M+m}\left\||B|^{2r} + |A|^{2r}\right\|,$$

whenever $0 < |B|^2 \leqslant m < M \leqslant |A|^2$ or $0 < |A|^2 \leqslant m < M \leqslant |B|^2$. The above inequality improves (1.5).

We can show a similar improvement with different condition for $A^*g^2(|X|)A$ and $B^*f^2(|X|)B$. Recall that the weighted operator arithmetic mean ∇_v and geometric mean $\sharp_v$, for $0 < v < 1$, positive invertible operator A, and positive operator B, are defined as follows:

$$A\nabla_v B = (1-v)A + vB \quad \text{and} \quad A\sharp_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}.$$

If $v = \frac{1}{2}$, we denote the arithmetic and geometric means, respectively, by ∇ and $\sharp$.

THEOREM 2.3. *Let* $A,B,X \in \mathbb{B}(\mathscr{H})$, f *and* g *be non-negative functions on* $[0,\infty)$ *which are continuous and satisfy the relation* $f(t)g(t) = t$ *for all* $t \in [0,\infty)$, *and let* h *be a non-negative increasing convex function on* $[0,\infty)$. *If for given* $m',M' > 0$,

$$0 < m' \leqslant B^* f^2(|X|)B \leqslant A^* g^2(|X|)A \leqslant M'$$

or

$$0 < m' \leqslant A^* g^2(|X|)A \leqslant B^* f^2(|X|)B \leqslant M',$$

then

$$h(\omega(A^*XB)) \leqslant \frac{1}{2\gamma}\left\|h\left(B^* f^2(|X|)B\right) + h\left(A^* g^2(|X^*|)A\right)\right\|,$$

where $\gamma := \left(1 - \frac{1}{8}\left(1 - \frac{1}{h'}\right)^2\right)^{-1} \geqslant 1$ *with* $h' = \frac{M'}{m'}$.

Proof. From [8, Corollary 3.15], we have

$$\exp_r\left(\frac{v(1-v)}{2}\left(1 - \frac{1}{h'}\right)^2\right) A\sharp_v B \leqslant A\nabla_v B,$$

for $A,B > 0$ with $m',M' > 0$ satisfying $0 < m' \leqslant A \leqslant B \leqslant M'$ or $0 < m' \leqslant B \leqslant A \leqslant M'$, where $\exp_r(x) := (1+rx)^{1/r}$, if $1+rx > 0$, and it is undefined otherwise. Since $\exp_r(x)$ is decreasing in $r \in [-1,0)$, the above inequality gives a tight lower bound when $r = -1$. After all, we have the scalar inequality: $\gamma\sqrt{ab} \leqslant \frac{a+b}{2}$, for $a,b > 0$ and $m',M' > 0$ such that $0 < m' \leqslant \min\{a,b\} \leqslant \max\{a,b\} \leqslant M'$. Applying this inequality with a similar argument as in Theorem 2.2, we obtain the desired result. □

We also obtain the similar remarks with Remark 2.2, we omit them.

As we have seen, Lemma 2.3 played an essential role in Proposition 2.1 and Theorem 2.2. In the following, we aim to improve Lemma 2.3.

PROPOSITION 2.2. *Let the assumptions of Lemma 2.3 hold. Then*

$$\|f((1-v)A + vB)\| \leqslant \|(1-v)f(A) + vf(B)\| - r\mu(f) \tag{2.14}$$

where $r = \min\{v, 1-v\}$, *and*

$$\mu(f) = \inf_{\|x\|=1}\left\{f(\langle Ax,x\rangle) + f(\langle Bx,x\rangle) - 2f\left(\left\langle\left(\frac{A+B}{2}\right)x,x\right\rangle\right)\right\}. \tag{2.15}$$

Proof. We assume $0 \leqslant v \leqslant \frac{1}{2}$. For each unit vector $x \in \mathscr{H}$,

$$f(\langle((1-v)A + vB)x,x\rangle) + r\mu(f) = f((1-v)\langle Ax,x\rangle + v\langle Bx,x\rangle) + r\mu(f)$$
$$= f\left((1-2v)\langle Ax,x\rangle + 2v\left\langle\left(\frac{A+B}{2}\right)x,x\right\rangle\right) + r\mu(f)$$

$$\begin{aligned}
&\leqslant (1-2v) f(\langle Ax,x\rangle) + 2vf\left(\left\langle \left(\frac{A+B}{2}\right) x,x\right\rangle\right) + r\mu(f) && (2.16)\\
&\leqslant (1-2v) f(\langle Ax,x\rangle) + 2vf\left(\left\langle \left(\frac{A+B}{2}\right) x,x\right\rangle\right)\\
&\quad + r\left(f(\langle Ax,x\rangle) + f(\langle Bx,x\rangle) - 2f\left(\left\langle \left(\frac{A+B}{2}\right) x,x\right\rangle\right)\right) && (2.17)\\
&= (1-v) f(\langle Ax,x\rangle) + vf(\langle Bx,x\rangle)\\
&\leqslant (1-v)\langle f(A)x,x\rangle + v\langle f(B)x,x\rangle && (2.18)\\
&= \langle ((1-v) f(A) + vf(B))x,x\rangle,
\end{aligned}$$

where (2.16) follows from convexity of f, the relation (2.15) implies (2.17), and (2.18) follows from Lemma 2.2.

If we apply similar arguments for $\frac{1}{2} \leqslant v \leqslant 1$, then we can write

$$f(\langle ((1-v)A + vB)x,x\rangle) \leqslant \langle ((1-v) f(A) + vf(B))x,x\rangle - r\mu(f).$$

We know that if $A \in \mathbb{B}(\mathscr{H})$ is a positive operator, then $\|A\| = \sup_{\|x\|=1} \langle Ax,x\rangle$. By using this, the continuity and the increase of f, we have

$$\begin{aligned}
f(\|(1-v)A + vB\|) &= f\left(\sup_{\|x\|=1} \langle ((1-v)A + vB)x,x\rangle\right) = \sup_{\|x\|=1} f(\langle ((1-v)A+vB)x,x\rangle)\\
&\leqslant \sup_{\|x\|=1} (\langle ((1-v) f(A) + vf(B))x,x\rangle) - r\mu(f)\\
&= \|(1-v) f(A) + vf(B)\| - r\mu(f).
\end{aligned}$$

On the other hand, if $X \in \mathbb{B}(\mathscr{H})$, and if f is a non-negative increasing function on $[0,\infty)$, then $f(\|X\|) = \|f(|X|)\|$, so we get the desired result. □

REMARK 2.3. With inequality (2.14) in hand, we can improve Proposition 2.1 and Theorem 2.2. For instance, under the assumptions of Proposition 2.1, we have

$$h\left(w^2(A^*XB)\right) \leqslant \left\|(1-v)h\left(\left(B^* f^2(|X|)B\right)^{\frac{1}{1-v}}\right) + vh\left(\left(A^* g^2(|X^*|)A\right)^{\frac{1}{v}}\right)\right\| - r\gamma(f),$$

where

$$\begin{aligned}
\gamma(f) = \inf_{\|x\|=1} \Bigg\{ & h\left(\left\langle \left(B^* f^2(|X|)B\right)^{\frac{1}{1-v}} x,x\right\rangle\right) + h\left(\left\langle \left(A^* g^2(|X^*|)A\right)^{\frac{1}{v}} x,x\right\rangle\right)\\
& -2h\left(\left\langle \left(\frac{\left(B^* f^2(|X|)B\right)^{\frac{1}{1-v}} + \left(A^* g^2(|X^*|)A\right)^{\frac{1}{v}}}{2}\right) x,x\right\rangle\right)\Bigg\}.
\end{aligned}$$

Now we present some inequalities for the numerical radius and operator norm, but under the effect of a superquadratic function. Recall that a function $f : [0,\infty) \to \mathbb{R}$ is said to be superquadratic provided that for all $s \geqslant 0$, there exists a constant $C_s \in \mathbb{R}$ such that

$$f(|t-s|)+C_s(t-s)+f(s) \leqslant f(t) \tag{2.19}$$

for all $t \geqslant 0$.

The following useful lemma is well known [1, Lemma 2.1].

LEMMA 2.5. *Suppose that f is superquadratic and non-negative. Then f is convex and increasing. Also, if C_s is as in (2.19), then $C_s \geqslant 0$.*

By adopting the above notions, we can refine the second inequality in (1.1).

THEOREM 2.4. *Let $A \in \mathbb{B}(\mathscr{H})$ and let f be a non-negative superquadratic function. Then*

$$f(w(A)) \leqslant \|f(|A|)\| - \inf_{\|x\|=1} \left\| f(||A|-w(A)|)^{\frac{1}{2}} x \right\|^2. \tag{2.20}$$

Proof. Letting $s = w(A)$ in the inequality (2.19), we get

$$f(|t-w(A)|)+C_{w(A)}(t-w(A))+f(w(A)) \leqslant f(t). \tag{2.21}$$

By applying functional calculus for the operator $|A|$ in (2.21) we get

$$f(||A|-w(A)|)+C_{w(A)}(|A|-w(A))+f(w(A)) \leqslant f(|A|). \tag{2.22}$$

Consequently,

$$\left\| f(||A|-w(A)|)^{\frac{1}{2}} x \right\|^2 + C_{w(A)}(\langle |A|x,x\rangle - w(A)) + f(w(A)) \leqslant \langle f(|A|)x,x\rangle, \tag{2.23}$$

for any unit vector $x \in \mathscr{H}$.

Now, by taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in (2.23), and using the fact $w(|A|) = \|A\| \geqslant w(A)$, we deduce the desired inequality (2.20). □

Applying Theorem 2.4 to the superquadratic function $f(t) = t^r$ $(r \geqslant 2)$, we reach the following corollary:

COROLLARY 2.2. *Let $A \in \mathbb{B}(\mathscr{H})$. Then for any $r \geqslant 2$,*

$$w^r(A) \leqslant \|A\|^r - \inf_{\|x\|=1} \left\| ||A|-w(A)|^{\frac{r}{2}} x \right\|^2.$$

In particular,

$$w(A) \leqslant \sqrt{\|A\|^2 - \inf_{\|x\|=1} \| ||A|-w(A)|x \|^2} \leqslant \|A\|.$$

3. An inequality related to f–connection of operators

In the forthcoming, we aim to extend the main result of [17].

In [17, Theorem 2.3], the author tried to prove the numerical radius version of operator arithmetic-geometric mean inequality

$$w^r\left((A\sharp B)X\right) \leqslant w\left(\frac{A^{\frac{rp}{2}}}{p}+\frac{(X^*BX)^{\frac{rq}{2}}}{q}\right)-\frac{1}{p}\inf_{\|x\|=1}\delta(x)$$

where $A,B,X \in \mathbb{B}(\mathscr{H})$ such that A,B are positive invertible operators, $p \geqslant q > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $r \geqslant \frac{2}{q}$, and $\delta(x)=\left(\langle Ax,x\rangle^{\frac{rp}{4}}-\langle X^*BXx,x\rangle^{\frac{rq}{4}}\right)^2$.

Of course, $\frac{A^{\frac{rp}{2}}}{p}+\frac{(X^*BX)^{\frac{rq}{2}}}{q}$ is positive. On the other hand, it is well-known to all that if X is positive operator then $w(X)=\|X\|$. On taking into account these considerations, it should be written to the following form:

$$w^r\left((A\sharp B)X\right) \leqslant \left\|\frac{A^{\frac{rp}{2}}}{p}+\frac{(X^*BX)^{\frac{rq}{2}}}{q}\right\|-\frac{1}{p}\inf_{\|x\|=1}\delta(x).$$

Of course, the geometric mean (resp. arithmetic mean) of two positive operators is also a positive operator. So Corollary 2.6, Corollary 2.7, Remark 2.8, and Corollary 2.10 in [17] should be written in the following way, respectively,

$$\|A\sharp B\|^r \leqslant \left\|\frac{A^{\frac{rp}{2}}}{p}+\frac{B^{\frac{rq}{2}}}{q}\right\|-\frac{1}{p}\inf_{\|x\|=1}\left\{\left(\langle Ax,x\rangle^{\frac{rp}{4}}-\langle Bx,x\rangle^{\frac{rq}{4}}\right)^2\right\},$$

$$\|A\sharp B\|^{2r} \leqslant \left\|\frac{A^{rp}}{p}+\frac{B^{rq}}{q}\right\|-\frac{1}{p}\inf_{\|x\|=1}\left\{\left(\langle Ax,x\rangle^{\frac{rp}{2}}-\langle Bx,x\rangle^{\frac{rq}{2}}\right)^2\right\},$$

$$\|A\sharp B\|^2 \leqslant \left\|\frac{A^2+B^2}{2}\right\|-\frac{1}{2}\inf_{\|x\|=1}\left\{\langle A-Bx,x\rangle^2\right\}, \text{ and}$$

$$\sqrt{2}\,\|A\sharp B\| \leqslant w_e(A,B) \leqslant \left\|A^2+B^2\right\|^{\frac{1}{2}}.$$

Here $w_e(A,B)=\sup\limits_{\|x\|=1}\left(|\langle Ax,x\rangle|^2+|\langle Bx,x\rangle|^2\right)^{\frac{1}{2}}$.

Let f be a continuous function defined on the real interval J containing the spectrum of $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, where B is a self-adjoint operator and A is a positive invertible operator. Then by using the continuous functional calculus, we can define f-connection σ_f as follows

$$A\sigma_f B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}. \tag{3.1}$$

Note that for the functions $(1-v)+vt$ and t^v, the definition in (3.1) leads to the arithmetic and geometric operator means, respectively.

Now, we give our numerical radius inequality concerning f-connection of operators.

THEOREM 3.1. *Let $A,B,X \in \mathbb{B}(\mathscr{H})$ such that A,B be two positive operators. Then*

$$w\left(\left(A\sigma_f B\right)X\right) \leqslant \frac{1}{2}\left\|X^*A^{\frac{1}{2}}f^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}X+A\right\|. \tag{3.2}$$

Proof. For any unit vector $x \in \mathscr{H}$, we have

$$\begin{aligned}
\left|\left\langle\left(A\sigma_f B\right)Xx,x\right\rangle\right| &= \left|\left\langle A^{\frac{1}{2}}f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}Xx,x\right\rangle\right| = \left|\left\langle f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}Xx,A^{\frac{1}{2}}x\right\rangle\right| \\
&\leqslant \left\|f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}Xx\right\|\left\|A^{\frac{1}{2}}x\right\| \\
&= \sqrt{\left\langle f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}Xx,f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}Xx\right\rangle\left\langle A^{\frac{1}{2}}x,A^{\frac{1}{2}}x\right\rangle} \\
&= \sqrt{\left\langle X^*A^{\frac{1}{2}}f^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}Xx,x\right\rangle\langle Ax,x\rangle} \\
&\leqslant \frac{1}{2}\left\langle X^*A^{\frac{1}{2}}f^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}X+Ax,x\right\rangle.
\end{aligned}$$

Now, the result follows by taking the supremum over $x \in \mathscr{H}$ with $\|x\|=1$. □
By choosing $f(t)=\sqrt{t}$, in Theorem 3.1 we reach the following result:

COROLLARY 3.1. *Let $A,B,X \in \mathbb{B}(\mathscr{H})$ such that A,B be two positive operators. Then*

$$w\left((A\sharp B)X\right) \leqslant \frac{1}{2}\left\|X^*BX+A\right\|.$$

REMARK 3.1. The interested reader can construct refinements of inequality (3.2) using improvements of weighted arithmetic-geometric mean inequality. We leave the details of this idea to the interested reader, as it is just an application of our result.

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REFERENCES

[1] S. ABRAMOVICH, G. JAMESON AND G. SINNAMON, *Inequalities for averages of convex and superquadratic functions*, J. Inequal. Pure Appl. Math., **5**(4) (2004), 1–14.
[2] A. ABU-OMAR AND F. KITTANEH, *A numerical radius inequality involving the generalized Aluthge transform*, Studia Math., **216**(1) (2013), 69–75.
[3] J. S. AUJLA AND F. C. SILVA, *Weak majorization inequalities and convex functions*, Linear Algebra Appl., **369** (2003), 217–233.
[4] S. S. DRAGOMIR, *Inequalities for the numerical radius of linear operators in Hilbert spaces*, Springer Briefs in Mathematics. Springer, Cham, 2013.
[5] S. S. DRAGOMIR, *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*, Sarajevo J Math., **5**(18) (2009), 269–278.
[6] S. S. DRAGOMIR, *Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces*, Linear Algebra Appl., **419** (2006), 256–264.
[7] M. EL-HADDAD AND F. KITTANEH, *Numerical radius inequalities for Hilbert space operators. II*, Studia Math., **182**(2) (2007), 133–140.

[8] S. FURUICHI, *Further improvements of Young inequality*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., **113** (2019), 255–266.

[9] S. FURUICHI, H. R. MORADI AND M. SABABHEH, *New sharp inequalities for operator means*, Linear Multilinear Algebra., **67**(8) (2019), 1567–1578.

[10] S. FURUICHI AND H. R. MORADI, *On further refinements for Young inequalities*, Open Math., **16** (2018), 1478–1482.

[11] K. E. GUSTAFSON, D. K. M. RAO, *Numerical range, the field of values of linear operators and matrices*, Springer-Verlag, Berlin, 1997.

[12] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math., **168**(1) (2005), 73–80.

[13] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math., **158** (2003), 11–17.

[14] F. KITTANEH, *Note on some inequalities for Hilbert space operators*, Publ. RIMS Kyoto Univ., **24** (1988), 283–293.

[15] B. MOND AND J. PEČARIĆ, *On Jensen's inequality for operator convex functions*, Houston J. Math., **21** (1995), 739–753.

[16] K. SHEBRAWI AND H. ALBADAWI, *Numerical radius and operator norm inequalities*, J. Inequal. Appl., 2009 (2009), 1–11.

[17] A. SHEIKHHOSSEINI, *A numerical radius version of the arithmetic–geometric mean of operators*, Filomat., **30**(8) (2016), 2139–2145.

[18] T. YAMAZAKI, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math., **178**(1) (2007), 83–89.

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