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THE SZEGŐ–MARKOV–BERNSTEIN INEQUALITIES AND BARYCENTRIC REPRESENTATIONS OF THE OSCULATORY INTERPOLATING OPERATORS FOR CLASSICAL ITERATED WEIGHTS

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Abstract. We study inequalities of Szegő-Markov-Bernstein types, barycentric representations of the Lagrange, Fejér and Hermite interpolating operators, and the Gauss quadrature formulae for all iterated weights $w_k(x) = A^k(x)w(x)$ of classical weight functions $w(x)$. In particular, we establish the explicit formulae for the best constants, extremal polynomials and Christoffel numbers, associated with the iterated weight functions of six basic classical weights of Hermite, Laguerre, Jacobi, generalized Bessel, Jacobi on $(0,+\infty)$ and pseudo-Jacobi kind. It should be noted that the results on Markov-Bernstein inequalites continue the investigations of the best constants and extremal polynomials by Guessab and Milovanović [J. Math. Anal. Appl. 182 (1994), pp. 244-249] and Agarwal and Milovanović [Appl. Math. Comput. 128 (2002), pp. 151-166], without any additional assumptions on classical weight functions. Moreover, the presented generic formulae for the Christoffel numbers of the iterated Gauss quadrature rules, together with the corresponding representations of the barycentric weights of Lagrange, Fejér and Hermite type, complete the recent results of Wang et al. and the authors, published in [Math. Comp. 81 (2012) and 83 (2014), pp. 861-877 and 2893-2914, respectively] and [Math. Comp. 86 (2017), pp. 2409-2427].

1. Introduction

Let $\{p_n(x)\}_{0\leqslant n<n_w}$ be a finite or infinite sequence of polynomials $p_n(x)$ of degree n, which are orthonormal with respect to the weighted $L^2_w(a,b)$-inner product

$$\int_a^b p_n(x)\,p_m(x)\,w(x)\,dx = \delta_{n,m}, \quad 0\leqslant n,m<n_w, \quad n_w\in\mathbb{N}\cup\{+\infty\}.$$

Here $\delta_{n,m}$ denotes the Kronecker delta. According to Chihara [6], Koekoek et al. [15], Lesky [22, 23] and Nikiforov and Uvarov [30], the weight function $w(x)$ and the corresponding orthogonal polynomials $p_n(x)$ are called classical provided that $w(x)$ is a positive solution of the Pearson differential equation

$$\frac{d}{dx}\left[A(x)\,w(x)\right] = B(x)\,w(x), \quad a<x<b,$$

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with boundary conditions

$$A(x)\,w(x)\,x^j\big|_{x=a,b} = 0, \quad 0 \leqslant j < n_w, \tag{1}$$

where the polynomial coefficients

$$A(x) = a_2x^2 + a_1x + a_0 \quad \text{and} \quad B(x) = b_1x + b_0$$

are such that $b_1 \neq 0$ and $A(x) > 0$ on (a,b).

It is of importance that the derivatives $D^k p_n(x)$ of classical orthogonal polynomials $p_n(x)$ are also classical [18, 19, 20, 27, 30] with respect to the iterated weight functions

$$w_k(x) = A^k(x)\,w(x), \quad k = 0,1,\ldots,n,$$

which are the solutions of differential equation of the Pearson type

$$\frac{d}{dx}\left[A(x)\,w_k(x)\right] = \left[B(x) + kA'(x)\right]w_k(x), \quad a < x < b, \tag{2}$$

with the boundary conditions of the type (1). The corresponding weighted $L^2_{w_k}(a,b)$-inner product $(p,q)_{w_k}$ and norm $\|p\|_{w_k} = \sqrt{(p,p)_{w_k}}$ are defined, on the space $\mathscr{P}_{n-k}$ of all polynomials of degree less or equal to $n-k$, by the formula

$$(p,q)_{w_k} = \int_a^b p(x)\,q(x)\,w_k(x)\,dx.$$

In Section 2 of this paper we shortly discuss properties of classical weights and orthogonal polynomials, which are necessary in Sections 3, 4 and 5. Next, in Section 3 we study generic properties [15, 17, 32] of the following extremal Szegő-type problem

$$\left|D^k p^*(y)\right|^2 = \max_{p\in\mathscr{P}_n}\left\{\left|D^k p(y)\right|^2 : \left\|D^k p\right\|_{w_k} = 1\right\}, \quad k = 0,1,\ldots,n, \tag{3}$$

for all iterated weights $w_k(x)$. More precisely, we show that the best constants $C_{n,k}(y) = \left|D^k p^*(y)\right|^2$ in the Szegő-type inequalities

$$\left|D^k p(y)\right|^2 \leqslant C_{n,k}(y)\left\|D^k p\right\|^2_{w_k}, \quad p \in \mathscr{P}_n,$$

are equal to

$$C_{n,k}(y) = \gamma_{k,n}\begin{vmatrix} D^{k+1}p_{n+1}(y) & D^k p_{n+1}(y) \\ D^{k+1}p_n(y) & D^k p_n(y)\end{vmatrix}.$$

Moreover, the extremal polynomials $p(x) = p^*(x)$, for which these inequalities become identities, satisfy the formulae

$$D^k p^*(x)\left|D^k p^*(y)\right| = \pm\gamma_{k,n}\frac{\begin{vmatrix} D^k p_{n+1}(x) & D^k p_{n+1}(y) \\ D^k p_n(x) & D^k p_n(y)\end{vmatrix}}{x-y},$$

where $y \in \mathbb{R}$ is arbitrary and constant factors $\gamma_{k,n}$ are as in Theorem 1.

It is of interest that the Szegő-type problem is strictly connected with proving both generic formulae for the Christoffel numbers $A_{k,\nu}$ of the Gauss quadrature rule

$$G_{n,k}(f) = \sum_{\nu=1}^{n-k} A_{k,\nu} f(x_{k,\nu}), \quad 1 \leqslant k < n,$$

at the roots $(x_{k,\nu})_{\nu=1}^{n-k}$ of the iterated orthogonal polynomials $D^k p_n(x)$ and the corresponding representations for the barycentric weights for the Lagrange, Fejér and Hermite interpolating operators; cf. Theorem 2 and Corollary 1 in Section 4. It is also connected with the problem of computing the operator norms

$$\left\|D^k\right\|_{w_j}^2 = \max_{p\in\mathscr{P}_n}\left\{\left\|D^k p\right\|_{w_j}^2 : \|p\|_w = 1\right\}, \quad k = 0,1,\ldots,n, \quad j = 0,1,\ldots,k,$$

of the differential operators

$$D^k : \mathscr{P}_n(w) \ni p \to D^k p \in \mathscr{P}_{n-k}(w_j).$$

Here $\mathscr{P}_n(w)$ denotes the space of all polynomials of degree not greater than n, equipped with the weighted $L_2(w)$-norm, or more generally with the weighted L_s-norm, where $1 \leqslant s \leqslant +\infty$. Usually, the last problem is replaced by the equivalent problem of computing the best constants $C_{n,k}(w_j) = \left\|D^k\right\|_{w_j}^2$ and characterizing the extremal polynomials $p(x) = p^*(x)$ for Markov-Bernstein-type inequalities of the form

$$\left\|D^k p\right\|_{w_j}^2 \leqslant C_{n,k}(w_j)\,\|p\|_w^2, \quad p \in \mathscr{P}_n. \tag{4}$$

We note that a complete solution of this difficult problem is known only for a few particular values of the indices k, j and s, whenever $w(x)$ is either classical or Freud-type weight; cf. Freud [9,10], Kroó [21], Guessab and Milovanović [13], Agarwal and Milovanović [2], Marcellán et al. [25], Milovanović [28], Nevai [29] and the bibliography quoted there. In particular, Kroó [21] proved that in the case of classical weight function $w(x) = 1$ on $(-1,1)$, the best constant $C_{n,1}(w)$ coincides with the squared largest positive root of the equation

$$\sum_{\nu=0}^{\lfloor (n+1)/2\rfloor} (-1)^\nu \frac{(n-2\nu+2)_{4\nu}}{4^\nu (2\nu)!} x^{-2\nu} = 0,$$

where $\lfloor u \rfloor$ and $(u)_k$ denote the floor function and the Pochhammer symbol, respectively.

The connection between above-mentioned problems of Szegő and Markov-Bernstein is pointed out at Section 5, where we extend the beautiful results of Agarwal-Milovanović [2] and Guessab-Milovanović [13] on the best L_2-constants $C_{n,k}(w_k)$ for the Hermite, Laguerre and Jacobi weights to all remaining classical weights, e.g., to generalized Bessel, Jacobi on $(0,+\infty)$ and pseudo-Jacobi weights. More precisely, in Theorem 3 we show, for all classical weight functions $w(x)$, that

$$\left\|D^k\right\|_{w_k}^2 = \left\|D^k p_n\right\|_{w_k}^2 = \sigma_{k,n-k},$$

where constant factors $\sigma_{k,n-k}$ are as in Lemma 1. This means that the best constants $C_{n,k}(w_k)$ are equal to $\sigma_{k,n-k}$ and $p^*(x) = p_n(x)$ are corresponding extremal polynomials for the Markov-Bernstein inequalities (4), whenever $j = k$.

2. Auxiliary properties of classical orthogonal polynomials

Among classical orthogonal polynomials there are exactly three infinite sequences of orthogonal polynomials of Hermite, Laguerre and Jacobi [37], and exactly three, less known, finite sequences of generalized Bessel, Jacobi on $(0,+\infty)$ and pseudo-Jacobi orthogonal polynomials, up to a linear change of variable; cf. [15, 17, 23, 24]. The lengths $n_w = \lfloor (1-b_1)/2 \rfloor$ of the finite polynomial sequences $\{p_n(x)\}_{0\leqslant n<n_w}$ depend only on the leading coefficients $b_1 = \alpha, 2-\alpha, 2(1-\alpha)$ of the polynomials $B(x)$, presented in Table 1. In the last row of Table 1 we use the following additional notation

$$a_1 = 2\frac{\mathscr{A}\mathscr{B}+\mathscr{C}\mathscr{D}}{\mathscr{A}^2+\mathscr{C}^2}, \quad a_0 = \frac{\mathscr{B}^2+\mathscr{D}^2}{\mathscr{A}^2+\mathscr{C}^2}, \quad \zeta = \frac{\mathscr{A}\mathscr{D}-\mathscr{B}\mathscr{C}}{\mathscr{A}^2+\mathscr{C}^2} > 0,$$
$$b_0 = (1-\alpha)a_1 + \beta\zeta, \quad E(x) = \frac{1}{\zeta}\left(x + \frac{1}{2}a_1\right),$$

to define the pseudo-Jacobi classical weights and polynomials. It should be noted that Koepf and Masjed-Jamei [16] have observed recently that these weights generalize the Student t-distribution, one of the most important distributions in the sampling problems of normal population. According to [16, 26], they also extend the F-distribution.

The classical orthogonal polynomials were studied in a large number of articles and monographs, see e.g. Agarwal and Milovanović [1, 2], Al-Salam [3], Bochner [5], Koekoek et al. [15], Lesky [22], Mastroianni and Milovanović [27], Nikiforov and Uvarov [30], Suetin [36], Valent and Van Assche [38], and the authors [31, 32, 33, 35]. In almost all these studies an essential role has been played by the three-term and Al-Salam-Chihara recurrence relations and the Sturm-Liouville differential equations [6, 14, 18, 19, 20]

$$\begin{aligned} q_0(x) = 1, \quad q_1(x) = x - c_0, \\ q_{n+1}(x) = (x-c_n)q_n(x) - d_n q_{n-1}(x), \quad n = 1,2,\ldots, \end{aligned} \tag{5}$$

$$A(x)q_n'(x) = (na_2x+\eta_n)q_n(x) + \rho_n q_{n-1}(x), \quad n = 1,2,\ldots, \tag{6}$$

$$\frac{d}{dx}\left[w_{k+1}(x)D^{k+1}q_n(x)\right] = \lambda_{n,k}w_k(x)D^kq_n(x), \quad a<x<b, \tag{7}$$

for the monic classical orthogonal polynomials $q_n(x)$ and their derivatives, where

Table 1: ([34]) The basic types of classical weights $w(x)$. All remaining classical weights $\widehat{w}(\widehat{x}) = w((\widehat{x}-\beta_0)/\beta_1)$ and polynomials $\widehat{A}(\widehat{x}) = A((\widehat{x}-\beta_0)/\beta_1)$ and $\widehat{B}(\widehat{x}) = \frac{1}{\beta_1}B((\widehat{x}-\beta_0)/\beta_1)$ can be obtained from the basic $w(x)$, $A(x)$ and $B(x)$ by an appropriate change of variable $\widehat{x} = \beta_1 x + \beta_0$, $\beta_1 > 0$.

Weights	$A(x)$	$B(x)$
Hermite, e^{-x^2} on $(-\infty,+\infty)$	1	$-2x$
Laguerre, $x^{\alpha}e^{-x}$ on $(0,+\infty)$	x	$-x+\alpha+1$ $\alpha > -1$
Jacobi, $(1-x)^{\alpha}(1+x)^{\beta}$ on $(-1,1)$	$-x^2+1$	$-(\alpha+\beta+2)x+\beta-\alpha$ $\alpha > -1,\ \beta > -1$
Generalized Bessel, $x^{\alpha-2}e^{-\frac{\beta}{x}}$ on $(0,+\infty)$	x^2	$\alpha x+\beta$ $\alpha < -1$ $\alpha \notin \{-2,-3,\ldots\}$ $\beta > 0$
Jacobi, $\frac{x^{\beta}}{(1+x)^{\alpha+\beta}}$ on $(0,+\infty)$	x^2+x	$(2-\alpha)x+\beta+1$ $\alpha \geqslant 3,\ \beta > -1$
Pseudo-Jacobi, $\frac{e^{\beta \arctan E(x)}}{A^{\alpha}(x)}$ on $(-\infty,+\infty)$	$x^2+a_1x+a_0$	$2(1-\alpha)x+b_0$ $\alpha \geqslant \frac{3}{2},\ \beta \in \mathbb{R}$

coefficients c_n, $d_n = h_n/h_{n-1}$, η_n, ρ_n and $\lambda_{n,k}$ are defined by

$$\begin{aligned} c_n &= -\frac{2na_1r_{n-1} - b_0(2a_2-b_1)}{r_{2n-2}r_{2n}}, \\ d_n &= nr_{n-2}\frac{s_{n-1}(r_{n-1}a_1 - a_2b_0) - a_0r_{2n-2}^2}{r_{2n-3}r_{2n-2}^2r_{2n-1}}, \\ \eta_n &= n\frac{(n-1)a_1a_2 + a_1b_1 - a_2b_0}{2(n-1)a_2+b_1}, \\ \rho_n &= -d_nr_{2n-1}, \\ \lambda_{n,k} &= (n-k)[(n+k-1)a_2+b_1]. \end{aligned} \tag{8}$$

Here we use the following notation:

$$h_n = \|q_n\|_w^2, \quad r_k = ka_2+b_1, \quad s_k = ka_1+b_0. \tag{9}$$

3. Inequalities of Szegő type

Throughout the rest of the paper we assume that $\{p_{\nu,w_k}(x)\}_{0\leqslant\nu\leqslant n-k}$ are finite or infinite sequences of classical polynomials of degree ν, orthonormal with respect to the classical iterated weight functions of the form

$$w_k(x) = A^k(x)\,w(x), \quad k = 0,1,\ldots,n, \quad 0 \leqslant n < n_w.$$

Moreover, we denote by $\mathscr{P}_{n-k} = \mathscr{P}_{n-k}(w_k)$ the $(n-k+1)$-dimensional Hilbert space of all polynomials of degree less or equal to $n-k$ equipped with the weighted inner product $(p,q)_{w_k}$. Then the function

$$K_{n,w_k}(x,y) = \sum_{\nu=0}^{n-k} p_{\nu,w_k}(x)\,p_{\nu,w_k}(y)$$

is the reproducing kernel of $\mathscr{P}_{n-k} = \mathscr{P}_{n-k}(w_k)$ [4, 7, 37]. In view of the Christoffel-Darboux identity [37] it follows that

$$K_{n,w_k}(x,y) = \frac{\alpha_{n-k,k}}{\alpha_{n-k+1,k}} \frac{\begin{vmatrix} p_{n-k+1,w_k}(x) & p_{n-k+1,w_k}(y) \\ p_{n-k,w_k}(x) & p_{n-k,w_k}(y) \end{vmatrix}}{x-y} \tag{10}$$

and

$$K_{n,w_k}(y,y) = \frac{\alpha_{n-k,k}}{\alpha_{n-k+1,k}} \begin{vmatrix} p'_{n-k+1,w_k}(y) & p_{n-k+1,w_k}(y) \\ p'_{n-k,w_k}(y) & p_{n-k,w_k}(y) \end{vmatrix}.$$

Here $\alpha_{m,k}$ is the leading coefficient of the orthonormal polynomial $p_{m,w_k}(x)$.

According to Szegő [37, Theorem 3.1.3], the solution of problem (3) is given by the polynomial $p^*(x)$ in $\mathscr{P}_n$ such that

$$D^k p^*(x) = \pm \frac{K_{n,w_k}(x,y)}{\sqrt{K_{n,w_k}(y,y)}}, \quad a < x < b, \tag{11}$$

for which

$$\left|D^k p^*(y)\right| = \sqrt{K_{n,w_k}(y,y)}. \tag{12}$$

His proof is based on the Parseval identity and Cauchy-Schwarz inequality, applied to the polynomial

$$D^k p(y) = \sum_{\nu=0}^{n-k} \left(D^k p, p_{\nu,w_k}\right)_{w_k} p_{\nu,w_k}(y).$$

Until now we have not used the fact that the orthonormal polynomials $p_{n,w_k}(x)$ were supposed to be classical. However this assumption is both necessary and sufficient in the next lemma. In other words, it is a generic property for the family of all classical orthogonal polynomials [15, 17, 31, 32]. For the simplicity, we shall write below $w(x)$ and $p_n(x)$ instead of $w_0(x)$ and $p_{n,w_0}(x)$.

LEMMA 1. *Let $\{p_n(x)\}_{0\leqslant n<n_w}$ be classical polynomials, orthonormal with respect to a classical weight $w(x)$ on (a,b). If $0\leqslant k\leqslant n$, $y\in\mathbb{R}$, $\sigma_{0,\nu}=1$ and*

$$\sigma_{k,\nu}=(-1)^k(\nu+1)_k\prod_{s=0}^{k-1}[(k+\nu+s-1)a_2+b_1],\quad k>1, \tag{13}$$

then the best constants in the Szegő inequalities

$$\left|D^kp(y)\right|^2\leqslant C_{n,k}(y)\left\|D^kp\right\|_{w_k}^2,\quad p\in\mathscr{P}_n,$$

are equal to

$$C_{n,k}(y)=K_{n,w_k}(y,y)=\sum_{\nu=0}^{n-k}\frac{\left[D^kp_{k+\nu}(y)\right]^2}{\sigma_{k,\nu}}.$$

Additionally, the extremal polynomials $p(x)=p^*(x)$, for which these inequalities become identities, are characterized by the equation

$$D^kp^*(x)\left|D^kp^*(y)\right|=\pm\sum_{\nu=0}^{n-k}\frac{D^kp_{k+\nu}(x)D^kp_{k+\nu}(y)}{\sigma_{k,\nu}}.$$

In particular, the constant factors $\sigma_{k,\nu}$ are as in Table 2 for all basic classical orthogonal polynomials.

Proof. Since we have

$$p_{\nu,w_k}(x)=\frac{D^kp_{k+\nu}(x)}{\|D^kp_{k+\nu}\|_{w_k}},\quad w_k(x)=A^k(x)w(x),$$

we conclude from formulae (11) and (12) that it remains to show that

$$\left\|D^kp_{k+\nu}\right\|_{w_k}^2=\sigma_{k,\nu} \tag{14}$$

and to evaluate the constants $\sigma_{k,\nu}$ in the case when $w(x)$ is an arbitrary classical weight. For this purpose, denote by $q_{r,w_s}(x)=\|q_{r,w_s}\|_{w_s}p_{r,w_s}(x)$ the monic classical polynomial of degree r, orthogonal with respect to the iterated weight $w_s(x)$. Since its derivative $Dq_{r,w_s}(x)$ is also classical orthogonal polynomial with respect to the weight $w_{s+1}(x)$ [30], it follows inductively from the uniqueness, up to a constant factor [7], of the orthogonalization process that

$$D^kq_{r,w_s}(x)=(r-k+1)_kq_{r-k,w_{s+k}}(x),\quad 1\leqslant k\leqslant r. \tag{15}$$

On the other hand, we can apply the orthogonality of $q_{r-1,w_{s+1}}(x)$ and integrate by parts to get

$$\begin{aligned}r\left\|q_{r-1,w_{s+1}}\right\|_{w_{s+1}}^2&=r\int_a^b\left[q_{r-1,w_{s+1}}(x)\right]^2w_{s+1}(x)\,dx\\&=\int_a^bx^{r-1}Dq_{r,w_s}(x)\,w_{s+1}(x)\,dx\\&=x^{r-1}q_{r,w_s}(x)\,w_{s+1}(x)\Big|_a^b-\int_a^bq_{r,w_s}(x)\left[x^{r-1}w_{s+1}(x)\right]'dx.\end{aligned}$$

Further, by the Pearson boundary conditions (1) for the classical weight $w_{s+1}(x)$, the right-hand side is equal to

$$-\int_a^b q_{r,w_s}(x)\left\{(r-1)A(x)+x\left[B(x)+sA'(x)\right]\right\}x^{r-2}w_s(x)\,dx$$
$$=\beta_{r,s}\int_a^b q_{r,w_s}(x)x^r w_s(x)\,dx=\beta_{r,s}\|q_{r,w_s}\|_{w_s}^2,$$

where $\beta_{r,s}=-[(r+2s-1)a_2+b_1]$. Now one can apply k times the recurrent formula

$$\left\|q_{r-1,w_{s+1}}\right\|_{w_{s+1}}^2=\frac{\beta_{r,s}}{r}\|q_{r,w_s}\|_{w_s}^2$$

in order to obtain

$$\left\|q_{\nu+k,w_0}\right\|_{w_0}^2=\prod_{s=0}^{k-1}\frac{\nu+k-s}{\beta_{\nu+k-s,s}}\cdot\left\|q_{\nu,w_k}\right\|_{w_k}^2=\frac{(\nu+1)_k^2}{\sigma_{k,\nu}}\left\|q_{\nu,w_k}\right\|_{w_k}^2. \tag{16}$$

Thus it follows from (15) and (16) that

$$D^k p_{\nu+k}(x)=\frac{D^k q_{\nu+k,w_0}(x)}{\left\|q_{\nu+k,w_0}\right\|_{w_0}}=\sqrt{\sigma_{k,\nu}}\,p_{\nu,w_k}(x), \tag{17}$$

which finishes the proof of (14). In order to calculate $\sigma_{k,\nu}$ for all classical basic weights of Hermite, Laguerre, Jacobi, generalized Bessel, Jacobi on $(0,+\infty)$ and pseudo-Jacobi, it is sufficient to substitute values of coefficients a_2 and b_1 of polynomials $A(x)$ and $B(x)$ from Table 1 into formula (13) for $\sigma_{k,\nu}$. □

By using the Christoffel-Darboux identity the solution of problem (3) can be considerably improved and written in the following equivalent form.

THEOREM 1. *Let $\{p_n(x)\}_{0\leqslant n<n_w}$ be classical orthonormal polynomials associated with a classical weight $w(x)$ on (a,b), and let $d_{n+1}=h_{n+1}/h_n$ be the coefficient of the three-term recurrence relation* (5) *with n replaced by $n+1$. If $0\leqslant k\leqslant n$, $y\in\mathbb{R}$ and $\sigma_{k,n-k}$ are as in Lemma* 1*, then the best constants in the Szegő inequalities*

$$\left|D^k p(y)\right|^2\leqslant C_{n,k}(y)\left\|D^k p\right\|_{w_k}^2,\quad p\in\mathscr{P}_n,$$

are equal to

$$C_{n,k}(y)=\gamma_{k,n}\begin{vmatrix}D^{k+1}p_{n+1}(y) & D^k p_{n+1}(y)\\ D^{k+1}p_n(y) & D^k p_n(y)\end{vmatrix},$$

where

$$\gamma_{k,n}=\frac{(n-k+1)\sqrt{d_{n+1}}}{(n+1)\,\sigma_{k,n-k}}.$$

These inequalities become identities for the extremal polynomials $p^*\in\mathscr{P}_n$ such that

$$D^k p^*(x)\left|D^k p^*(y)\right|=\pm\gamma_{k,n}\frac{\begin{vmatrix}D^k p_{n+1}(x) & D^k p_{n+1}(y)\\ D^k p_n(x) & D^k p_n(y)\end{vmatrix}}{x-y}.$$

In particular, values of the constant factors $\gamma_{k,n}$, $\sigma_{k,n-k}$ *and* d_{n+1} *are as in Table 2 and in Table 3 [34] for all basic classical orthogonal polynomials.*

Proof. In view of (16) the leading coefficient $\alpha_{m,k}$ of the orthonormal polynomial $p_{m,w_k}(x)$ is equal to

$$\alpha_{m,k} = \left\|q_{m,w_k}\right\|_{w_k}^{-1} = \frac{(m+1)_k}{\sqrt{\sigma_{k,m}}} \left\|q_{m+k,w_0}\right\|_{w_0}^{-1}, \tag{18}$$

where the factor $\sigma_{k,m}$ is as in (13). Moreover, it follows from (17) that

$$\left\|D^k p_{k+\nu}\right\|_{w_k}^2 = \sigma_{k,\nu} \left\|p_{\nu,w_k}\right\|_{w_k}^2 = \sigma_{k,\nu}.$$

Hence we introduce the orthonormal polynomials $D^k p_{k+\nu}(x) / \sqrt{\sigma_{k,\nu}}$ into the Christoffel-Darboux identity (10) in order to get

$$\sum_{\nu=0}^{n-k} \frac{D^k p_{k+\nu}(x) D^k p_{k+\nu}(y)}{\sigma_{k,\nu}} = \frac{\alpha_{n-k,k}}{\alpha_{n-k+1,k}} \frac{\begin{vmatrix} D^k p_{n+1}(x) & D^k p_{n+1}(y) \\ D^k p_n(x) & D^k p_n(y) \end{vmatrix}}{\sqrt{\sigma_{k,n-k+1}\sigma_{k,n-k}}(x-y)}.$$

By (18) the constant factors on the right-hand side simplify to

$$\frac{(n-k+1)_k}{(n-k+2)_k \sigma_{k,n-k}} \frac{\left\|q_{n+1,w_0}\right\|_{w_0}}{\left\|q_{n,w_0}\right\|_{w_0}} = \frac{(n-k+1)}{(n+1)\sigma_{k,n-k}} \sqrt{d_{n+1}} = \gamma_{k,n}.$$

Additionally, if we pass to the limit $x \longrightarrow y$ and use the l'Hospital's rule, then we obtain

$$\sum_{\nu=0}^{n-k} \frac{\left[D^k p_{k+\nu}(y)\right]^2}{\sigma_{k,\nu}} = \gamma_{k,n} \begin{vmatrix} D^{k+1} p_{n+1}(y) & D^k p_{n+1}(y) \\ D^{k+1} p_n(y) & D^k p_n(y) \end{vmatrix}.$$

Thus the proof of theorem is a direct consequence of Lemma 1. □

4. Iterated quadrature rules and barycentric weights

Throughout this section we shall assume that $\left(x_{k,\nu}\right)_{\nu=1}^{n-k}$ are the zeros of the monic classical orthogonal polynomials

$$q_{n-k,w_k}(x) = \frac{\sqrt{\sigma_{k,n-k} h_n}}{(n-k+1)_k} p_{n-k,w_k}(x) = \frac{\sqrt{h_n}}{(n-k+1)_k} D^k p_n(x), \quad 0 \leqslant k < n, \tag{19}$$

of degree $n-k < n_w/2$, where $h_n = \|q_n\|_w^2$, $w_0(x) = w(x)$ and $q_{n,w_0}(x) = q_n(x)$. Note that the inequality $n-k < n_w/2$ means that $\mathscr{P}_{2n}$ is a subspace of $L_w^2(a,b)$.

Now we can proceed to investigate the iterated Gauss quadrature formulae with the nodes $x_{k,1} < x_{k,2} < \ldots < x_{k,n-k}$. For this purpose we first observe that the reproducing kernels $K_{n,w_k}(x,y)$ satisfy

$$K_{n,w_k}\left(x, x_{k,\nu}\right) = K_{n-1,w_k}\left(x, x_{k,\nu}\right)$$

Table 2: The parameters $\gamma_{k,n}$ of the solutions of problem (3) for the classical weight functions of Hermite, Laguerre, Jacobi, generalized Bessel, Jacobi on $(0,+\infty)$ and pseudo-Jacobi type. Here $(\nu)_0 = 1$, $(\nu)_k = \nu(\nu+1)\cdots(\nu+k-1)$ denotes the Pochhammer symbol, and $\sigma_{k,\nu} = \left\|D^k p_{\nu+k}\right\|^2_{w_k}$ and $d_n = h_n/h_{n-1}$ are as in Lemma 1 and Table 3 [34], respectively. Moreover $\chi_{k,n} = ka_2\left[(n-2)a_2+b_1\right]^{-1}$ are parameters of generic formulae for Christoffel numbers given in Theorem 2.

a_2	b_1	$\sigma_{k,\nu}$	$\gamma_{k,n}$	$\chi_{k,n}$
0	-2	$2^k(\nu+1)_k$	$\frac{\sqrt{d_{n+1}}}{2^k(n-k+2)_k}$	0
0	-1	$(\nu+1)_k$	$\frac{\sqrt{d_{n+1}}}{(n-k+2)_k}$	0
-1	$-(\alpha+\beta+2)$	$(\nu+1)_k(\nu+k+\alpha+\beta+1)_k$	$\frac{\sqrt{d_{n+1}}}{(n-k+2)_k(n+\alpha+\beta+1)_k}$	$\frac{k}{n+\alpha+\beta}$
1	α	$(-1)^k(\nu+1)_k(\nu+k+\alpha-1)_k$	$\frac{(-1)^k\sqrt{d_{n+1}}}{(n-k+2)_k(n+\alpha-1)_k}$	$\frac{k}{n+\alpha-2}$
1	$2-\alpha$	$(-1)^k(\nu+1)_k(\nu+k+1-\alpha)_k$	$\frac{(-1)^k\sqrt{d_{n+1}}}{(n-k+2)_k(n-\alpha+1)_k}$	$\frac{k}{n-\alpha}$
1	$2-2\alpha$	$(-1)^k(\nu+1)_k(\nu+k+1-2\alpha)_k$	$\frac{(-1)^k\sqrt{d_{n+1}}}{(n-k+2)_k(n-2\alpha+1)_k}$	$\frac{k}{n-2\alpha}$

and

$$K_{n,w_k}\left(x_{k,\nu},x_{k,\nu}\right) = K_{n-1,w_k}\left(x_{k,\nu},x_{k,\nu}\right) = \gamma_{k,n-1}D^{k+1}p_n\left(x_{k,\nu}\right)D^k p_{n-1}\left(x_{k,\nu}\right),$$

at the zeros $y = x_{k,\nu}$. These formulae enable to derive the representations

$$A_{k,\nu} = \frac{1}{\gamma_{k,n-1}D^{k+1}p_n\left(x_{k,\nu}\right)D^k p_{n-1}\left(x_{k,\nu}\right)} \tag{20}$$

for the Christoffel numbers

$$A_{k,\nu} = \int_a^b l_{k,\nu}(x)\,w_k(x)\,dx = \int_a^b l^2_{k,\nu}(x)\,w_k(x)\,dx$$

of the iterated Gauss quadrature rules, determined uniquely by the conditions

$$\int_a^b p(x)\,w_k(x)\,dx = G_{n,k}(p) := \sum_{\nu=1}^{n-k} A_{k,\nu}\,p\left(x_{k,\nu}\right), \quad p \in \mathscr{P}_{2(n-k)-1}.$$

Here $l_{k,\nu}(x)$ are the fundamental Langrange polynomials defined by

$$l_{k,\nu}(x) = \frac{q_{n-k,w_k}(x)}{\left(x-x_{k,\nu}\right)q'_{n-k,w_k}\left(x_{k,\nu}\right)}.$$

For the completeness, we note that identities (19) for the monic orthogonal polynomials $q_{n-k,w_k}(x)$ follow easily from (14) and the identities

$$\left\|q_{n-k,w_k}\right\|_{w_k}^2 = \frac{\sigma_{k,n-k}}{(n-k+1)_k^2}\left\|q_n\right\|_w^2 = \frac{\sigma_{k,n-k}}{(n-k+1)_k^2}h_n, \tag{21}$$

which are implied by (16) and (9). Moreover, formulae (20) follow from the identity $A_{k,\nu}K_{n,w_k}\left(x_{k,\nu},x_{k,\nu}\right)=1$, presented in Theorem 3.4.2 of Szegő [37]. By (19) and (20) we can also express the Christoffel numbers in the form

$$A_{k,\nu} = \frac{\sqrt{h_n h_{n-1}}}{\gamma_{k,n-1}D^{k+1}q_n\left(x_{k,\nu}\right)D^k q_{n-1}\left(x_{k,\nu}\right)},$$

dependent on the derivatives and norms of the monic polynomials, orthogonal with respect to a classical weight $w(x)$. It should be noted that values of h_n are listed in Table 2 of our paper [34] for all basic classical weights.

Now we can extend our recent generic results, presented for the Christoffel numbers $A_\nu = A_{0,\nu}$ of the Gauss quadrature rules with respect to an arbitrary classical weight $w(x)$ in the paper [34]. For this purpose, we have to express the parameters d_{n-k,w_k}, ρ_{n-k,w_k}, h_{n-k,w_k} and $r_{2(n-k)-1,w_k}$ of the three-terms and Al-Salam-Chihara recurrence relations (5) and (6), for the monic orthogonal polynomial associated with the weight $w_k(x)$, in terms of the corresponding parameters d_n, ρ_n, h_n and r_{2n-1}, which are defined by formulae (8) and (9).

LEMMA 2. *If $1 \leqslant k < n$, then we have*

$$h_{n-k,w_k} = \frac{\sigma_{k,n-k}}{(n-k+1)_k^2}h_n, \quad d_{n-k,w_k} = \frac{(n-k)^2\,\sigma_{k,n-k}}{n^2\sigma_{k,n-k-1}}d_n$$

and

$$\rho_{n-k,w_k} = \frac{\left(1-\frac{k}{n}\right)^2\sigma_{k,n-k}}{\sigma_{k,n-k-1}}\rho_n.$$

Proof. Since $h_{n-k,w_k} = \left\|q_{n-k,w_k}\right\|_{w_k}^2$ and $d_{n-k,w_k} = h_{n-k,w_k}/h_{n-k-1,w_k}$, the first two formulae are direct consequences of equations (21). Further we have

$$B(x)+kA'(x) = (b_1+2ka_2)x+b_0+ka_1$$

in the Pearson differential equation (2) for the weight $w_k(x)$. Hence we obtain $r_{2(n-k)-1,w_k} = r_{2n-1}$ and

$$\rho_{n-k,w_k} = -d_{n-k,w_k}r_{2n-1} = -\frac{(n-k)^2\,\sigma_{k,n-k}}{n^2\sigma_{k,n-k-1}}d_n r_{2n-1},$$

which finishes the proof. □

The following theorem completes Theorem 3.1 [34] for iterated classical weights. It establishes two generic formulae for the Christoffel numbers, which are especially convenient in numerical computations [11, 12, 40, 41]. The second formula uses the constant factors $\chi_{k,n}$ defined by

$$\chi_{k,n} = \frac{n-k}{n}\frac{\sigma_{k,n-k}}{\sigma_{k,n-k-1}} - 1 = \frac{ka_2}{(n-2)a_2+b_1}. \tag{22}$$

Their values have been listed in Table 2 for all six basic classical weight functions.

THEOREM 2. *Let* $(x_{k,\nu})_{\nu=1}^{n-k}$ *be the zeros of a derivative* $D^k q_n(x)$, $1 \leqslant k < n$, *of the monic polynomial* $q_n(x)$, *orthogonal with respect to a classical weight* $w(x)$. *Then the Christoffel numbers* $A_{k,\nu}$ *of the Gauss quadrature formula* $G_{n,k}(p)$ *satisfy*

$$A_{k,\nu} = \frac{\rho_n h_{n-1}\sigma_{k,n-k}}{A(x_{k,\nu})\left[D^{k+1}q_n(x_{k,\nu})\right]^2} = \frac{n\sigma_{k,n-k-1}h_{n-1}A(x_{k,\nu})}{(n-k)(1+\chi_{k,n})\rho_n\left[D^k q_{n-1}(x_{k,\nu})\right]^2}.$$

In the case of all six basic classical weight functions the parameters ρ_n, h_{n-1}, $\sigma_{k,\nu}$ *and* $\chi_{k,n}$ *are listed in Tables 2 from this paper and [34].*

Proof. Apply Theorem 3.1 [34] to the classical weight function $w_k(x)$ to get the formulae

$$A_{k,\nu} = \frac{\rho_{n-k,w_k}h_{n-k-1,w_k}}{A(x_{k,\nu})\left[Dq_{n-k,w_k}(x_{k,\nu})\right]^2} = \frac{h_{n-k-1,w_k}A(x_{k,\nu})}{\rho_{n-k,w_k}\left[q_{n-k-1,w_k}(x_{k,\nu})\right]^2}.$$

Next, transform the first formula to the desired form by using Lemma 2 and the identities

$$q_{n-k,w_k}(x) = \frac{\sqrt{h_n}}{(n-k+1)_k}D^k p_n(x) = \frac{D^k q_n(x)}{(n-k+1)_k}.$$

In the same way, the second formula can be transformed to

$$A_{k,\nu} = \frac{n^2\sigma_{k,n-k-1}^2 h_{n-1}A(x_{k,\nu})}{(n-k)^2\sigma_{k,n-k}\rho_n\left[D^k q_{n-1}(x_{k,\nu})\right]^2}.$$

Hence, it is sufficient to apply formula (22) in order to finish the proof. □

According to Wang et al. [40,41] and the authors [34], it is of interest that Christoffel numbers $A_{k,\nu}$ are extremly useful during numerical computations of the barycentric weights of the Lagrange, Fejér and Hermite type:

$$\mathscr{L}_{n,k}f(x) = \frac{D^k q_n(x)}{(n-k+1)_k}\sum_{\nu=1}^{n-k} f(x_{k,\nu})\frac{B_{k,\nu}}{x-x_{k,\nu}},$$

$$\mathscr{F}_{n,k}f(x) = w_1(x)\frac{\left[D^k q_n(x)\right]^2}{(n-k+1)_k^2}\sum_{\nu=1}^{n-k} f(x_{k,\nu})\frac{C_{k,\nu}}{(x-x_{k,\nu})^2},$$

$$\mathscr{H}_{n,k}f(x) = \frac{\left[D^k q_n(x)\right]^2}{(n-k+1)_k^2}\sum_{\nu=1}^{n-k}\left[\frac{f(x_{k,\nu})D_{k,\nu}^{(0)}}{(x-x_{k,\nu})^2} + \frac{f'(x_{k,\nu})D_{k,\nu}^{(0)} + f(x_{k,\nu})D_{k,\nu}^{(1)}}{x-x_{k,\nu}}\right].$$

This is a consequence of the following corollary, which refines Theorem 3.2 [34] in the case of iterated classical weight functions.

COROLLARY 1. *Let $x_{k,1} < x_{k,2} < \ldots < x_{k,n-k}$, $1 \leqslant k < n$, be zeros of the derivative $D^k q_n(x)$ of the monic polynomial $q_n(x)$, orthogonal with respect to any classical weight $w(x)$. Then the barycentric weights $B_{k,\nu}$, $C_{k,\nu}$, $D^{(0)}_{k,\nu}$ and $D^{(1)}_{k,\nu}$ of Lagrange, Fejér and Hermite interpolating operators have the following representations:*

$$B_{k,\nu} = (-1)^{\nu-1}\sqrt{\frac{(n-k+1)_k^2 A(x_{k,\nu})}{\sigma_{k,n-k}\rho_n h_{n-1}}}A_{k,\nu},$$
$$C_{k,\nu} = \frac{(n-k+1)_k^2}{\sigma_{k,n-k}\rho_n h_{n-1} w_k(x_{k,\nu})}A_{k,\nu},$$
$$D^{(0)}_{k,\nu} = \frac{(n-k+1)_k^2 A(x_{k,\nu})}{\sigma_{k,n-k}\rho_n h_{n-1}}A_{k,\nu},$$
$$D^{(1)}_{k,\nu} = \frac{(n-k+1)_k^2\left[B(x_{k,\nu}) + kA'(x_{k,\nu})\right]}{\sigma_{k,n-k}\rho_n h_{n-1}}A_{k,\nu}.$$

Here the constant factors ρ_n, h_{n-1}, $\sigma_{k,\nu}$ and $\chi_{k,n}$ are as in Theorem 2.

Proof. Following the proof of Theorem 2, we have to use only Lemma 2, Theorem 2 and Theorem 3.2 [34]. □

5. Inequalities of Markov-Bernstein type

Before we compute the squares of weighted operator norms

$$\left\|D^k\right\|^2_{w_k} := \max_{p\in\mathscr{P}_n}\left\{\left\|D^k p\right\|^2_{w_k} : \|p\|_w = 1\right\}$$

of the differential operators

$$D^k : \mathscr{P}_n(w) \ni p \to D^k p \in \mathscr{P}_{n-k}(w_k),$$

we note that Agarwal, Guessab and Milovanović [2, 13] proved the formula

$$\left\|D^k\right\|^2_{w_k} = \lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,k-1}$$

for the iterated weights $w_k(x)$, associated with the classical basic weights $w(x)$ of Hermite, Laguerre and Jacobi type. Here $\lambda_{n,k}$ are the eigenvalues of the Sturm-Liouville differential equation (7).

In the following theorem we extend their result to an arbitrary classical weight function $\widehat{w}(\widehat{x})$ on $\left(\widehat{a},\widehat{b}\right)$. For this purpose suppose that a classical weight $\widehat{w}(\widehat{x})$,

$\widehat{a} < x < \widehat{b}$, is transformed to the basic classical weight $w(x)$, $a < x < b$, by changing variable $\widehat{x} = \beta_1 x + \beta_0$, $\beta_1 > 0$. It is clear that this change of variable preserves the Pearson differential equation and orthonormality of the corresponding basic polynomials $p_\nu(x)$, provided that the normalizations

$$\widehat{B}(\widehat{x}) = \frac{1}{\beta_1} B\left(\frac{\widehat{x}-\beta_0}{\beta_1}\right) \quad \text{and} \quad \widehat{p}_\nu(\widehat{x}) = \frac{1}{\sqrt{\beta_1}} p_\nu\left(\frac{\widehat{x}-\beta_0}{\beta_1}\right) \tag{23}$$

are preassumed. Indeed, these normalizations guarantee that

$$\frac{d}{d\widehat{x}}\left(\widehat{A}(\widehat{x})\,\widehat{w}(\widehat{x})\right) = \widehat{B}(\widehat{x})\,\widehat{w}(\widehat{x})$$

and

$$\|\widehat{p}_\nu\|_{\widehat{w}}^2 = \frac{1}{\beta_1}\int_{\widehat{a}}^{\widehat{b}} p_\nu^2\left(\frac{\widehat{x}-\beta_0}{\beta_1}\right) w\left(\frac{\widehat{x}-\beta_0}{\beta_1}\right) d\widehat{x} = \int_a^b p_\nu^2(x)\,w(x)\,dx = 1.$$

THEOREM 3. *Let $w_k(x) = A^k(x)\,w(x)$, $0 \leqslant k \leqslant n$, be an iterated weight corresponding to an arbitrary classical weight $w(x) = w_0(x)$ on (a,b). If $\{p_n(x)\}_{0\leqslant n<n_w}$ are classical orthonormal polynomials associated with the weight $w(x)$, then the best constants in the Markov-Bernstein inequalities*

$$\left\|D^k p\right\|_{w_k}^2 \leqslant C_{n,k}(w_k)\,\|p\|_w^2, \quad p \in \mathscr{P}_n,$$

are equal to

$$C_{n,k}(w_k) = \left\|D^k\right\|_{w_k}^2 = \left\|D^k p_n\right\|_{w_k}^2 = \sigma_{k,n-k},$$

where $\sigma_{k,n-k}$ are defined as in Lemma 1. Additionally, the constants $\sigma_{k,n-k}$ are as in Table 2 in the case of all six basic classical weight functions of the Hermite, Laguerre, Jacobi, generalized Bessel, Jacobi on $(0,+\infty)$ and pseudo-Jacobi type.

Proof. Since the weight $w(x)$ is classical, it follows that the polynomials $\left\{D^k p_\nu(x)\right\}_{k\leqslant\nu\leqslant n}$ are orthogonal with respect to the weight $w_k(x)$. Hence we get

$$\left\|D^k p\right\|_{w_k}^2 = \sum_{\nu=k}^{n}(p,p_\nu)_w^2\left\|D^k p_\nu\right\|_{w_k}^2 \leqslant \max_{k\leqslant\nu\leqslant n}\left\|D^k p_\nu\right\|_{w_k}^2, \quad p \in \mathscr{P}_n,$$

whenever

$$\|p\|_w^2 = \sum_{\nu=0}^{n}(p,p_\nu)_w^2 = 1.$$

On the other hand, it has been established in the proof of Theorem 1.1 from [35] that the sequence $\left\{\|D^k p_\nu\|_{w_k}^2\right\}_{k\leqslant\nu<n_w}$ is strictly increasing, whenever $w(x)$ is an arbitrary basic classical weight function. Consequently, we conclude that

$$\left\|D^k p\right\|_{w_k}^2 \leqslant \left\|D^k p_n\right\|_{w_k}^2, \tag{24}$$

with the equality holding when $p(x) = p_n(x)$. Hence the theorem is a direct consequence of Lemma 1 for all six basic classical weights $w(x)$. Therefore it remains to consider an arbitrary classical weight function $\widehat{w}(\widehat{x})$ on the interval $\left(\widehat{a},\widehat{b}\right)$, which is not basic. Then there exist the unique basic weight $w(x)$ on (a,b) and the linear transformation $\widehat{x} = \beta_1 x + \beta_0$, $\beta_1 > 0$, of (a,b) onto $\left(\widehat{a},\widehat{b}\right)$ such that

$$\widehat{w}(\widehat{x}) = w\left(\frac{\widehat{x}-\beta_0}{\beta_1}\right).$$

Hence we obtain

$$\begin{aligned}\left\|D^k \widehat{p}_\nu\right\|^2_{\widehat{w}_k} &= \frac{1}{\beta_1}\int_{\widehat{a}}^{\widehat{b}} \left[\frac{d^k}{d\widehat{x}^k} p_\nu\left(\frac{\widehat{x}-\beta_0}{\beta_1}\right)\right]^2 w_k\left(\frac{\widehat{x}-\beta_0}{\beta_1}\right) d\widehat{x} \\ &= \frac{1}{\beta_1^{2k}}\int_a^b \left[\frac{d^k}{dx^k} p_\nu(x)\right]^2 w_k(x)\,dx = \frac{1}{\beta_1^{2k}}\left\|D^k p_\nu\right\|^2_{w_k}.\end{aligned}$$

Thus the sequence $\left\{\left\|D^k \widehat{p}_\nu\right\|^2_{\widehat{w}_k}\right\}_{k\leqslant \nu < n_w}$ strictly increases. Consequently, we can apply formulae (23) and (24) in order to get

$$\left\|D^k p\right\|^2_{\widehat{w}_k} \leqslant \left\|D^k \widehat{p}_n\right\|^2_{\widehat{w}_k} = \widehat{\sigma}_{k,n-k} = \frac{1}{\beta_1^{2k}}\sigma_{k,n-k}, \quad p \in \mathscr{P}_n,$$

which completes the proof of the theorem. □

If $0 \leqslant j < k$ is arbitrary, then the problem of calculation of the norms $\left\|D^k\right\|_{w_j}$ is much more difficult, except the Hermite iterated weights $w_j(x) = e^{-x^2}$. In the case of the Legendre weight $w(x) \equiv 1$, it has been already observed by Kroó [21] for $j = 0$ and $k = 1$. In fact this extremaly difficult problem remains open and its solving requires different techniques than those used above. Of course, in the Hermite case it follows from Theorem 3 that the norms $\left\|D^k\right\|^2_{w_j}$ are independent of j and that we have

$$\left\|D^k\right\|^2_{w_j} = \sigma_{k,n-k} = 2^k (n-k+1)_k.$$

Here the norms are attained by the Hermite orthonormal polynomials $p_n(x)$. It has been already proved by Dörfler [8]. On the other hand, we note that Varma [39] characterized Hermite orthogonal polynomials by a Pythagorean-type inequality in terms of the norms of their first and second derivatives. It was generalized by Agarwal and Milovanović to the classical polynomials of Hermite, Laguerre and Jacobi in the paper [1].

REFERENCES

[1] R. P. AGARWAL AND G. V. MILOVANOVIĆ, *A characterization of the classical orthogonal polynomials*, in: P. Nevai, A. Pinkus (Eds.), Progress in Approximation Theory, Academic Press, New York, 1991, pp. 1–4.

[2] R. P. AGARWAL AND G. V. MILOVANOVIĆ, *Extremal problems, inequalities, and classical orthogonal polynomials*, Appl. Math. Comput., **128**, (2002), 151–166.

[3] W. A. AL-SALAM, *Characterization theorems for orthogonal polynomials*, in: P. Nevai (Ed.), Orthogonal Polynomials: Theory and Practice, Vol. **294** of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer Academic Publishers, Dordrecht, 1990, pp. 1–24.

[4] N. ARONSZAJN, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., **68**, (1950), 337–404.

[5] S. BOCHNER, *Über Sturm-Liouvillesche Polynomsysteme*, Math. Z., **29**, (1929), 730–736.

[6] T. S. CHIHARA, *An Introduction to Orthogonal Polynomials*, Mathematics and its Applications, Vol. **13**, Gordon and Breach, New York, 1978.

[7] P. J. DAVIS, *Interpolation and Approximation*, Dover Publications, Inc., New York, 1975.

[8] P. DÖRFLER, *New inequalities of Markov type*, SIAM J. Math. Anal., **18**, (1987), 490–494.

[9] G. FREUD, *Markov-Bernstein type inequalities in $L_p(-\infty,\infty)$*, in: G. G. Lorentz, C. K. Chui, L. L. Schumaker (Eds.), Approximation Theory, II, Academic Press, New York, 1976, pp. 369–377.

[10] G. FREUD, *On Markov-Bernstein-type inequalities and their applications*, J. Approx. Theory, **19**, (1977), 22–37.

[11] A. GLASER, X. LIU AND V. ROKHLIN, *A fast algorithm for the calculation of the roots of special functions*, SIAM J. Sci. Comput., **29**, (2007), 1420–1438.

[12] M. GU AND S. C. EISENSTAT, *A divide-and-conquer algorithm for the symmetric tridiagonal eigenproblem*, SIAM J. Matrix Anal. Appl., **16**, (1995), 172–191.

[13] A. GUESSAB AND G. V. MILOVANOVIĆ, *Weighted L^2-analogues of Bernstein's inequality and classical orthogonal polynomials*, J. Math. Anal. Appl., **182**, (1994), 244–249.

[14] W. HAHN, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Math. Z., **39**, (1935), 634–638.

[15] R. KOEKOEK, P. A. LESKY AND R. F. SWARTTOUW, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.

[16] W. KOEPF AND M. MASJED-JAMEI, *A generalization of Student's t-distribution from the viewpoint of special functions*, Integral Transforms Spec. Funct., **17**, (2006), 863–875.

[17] W. KOEPF AND M. MASJED-JAMEI, *A generic polynomial solution for the differential equation of hypergeometric type and six sequences of orthogonal polynomials related to it*, Integral Transforms Spec. Funct., **17**, (2006), 559–576.

[18] H. L. KRALL, *On derivatives of orthogonal polynomials*, Bull. Amer. Math. Soc., **42**, (1936), 423–428.

[19] H. L. KRALL, *On higher derivatives of orthogonal polynomials*, Bull. Amer. Math. Soc., **42**, (1936), 867–870.

[20] H. L. KRALL, *On derivatives of orthogonal polynomials. II*, Bull. Amer. Math. Soc., **47**, (1941), 261–264.

[21] A. KROÓ, *On the exact constant in the L_2 Markov inequality*, J. Approx. Theory, **151**, (2008), 208–211.

[22] P. LESKY, *Die Charakterisierung der klassischen orthogonalen Polynome durch Sturm-Liouvillesche Differentialgleichungen*, Arch. Rational Mech. Anal., **10**, (1962), 341–351.

[23] P. A. LESKY, *Endliche und unendliche Systeme von kontinuierlichen klassischen Orthogonalpolynomen*, Z. Angew. Math. Mech., **76**, (1996), 181–184.

[24] P. A. LESKY, *Eine Charakterisierung der klassichen kontinuierlichen-, diskreten- und q-Orthogonalpolynome*, Shaker Verlag, Aachen, 2005.

[25] F. MARCELLÁN, Y. QUINTANA AND J. M. RODRÍGUEZ, *Markov-type inequalities and duality in weighted Sobolev spaces*, preprint.

[26] M. MASJED-JAMEI, *Classical orthogonal polynomials with weight function $((ax+b)^2+(cx+d)^2)^{-p}\exp(q\arctan((ax+b)/(cx+d)))$, $x\in(-\infty,\infty)$ and a generalization of T and F distributions*, Integral Transforms Spec. Funct., **15**, (2004), 137–153.

[27] G. MASTROIANNI AND G. V. MILOVANOVIĆ, *Interpolation Processes, Basic Theory and Applications*, Springer-Verlag, Berlin, 2008.

[28] G. V. MILOVANOVIĆ, *Various extremal problems of Markov's type for algebraic polynomials*, Facta Univ. Ser. Math. Inform. **2** (1987), 7–28.

[29] P. NEVAI, *Géza Freud, orthogonal polynomials and Christoffel functions. A case study*, J. Approx. Theory, **48**, (1986), 3–167.

[30] A. F. NIKIFOROV AND V. B. UVAROV, *Special Functions of Mathematical Physics*, Birkhäuser, Basel-Boston, 1988.

[31] P. RUTKA AND R. SMARZEWSKI, *Complete solution of the electrostatic equilibrium problem for classical weights*, Appl. Math. Comput., **218**, (2012), 6027–6037.

[32] P. RUTKA AND R. SMARZEWSKI, *Extremal interpolatory problem of Fejér type for all classical weight functions*, Electron. Trans. Numer. Anal., **39**, (2012), 46–59.

[33] P. RUTKA AND R. SMARZEWSKI, *Multivariate inequalities of Chernoff type for classical orthogonal polynomials*, J. Math. Anal. Appl., **388**, (2012), 78–85.

[34] P. RUTKA AND R. SMARZEWSKI, *Explicit barycentric formulae for osculatory interpolation at roots of classical orthogonal polynomials*, Math. Comp., **86**, (2017), 2409–2427.

[35] R. SMARZEWSKI AND P. RUTKA, *Inequalities of Chernoff type for finite and infinite sequences of classical orthogonal polynomials*, Proc. Amer. Math. Soc., **138**, (2010), 1305–1315.

[36] P. K. SUETIN, *Classical Orthogonal Polynomials* [in Russian], Nauka, Moscow, 1979.

[37] G. SZEGŐ, *Orthogonal Polynomials*, 4th Edition, Vol. XXIII of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence RI, 1975.

[38] G. VALENT AND W. VAN ASSCHE, *The impact of Stieltjes' work on continued fractions and orthogonal polynomials: additional material*, J. Comput. Appl. Math., **65**, (1995), 419–447.

[39] A. K. VARMA, *A new characterization of Hermite polynomials*, Acta Math. Hungar. **49** (1987), 169–172.

[40] H. WANG, D. HUYBRECHS AND S. VANDEWALLE, *Explicit barycentric weights for polynomial interpolation in the roots or extrema of classical orthogonal polynomials*, Math. Comp., **83**, (2014), 2893–2914.

[41] H. WANG AND S. XIANG, *On the convergence rates of Legendre approximation*, Math. Comp., **81**, (2012), 861–877.

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