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ON RETARDED NONLINEAR INTEGRAL INEQUALITIES OF GRONWALL AND APPLICATIONS

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Abstract. By some new analysis techniques, we generalize the results presented by Pachpatte in [8] to nonlinear retarded inequalities, and also investigate some new forms. Some examples are presented to illustrate our results at the end.

1. Introduction

In 1919, Gronwall [4] discovered a new inequality which plays a fundamental role in the development of the theory of differential equation and proved if f and u are real-valued nonnegative continuous functions defined on $\mathbb{R}_+$, with a positive constant u_0, then

$$u(t) \leqslant u_0 + \int_0^t f(s)u(s)ds, \tag{1.1}$$

implies

$$u(t) \leqslant u_0 \exp\left(\int_0^t f(s)ds\right).$$

In 1943, Bellman generalized the inequality (1.1) to inequality with a function $a(t)$ instate of the constant u_0 and proved that: If $u,\ f,\ a, \in C(\mathbb{R}_+,\mathbb{R}_+)$ and $a(t)$ is a nondecreasing, then the inequality

$$u(t) \leqslant a(t) + \int_0^t f(s)u(s)ds, \tag{1.2}$$

implies

$$u(t) \leqslant a(t)\exp\left(\int_0^t f(s)u(s)ds\right).$$

Many results on its generalization can be found for example in [19, 14, 16, 12, 10, 11, 20, 21, 15, 6, 13]. Among them is Bihari's [11] in 1956, where he extended inequality (1.1) to the nonlinear inequality

$$u(t) \leqslant u_0 + \int_0^t f(s)\omega(u(s))ds. \tag{1.3}$$

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In 1957, Ou-Iang [2] investigated one of the most useful nonlinear inequalities in the development of the theory of differential equations. In particular, he proved if u, f are nonnegative continuous functions on $\mathbb{R}_+, u_0 \geqslant 0$ is a constant and

$$u^2(t) \leqslant u_0^2 + 2\int_0^t f(s)u(s)ds, \quad t \in \mathbb{R}_+, \tag{1.4}$$

then

$$u(t) \leqslant u_0 + \int_0^t f(s)ds, \quad t \in \mathbb{R}_+.$$

It is interesting to note that Ou-Iang inequalities and their generalizations have proved to be useful tools in oscillation theory, boundedness theory, stability theory, and other applications of differential and difference equations.

In 1995, Pachpatte [3] obtained the following generalization of the Ou-Iang inequality (1.4) as follows: If u, f and g are nonnegative continuous functions defined on $\mathbb{R}_+$ and c is a nonnegative constant, then

$$u^2(t) \leqslant c^2 + 2\int_0^t [f(s)u^2(s) + g(s)u(s)]ds, \tag{1.5}$$

for $t \in \mathbb{R}_+$, implies

$$u(t) \leqslant \varepsilon(t)\exp\left(\int_0^t f(s)ds\right),$$

for $t \in \mathbb{R}_+$, where

$$\varepsilon(t) = c + \int_0^t g(s)ds, \quad \forall t \in \mathbb{R}_+.$$

In 2000, Pachpatte [8] studied several non-retarded integral inequalities arising in the theory of differential equations and difference equations related to the inequality (1.4)

$$u^p(t) \leqslant a(t) + b(t)\int_0^t [g(s)u^p(s) + h(s)u(s)]ds, \forall t \in \mathbb{R}_+, \tag{1.6}$$

where u, a, b, g, h, are real-valued nonnegative continuous functions defined on $\mathbb{R}_+$, and $p > 1$ is a real constant.

Further, in 2000, Lipovan [6] studied the retarded case of the inequality (1.3) by replacing t by a function $\alpha(t)$

$$u(t) \leqslant u_0 + \int_{\alpha(t_0)}^{\alpha(t)} f(s)\omega(u(s))ds, \quad t_0 \leqslant t \leqslant t_1. \tag{1.7}$$

In 2005, Ravi Agarwal et al. improved the results obtained by Lipovan [6] where they generalized the inequality (1.7) to the general form

$$u(t) \leqslant a(t) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(t,s)\omega_i(u(s))ds, \quad t_0 \leqslant t \leqslant t_1. \tag{1.8}$$

where $a(t)$ is a function and ω_i's may be distinct.

For more contributions of Gronwall-type inequalities, we refer the reader to the paper [1] which considers the development of Gronwall-type discrete and continuous retarded integral inequalities. In recent years, the study of retarded inequalities has received a lot of attention.

In 2014, Hassan El-Owaidy, Abdeldaim and El-Deeb [15] proved the inequality in the following form:

$$u(t) \leqslant f(t) + \int_a^{\alpha_1(t)} g(s) w_1(u(s)) ds + \int_a^{\alpha_2(t)} h(s) w_2(u(s)) ds, \forall\, t \in I_1 = [a,b]. \quad (1.9)$$

In the same paper, the authors [15] also established the following inequality:

$$u(t) \leqslant f(t) + \int_a^{\alpha(t)} h(s) w(u(s)) ds + \int_a^{\alpha(t)} k(t,s) w(u(s)) ds, \forall t \in I_1,$$

where $u,\ g,\ h \in \mathscr{C}(I_1,\mathbb{R}_+)$, $\alpha,\ f\ \in \mathscr{C}^1(I_1,I_1)$ be nondecreasing functions, with $\alpha_i(t) \leqslant t$, $\alpha_i(a) = a$, $\alpha_i'(t) \geqslant 0$, $i = 1,2$, and $w_i \in (\mathbb{R}_+,\mathbb{R}_+)$ nondecreasing function, and $k(t,s) \in \mathscr{C}(I_1 \times I_1, \mathbb{R}_+)$ with $\frac{\partial k}{\partial t}(t,s) \in \mathscr{C}(I_1 \times I_1, \mathbb{R}_+)$.

In 2015, Abdeldaim and El-Deeb [14] discussed the inequality of the form

$$u(t) \leqslant u_0 + \int_0^{\alpha(t)} f(s)\varphi(u(s)) \left[\varphi(u(s)) + \int_0^s g(\lambda)\varphi(u(\lambda)) d\lambda \right] ds,$$

for all $t \in \mathbb{R}_+$, where $\varphi,\ \varphi',\ \alpha \in \mathscr{C}^1(\mathbb{R}_+,\mathbb{R}_+)$ are increasing functions, with $\varphi'(t) \leqslant k$, $\varphi > 0,\ \alpha(t) \leqslant t,\ \alpha(0) = 0$, for all $t \in \mathbb{R}_+,\ k,\ u_0$ are positive constants.

In the same paper, Abdeldaim and El-Deeb [14] proved a new inequality with a different kernel

$$\varphi_1(u(t)) \leqslant u_0 + \int_0^{\alpha(t)} f(s)\varphi_2(u(s)) \left[u(s) + \int_0^s g(\lambda)\varphi_1(u(\lambda)) d\lambda \right]^p ds, \forall t \in \mathbb{R}_+,$$

where $\varphi_1, \varphi_2, \alpha \in \mathscr{C}^1(\mathbb{R}_+,\mathbb{R}_+)$ are increasing functions with $\alpha(t) \leqslant t, \varphi_i(t) > 0, i = 1,2, \alpha(0) = 0$ and $\varphi_1'(t) = \varphi_2(t)$, $p > 1$ and u_0 is a positive constant.

One of the generalizations of Gronwall-type inequalities has been proved by the authors [19]

$$\varphi_1(u(t)) \leqslant u_0 + \int_0^{\alpha(t)} g(s)\varphi_1(u(s)) ds + \int_0^{\alpha(t)} h(s)\varphi_2(u(s)) ds, \forall t \in I,$$

with $\alpha(t) \leqslant t,\ \varphi_i(t) > 0,\ i = 1,2,\ \alpha(0) = 0,\ \varphi_1'(t) = \varphi_2(t)$, and $\varphi_1^{-1}(t)$ is a submultiplicative function and u_0 is a positive constant.

Now, we state the basic theorem that will be needed in the proofs of the main results.

THEOREM 1.1. ([9]) *If* $x \geqslant 0$, $y \geqslant 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$, *then*

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leqslant \frac{x}{p} + \frac{y}{q}. \quad (1.10)$$

Our results will be based on the mentioned results of Pachpatte [8]. The main aim of this paper is to extend some of these results to retarded integral inequalities depending on the application of the Hölder inequality and some analysis techniques. The paper is organized in the following way: In Section 2, we state and prove the main results. In Section 3, we introduce some applications of our results to obtain the estimates of the solutions of certain integral equations for which inequalities obtained in the literature thus far do not apply directly.

Our main result is given in the following section.

2. Main results

In what follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0,\infty)$, $I = (0,\infty)$ are given subset of $\mathbb{R}$, and $\mathscr{C}(S_1, S_2)$ denotes the class of all continuous functions defined on set S_1 with range in the set S_2.

THEOREM 2.1. *Let* $u,a,b,g,h \in \mathscr{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\alpha(t) \in \mathscr{C}^1(\mathbb{R}_+,\mathbb{R}_+)$, *with* $\alpha(t) \leqslant t$, $\alpha(0) = 0$ *and* $p > 1$ *be a constant. If*

$$u^p(t) \leqslant a(t) + b(t) \int_0^{\alpha(t)} \Big[g(s)u^p(s) + h(s)u(s) \Big] ds, \tag{2.1}$$

for all $t \in \mathbb{R}_+$, *then*

$$\begin{aligned} u(t) \leqslant \Bigg\{ a(t) + b(t) \exp\left[\int_0^{\alpha(t)} b(r)\left(g(r) + \frac{h(r)}{p} \right) dr \right] \\ \times \int_0^{\alpha(t)} \left[g(s)a(s) + h(s)\left(\frac{p-1}{p} + \frac{a(s)}{p} \right) \right] \\ \times \exp\left[-\int_0^s b(r)\left(g(r) + \frac{h(r)}{p} \right) dr \right] ds \Bigg\}^{\frac{1}{p}}, \end{aligned} \tag{2.2}$$

for all $t \in \mathbb{R}_+$.

Proof. We take the function $z(t)$ by

$$z(t) = \int_0^{\alpha(t)} \Big[g(s)u^p(s) + h(s)u(s) \Big] ds, \qquad \forall t \in \mathbb{R}_+. \tag{2.3}$$

That $z(t) \geqslant 0$ nondecreasing on $\mathbb{R}_+$ with $z(0) = 0$. Then (2.1), can be written as

$$u^p(t) \leqslant a(t) + b(t)z(t), \qquad \forall t \in \mathbb{R}_+. \tag{2.4}$$

From (2.4) and using Theorem 1.1, we have

$$\begin{aligned} u(t) &\leqslant \Big(a(t) + b(t)z(t) \Big)^{\frac{1}{p}} \Big(1 \Big)^{\frac{p-1}{p}}, \\ &\leqslant \frac{p-1}{p} + \frac{a(t)}{p} + \frac{b(t)}{p} z(t), \qquad \forall t \in \mathbb{R}_+. \end{aligned} \tag{2.5}$$

Differentiating (2.3) and using (2.4) and (2.5), we get

$$\begin{aligned} z'(t) &\leqslant b(\alpha(t))\left(g(\alpha(t))+\frac{h(\alpha(t))}{p}\right)z(\alpha(t))\alpha'(t) \\ &\quad +\left[g(\alpha(t))a(\alpha(t))+h(\alpha(t))\left(\frac{p-1}{p}+\frac{a(\alpha(t))}{p}\right)\right]\alpha'(t) \\ &\leqslant b(\alpha(t))\left(g(\alpha(t))+\frac{h(\alpha(t))}{p}\right)z(t)\alpha'(t) \\ &\quad +\left[g(\alpha)(t)a(\alpha(t))+h(\alpha(t))\left(\frac{p-1}{p}+\frac{a(\alpha(t))}{p}\right)\right]\alpha'(t), \end{aligned} \tag{2.6}$$

for all $t\in\mathbb{R}_+$. The inequality (2.6) gives us the following bound

$$\begin{aligned} z(t) &\leqslant \exp\left[\int_0^{\alpha(t)} b(r)\left(g(r)+\frac{h(r)}{p}\right)dr\right] \\ &\quad \times\int_0^{\alpha(t)}\left[g(s)a(s)+h(s)\left(\frac{p-1}{p}+\frac{a(s)}{p}\right)\right] \\ &\quad \times\exp\left[-\int_0^{s} b(r)\left(g(r)+\frac{h(r)}{p}\right)dr\right]ds, \forall t\in\mathbb{R}_+. \end{aligned} \tag{2.7}$$

The required inequality (2.2) follows from (2.7) and (2.4). The proof is completed.

REMARK 2.1. If $\alpha(t)=t$, then Theorem 2.1 reduces to [8, Theorem 1 part (a_1)].

THEOREM 2.2. *Let* $u,b,g,h\in\mathscr{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\alpha(t)\in\mathscr{C}^1(\mathbb{R}_+,\mathbb{R}_+)$, *with* $\alpha(t)\leqslant t$, $\alpha(0)=0$, $c(t)\in\mathscr{C}(\mathbb{R}_+,I)$ *be a nondecreasing function and* $p>1$ *be a constant. If*

$$u^p(t)\leqslant c^p(t)+b(t)\int_0^{\alpha(t)}\left[g(s)u^p(s)+h(s)u(s)\right]ds, \forall t\in\mathbb{R}_+, \tag{2.8}$$

then

$$\begin{aligned} u(t) &\leqslant c(t)\left\{1+b(t)\exp\left[\int_0^{\alpha(t)} b(r)\left(g(r)+\frac{h(r)c^{1-p}(r)}{p}\right)dr\right]\right. \\ &\quad \times\int_0^{\alpha(t)}\left[g(s)+h(s)c^{1-p}(s)\right] \\ &\quad \left.\times\exp\left[-\int_0^{\alpha(t)} b(r)\left(g(r)+\frac{h(r)c^{1-p}(r)}{p}\right)dr\right]ds\right\}^{\frac{1}{p}}, \end{aligned} \tag{2.9}$$

for all $t\in\mathbb{R}_+$.

Proof. Since $c(t)>0$ and nondecreasing on $\mathbb{R}_+$, then from (2.8), we note that

$$\left(\frac{u(t)}{c(t)}\right)^p\leqslant 1+b(t)\int_0^{\alpha(t)}\left[g(s)\left(\frac{u(t)}{c(t)}\right)^p+h(s)c^{1-p}(s)\left(\frac{u(t)}{c(t)}\right)\right]ds, \tag{2.10}$$

for all $t\in\mathbb{R}_+$. By using the inequality which proved in Theorem 2.3, we get the required inequality in (2.9). The proof is completed.

REMARK 2.2. If $\alpha(t)=t$, then Theorem 2.2 reduces to [8, Theorem 1 part (a_2)].

THEOREM 2.3. *Let* $u,a,b,g,h \in \mathscr{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\alpha(t)\in\mathscr{C}^1(\mathbb{R}_+,\mathbb{R}_+)$, *with* $\alpha(t)\leqslant t$, $\alpha(0)=0$ *and* $p>q\geqslant 1$ *be constants. If*

$$u^p(t) \leqslant a(t)+b(t)\int_0^{\alpha(t)}\Big[g(s)u^p(s)+h(s)u^q(s)\Big]ds, \tag{2.11}$$

for all $t\in\mathbb{R}_+$, *then*

$$\begin{aligned} u(t) \leqslant \Bigg\{ & a(t)+b(t)\exp\left[\int_0^{\alpha(t)} b(r)\Big(g(r)+\frac{q}{p}h(r)\Big)dr\right] \\ & \times\int_0^{\alpha(t)}\left[g(s)a(s)+h(s)\Big(\frac{p-q}{p}+\frac{q}{p}a(s)\Big)\right] \\ & \times\exp\left[-\int_0^{s} b(r)\Big(g(r)+\frac{q}{p}h(r)\Big)dr\right]ds\Bigg\}^{\frac{1}{p}}, \end{aligned} \tag{2.12}$$

for all $t\in\mathbb{R}_+$.

Proof. We take the function $z(t)$ by

$$z(t)=\int_0^{\alpha(t)}\Big[g(s)u^p(s)+h(s)u^q(s)\Big]ds, \qquad \forall t\in\mathbb{R}_+. \tag{2.13}$$

That $z(t)\geqslant 0$ nondecreasing on $\mathbb{R}_+$ with $z(0)=0$. Then (2.11) can be written as

$$u^p(t)\leqslant a(t)+b(t)z(t), \qquad \forall t\in\mathbb{R}_+. \tag{2.14}$$

From (2.14) and using Theorem 1.1, we have

$$\begin{aligned} u^q(t) &\leqslant \Big(a(t)+b(t)z(t)\Big)^{\frac{q}{p}}\Big(1\Big)^{\frac{p-q}{p}} \\ &\leqslant \frac{p-q}{p}+\frac{q}{p}a(t)+\frac{q}{p}b(t)z(t), \qquad \forall t\in\mathbb{R}_+. \end{aligned} \tag{2.15}$$

Differentiating (2.13) and using (2.14) and (2.15), we get

$$\begin{aligned} z'(t) \leqslant\ & b(\alpha(t))\Big(g(\alpha(t))+\frac{q}{p}h(\alpha(t))\Big)z(\alpha(t))\alpha'(t) \\ & +\left[g(\alpha(t))a(\alpha(t))+h(\alpha(t))\Big(\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))\Big)\right]\alpha'(t) \\ \leqslant\ & b(\alpha(t))\Big(g(\alpha(t))+\frac{q}{p}h(\alpha(t))\Big)z(t)\alpha'(t) \\ & +\left[g(\alpha)(t)a(\alpha(t))+h(\alpha(t))\Big(\frac{p-q}{p}+\frac{q}{p}a(t)\Big)\right]\alpha'(t), \end{aligned} \tag{2.16}$$

for all $t \in \mathbb{R}_+$. The inequality (2.16) gives us the following bound

$$z(t) \leqslant \exp\left[\int_0^{\alpha(t)} b(r)\left(g(r)+\frac{q}{p}h(r)\right)dr\right] \times \int_0^{\alpha(t)} \left[g(s)a(s)+h(s)\left(\frac{p-q}{p}+\frac{q}{p}a(s)\right)\right] \times \exp\left[-\int_0^{s} b(r)\left(g(r)+\frac{q}{p}h(r)\right)dr\right]ds, \forall t \in \mathbb{R}_+. \tag{2.17}$$

We get the required inequality (2.12) from (2.17) and (2.14). The proof is completed.

REMARK 2.3. If $\alpha(t)=t$ and $q=1$, then Theorem 2.3 reduces to [8, Theorem 1 part (a_1)].

REMARK 2.4. If $q=1$, then Theorem 2.3 reduces to Theorem 2.1.

THEOREM 2.4. *Let $u,b,g,h \in \mathscr{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\alpha(t) \in \mathscr{C}^1(\mathbb{R}_+,\mathbb{R}_+)$, with $\alpha(t) \leqslant t$, $\alpha(0)=0$, $c(t) \in \mathscr{C}(\mathbb{R}_+,I)$ be a nondecreasing function and $p>q \geqslant 1$ be a constants. If*

$$u^p(t) \leqslant c^p(t)+b(t)\int_0^{\alpha(t)} \left[g(s)u^p(s)+h(s)u^q(s)\right]ds, \tag{2.18}$$

for all $t \in \mathbb{R}_+$, then

$$u(t) \leqslant c(t)\left\{1+b(t)\exp\left[\int_0^{\alpha(t)} b(r)\left(g(r)+\frac{q}{p}h(r)c^{q-p}(r)\right)dr\right] \times \int_0^{\alpha(t)} \left[g(s)+h(s)c^{q-p}(s)\right] \times \exp\left[-\int_0^{\alpha(t)} b(r)\left(g(r)+\frac{q}{p}h(r)c^{q-p}(r)\right)dr\right]ds\right\}^{\frac{1}{p}}, \tag{2.19}$$

for all $t \in \mathbb{R}_+$.

Proof. Since $c(t)>0$ and nondecreasing on $\mathbb{R}_+$. From (2.18), we observe that

$$\left(\frac{u(t)}{c(t)}\right)^p \leqslant 1+b(t)\int_0^{\alpha(t)} \left[g(s)\left(\frac{u(t)}{c(t)}\right)^p + h(s)c^{q-p}(s)\left(\frac{u(t)}{c(t)}\right)^q\right]ds, \tag{2.20}$$

for all $t \in \mathbb{R}_+$. By using the inequality which proved in Theorem 2.3, we get the required inequality in (2.19). The proof is completed.

REMARK 2.5. If $\alpha(t)=t$ and $q=1$, then Theorem 2.4 reduces to [8, Theorem 1 part (a_2)].

REMARK 2.6. If $q=1$, then Theorem 2.4 reduces to Theorem 2.2.

THEOREM 2.5. *Let $u,a,b,g,h \in \mathscr{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\alpha(t) \in \mathscr{C}^1(\mathbb{R}_+,\mathbb{R}_+)$, with $\alpha(t) \leqslant t$, $\alpha(0) = 0$, and $k(t,s) \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, I)$ with $\frac{\partial}{\partial t}k(t,s) \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, I)$, and $p > 1$ be a constant. If*

$$u^p(t) \leqslant a(t) + b(t)\int_0^{\alpha(t)} k(t,s)\left[g(s)u^p(s) + h(s)u(s)\right]ds, \tag{2.21}$$

for all $t \in \mathbb{R}_+$, then

$$u(t) \leqslant \left\{a(t) + b(t)\exp\left[\int_0^{\alpha(t)} A_1(r)dr\right]\int_0^{\alpha(t)} B_1(s)\exp\left[-\int_0^{s} A_1(r)dr\right]ds\right\}^{\frac{1}{p}}, \tag{2.22}$$

for all $t \in \mathbb{R}_+$, where

$$\begin{aligned} A_1(t) = {} & k(t,\alpha(t))b(\alpha(t))\left[g(\alpha(t)) + \frac{h(\alpha(t))}{p}\right]\alpha'(t) \\ & + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)b(s)\left(g(s) + \frac{h(s)}{p}\right)ds, \end{aligned} \tag{2.23}$$

for all $t \in \mathbb{R}_+$.

$$\begin{aligned} B_1(t) = {} & k(t,\alpha(t))\left[g(\alpha(t))a(\alpha(t)) + h(\alpha(t))\left(\frac{p-1}{p} + \frac{a(\alpha(t))}{p}\right)\right]\alpha'(t) \\ & + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)\left(g(s)a(s) + h(s)\left(\frac{p-1}{p} + \frac{a(s)}{p}\right)\right)ds, \end{aligned} \tag{2.24}$$

for all $t \in \mathbb{R}_+$.

Proof. We take the function $z(t)$ by

$$z(t) = \int_0^{\alpha(t)} k(t,s)\left[g(s)u^p(s) + h(s)u(s)\right]ds, \forall t \in \mathbb{R}_+. \tag{2.25}$$

with $z(0) = 0$, that is $z(t) \geqslant 0$, and nondecreasing on the interval $\mathbb{R}_+$. As in the steps of proof Theorem 2.3 from (2.13), we observe that the inequalities (1.10) and (2.14) satisfy. By differentiating (2.25) and from the inequalities (1.10) and (2.14), we deduce

$$\begin{aligned} z'(t) = {} & k(t,\alpha(t))\left[g(\alpha(t))u^p(\alpha(t)) + h(\alpha(t))u(\alpha(t))\right]\alpha'(t) \\ & + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)\left[g(s)u^p(s) + h(s)u(s)\right]ds. \end{aligned} \tag{2.26}$$

For all $t \in \mathbb{R}_+$. From Theorem 1.1 and using (2.4) and (2.5) in (2.26)

$$\begin{aligned} z'(t) \leqslant k(t,\alpha(t)) \Big[g(\alpha(t)) \Big(a(\alpha(t)) + b(\alpha(t)) z(\alpha(t)) \Big) \\ + h(\alpha(t)) \Big(\frac{p-1}{p} + \frac{a(\alpha(t))}{p} + \frac{b(\alpha(t))}{p} z(\alpha(t)) \Big) \Big] \alpha'(t) \\ + \int_0^{\alpha(t)} \frac{\partial}{\partial t} k(t,s) \Big[g(s)(a(s)+b(s)z(s)) \\ + h(s) \Big(\frac{p-1}{p} + \frac{a(s)}{p} + \frac{b(s)}{p} z(s) \Big) \Big] ds, \end{aligned} \tag{2.27}$$

for all $t \in \mathbb{R}_+$. Using the fact that $z(\alpha) \leqslant z(t)$, from (2.27), we get

$$\begin{aligned} z'(t) \leqslant & \Big[k(t,\alpha(t)) \Big[b(\alpha(t))(g(\alpha(t)) + \frac{h(\alpha(t))}{p}) \Big] \alpha'(t) \\ & + \int_0^{\alpha(t)} \frac{\partial}{\partial t} k(t,s) b(s) \Big(g(s) + \frac{h(s)}{p} \Big) ds \Big] z(t) \\ & + k(t,\alpha(t)) \Big[g(\alpha(t)) a(\alpha(t)) + h(\alpha(t)) (\frac{p-1}{p} + \frac{a(\alpha(t))}{p}) \Big] \alpha'(t) \\ & + \int_0^{\alpha(t)} \frac{\partial}{\partial t} k(t,s) \Big[g(s)a(s) + h(s) (\frac{p-1}{p} + \frac{a(s)}{p}) \Big] ds, \\ = & A_1(t) z(t) + B_1(t), \forall t \in \mathbb{R}_+. \end{aligned} \tag{2.28}$$

The inequality (2.28) gives us the following bound for $z(t)$

$$z(t) \leqslant \exp \Big[\int_0^{\alpha(t)} A_1(r) dr \Big] \int_0^{\alpha(t)} B_1(s) \exp \Big[- \int_0^s A_1(r) dr \Big] ds. \tag{2.29}$$

For all $t \in \mathbb{R}_+$. Using (2.29) in $u^p(t) \leqslant a(t) + b(t) z(t)$, we get the required inequality in (2.22). The proof is completed.

REMARK 2.7. If $\alpha(t) = t$ and, then Theorem 2.5 reduces to [8, Theorem 1 part (a_3)].

THEOREM 2.6. *Let $u, a, b, g, h \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha(t) \in \mathscr{C}^1(\mathbb{R}_+, \mathbb{R}_+)$, with $\alpha(t) \leqslant t$, $\alpha(0) = 0$ and $c(t) \in \mathscr{C}(\mathbb{R}_+, I)$, $k(t,s) \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, I)$ with $\frac{\partial}{\partial t} k(t,s) \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, I)$, and $p > 1$ be a constant. If*

$$u^p(t) \leqslant c^p(t) + b(t) \int_0^{\alpha(t)} k(t,s) \Big[g(s) u^p(s) + h(s) u(s) \Big] ds, \tag{2.30}$$

for all $t \in \mathbb{R}_+$, then

$$u(t) \leqslant c(t) \Big\{ 1 + b(t) \exp \Big[\int_0^{\alpha(t)} A(r) dr \Big] \int_0^{\alpha(t)} B(s) \exp \Big[- \int_0^s A(r) dr \Big] ds \Big\}^{\frac{1}{p}}, \tag{2.31}$$

for all $t \in \mathbb{R}_+$, *where*

$$A(t) = k(t,\alpha(t))b(\alpha(t))\left(g(\alpha(t)) + \frac{h(\alpha(t))c^{1-p}(\alpha(t))}{p}\right)\alpha'(t) + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)(b(s)\left(g(s) + \frac{h(s)c^{1-p}(s)}{p}\right)ds,$$

for all $t \in \mathbb{R}_+$.

$$B(t) = k(t,\alpha(t))\left[g(\alpha(t)) + h(\alpha(t))c^{1-p}(\alpha(t))\right]\alpha'(t) + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)\left[g(s) + h(s)c^{1-p}(s)\right]ds,$$

for all $t \in \mathbb{R}_+$.

Proof. Since $c(t) > 0$ and nondecreasing on $\mathbb{R}_+$, from (2.30), we observe that

$$\left(\frac{u(t)}{c(t)}\right)^p \leqslant 1 + b(t)\int_0^{\alpha(t)} k(t,s)\left[g(s)\left(\frac{u(t)}{c(t)}\right)^p + h(s)c^{1-p}(s)\left(\frac{u(t)}{c(t)}\right)\right]ds,$$

for all $t \in \mathbb{R}_+$. Applying the inequality given in Theorem 2.5 implies the desired result in (2.31). The proof is completed.

THEOREM 2.7. *Let* $u,a,b \in \mathscr{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\alpha(t) \in \mathscr{C}^1(\mathbb{R}_+,\mathbb{R}_+)$, *with* $\alpha(0) = 0, \alpha(t) \leqslant t$ *and* $p > 1$ *be a constant and* $f \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+,\mathbb{R}_+)$, *such that*

$$0 \leqslant f(t,x) - f(t,y) \leqslant m(t,y)(x-y), \forall t \in \mathbb{R}_+, \tag{2.32}$$

and $x \geqslant 0$, $y \geqslant 0$, *where* $m \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+,\mathbb{R}_+)$. *If*

$$u^p(t) \leqslant a(t) + b(t)\int_0^{\alpha(t)} \left[f(s,u(s))\right]ds, \tag{2.33}$$

for all $t \in \mathbb{R}_+$, *then*

$$u(t) \leqslant \left\{a(t) + b(t)\exp\left[\int_0^{\alpha(t)} m(r,\frac{p-1}{p} + \frac{a(r)}{p})\frac{b(r)}{p}dr\right] \times \int_0^{\alpha(t)} \left[f(s,\frac{p-1}{p} + \frac{a(s)}{p})\right] \times \exp\left[-\int_0^s m(r,\frac{p-1}{p} + \frac{a(r)}{p})\frac{b(r)}{p}dr\right]ds\right\}^{\frac{1}{p}}, \tag{2.34}$$

for all $t \in \mathbb{R}_+$.

Proof. We take the function $z(t)$ by

$$z(t) = \int_0^{\alpha(t)} \Big(f(s,u(s)) \Big) ds, \forall t \in \mathbb{R}_+. \tag{2.35}$$

That $z(t) \geqslant 0$ nondecreasing on $\mathbb{R}_+$ with $z(0) = 0$. Then as in the proof of Theorem (2.1) from (2.33) we deduce the inequality (2.4) and (2.5) hold. from (2.35) and (2.5) and the hypothesis (2.32) it follows that

$$\begin{aligned}
z'(t) &= \left(f\left(\alpha(t), u(\alpha(t)) \right) \right) \alpha'(t) \\
&\leqslant \left[f\left(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} + \frac{b(\alpha(t))}{p} z(\alpha(t)) \right) \right] \alpha'(t) \\
&\leqslant f\left(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} + \frac{b(\alpha(t))}{p} z(t) \right) \alpha'(t) \\
&\quad - f\left(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} \right) \alpha'(t) + f\left(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} \right) \alpha'(t) \\
&\leqslant m\left(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} \right) \frac{b(\alpha(t))}{p} z(t) \alpha'(t) \\
&\quad + f\left(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} \right) \alpha'(t).
\end{aligned} \tag{2.36}$$

For all $t \in \mathbb{R}_+$. The inequality (2.36) gives us the following bound

$$\begin{aligned}
z(t) \leqslant \exp&\left[\int_0^{\alpha(t)} m\left(r, \frac{p-1}{p} + \frac{a(r)}{p} \right) \frac{b(r)}{p} dr \right] \\
&\times \int_0^{\alpha(t)} \left[f\left(s, \frac{p-1}{p} + \frac{a(s)}{p} \right) \right] \\
&\times \exp\left[- \int_0^s m\left(r, \frac{p-1}{p} + \frac{a(r)}{p} \right) \frac{b(r)}{p} dr \right] ds.
\end{aligned} \tag{2.37}$$

For all $t \in \mathbb{R}_+$. From (2.36) and (2.4) the desired inequality in (2.34) follows. The proof is completed.

REMARK 2.8. If $\alpha(t) = t$, then Theorem 2.7 reduces to [8, Theorem 2 part (b_1)].

THEOREM 2.8. *Let $u, a, b \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha(t) \in \mathscr{C}^1(\mathbb{R}_+, \mathbb{R}_+)$, with $\alpha(t) \leqslant t$, $\alpha(0) = 0$, and $p > 1$ be a constant and $f \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, with $\Phi \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}_+)$ be a strictly increasing function, $\phi(0) = 0$ such that*

$$0 \leqslant f(t,x) - f(t,y) \leqslant m(t,y)\phi^{-1}(x-y), \tag{2.38}$$

for all $t \in \mathbb{R}_+$, and $x \geqslant y \geqslant 0$, where $m \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and ϕ^{-1}, and

$$\phi^{-1}(xy) \leqslant \phi^{-1}(x)\phi^{-1}(y), \tag{2.39}$$

for all $x, y \in \mathbb{R}_+$, *where* ϕ^{-1} *is the inverse function of* ϕ. *If*

$$u^p(t) \leqslant a(t) + b(t)\phi \int_o^{\alpha(t)} \Big[f(s,u(s)) \Big] ds, \tag{2.40}$$

for all $t \in \mathbb{R}_+$, *then*

$$\begin{aligned} u(t) \leqslant \Bigg\{ a(t) + b(t)\phi \Bigg[\exp\Bigg[\int_0^{\alpha(t)} m(r, \frac{p-1}{p} + \frac{a(r)}{p})\phi^{-1}(\frac{b(r)}{p}) dr \Bigg] \\ \times \int_0^{\alpha(t)} \Bigg[f(s, \frac{p-1}{p} + \frac{a(s)}{p}) \Bigg] \\ \times \exp\Bigg[- \int_0^s m(r, \frac{p-1}{p} + \frac{a(r)}{p})\phi^{-1}(\frac{b(r)}{p}) dr \Bigg] ds \Bigg] \Bigg\}^{\frac{1}{p}}, \end{aligned} \tag{2.41}$$

for all $t \in \mathbb{R}_+$.

Proof. Repeats the steps as in the proof of theorem 2.1, then from the inequalities (2.4) and (2.5), we deduce

$$u^p(t) \leqslant a(t) + b(t)\phi(z(t)), \tag{2.42}$$

and

$$u(t) \leqslant \frac{p-1}{p} + \frac{a(t)}{p} + \frac{b(t)}{p}\phi(z(t)), \qquad \forall t \in \mathbb{R}_+. \tag{2.43}$$

From (2.35) , (2.43), it follows that

$$\begin{aligned} z'(t) &= \Bigg(f\Bigg(\alpha(t), u(\alpha(t)) \Bigg) \Bigg) \alpha'(t) \\ &\leqslant \Bigg[f\Bigg(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} + \frac{b(\alpha(t))}{p}\phi(z(\alpha(t))) \Bigg) \Bigg] \alpha'(t) \\ &\leqslant f\Bigg(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} + \frac{b(\alpha(t))}{p}\phi(z(t)) \Bigg) \alpha'(t) \\ &\quad - f\Bigg(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} \Bigg) \alpha'(t) + f\Bigg(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} \Bigg) \alpha'(t). \end{aligned} \tag{2.44}$$

For all $t \in \mathbb{R}_+$. Using the condition (2.38) in (2.44), we obtain

$$\begin{aligned} z'(t) \leqslant m\Bigg(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} \Bigg) \phi^{-1}\Bigg(\frac{b(\alpha(t))}{p}\phi(z(t)) \Bigg) \alpha'(t) \\ + f\Bigg(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p} \Bigg) \alpha'(t). \end{aligned} \tag{2.45}$$

For all $t \in \mathbb{R}_+$. By using the condition (2.39) in (2.45), we obtain

$$z'(t) \leqslant m\left(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p}\right)\phi^{-1}\left(\frac{b(\alpha(t))}{p}\right)z(t)\alpha'(t) \\ +f\left(\alpha(t), \frac{p-1}{p} + \frac{a(\alpha(t))}{p}\right)\alpha'(t). \tag{2.46}$$

For all $t \in \mathbb{R}_+$. The inequality (2.46) gives us the following bound

$$z(t) \leqslant \exp\left[\int_0^{\alpha(t)} m\left(r, \frac{p-1}{p} + \frac{a(r)}{p}\right)\phi^{-1}\left(\frac{b(r)}{p}\right)dr\right] \\ \times \int_0^{\alpha(t)} \left[f\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right)\right] \\ \times \exp\left[-\int_0^s m\left(r, \frac{p-1}{p} + \frac{a(r)}{p}\right)\phi^{-1}\left(\frac{b(r)}{p}\right)dr\right]ds. \tag{2.47}$$

For all $t \in \mathbb{R}_+$. We get the required inequality (2.41) from (2.42) and (2.47). The proof is completed.

REMARK 2.9. If $\alpha(t) = t$, then Theorem 2.8 reduces to [8, Theorem 2 part (b_2)].

3. Some Applications

In this section, we indicate some applications of our results to get the estimates of the solutions of certain retarded integral equations for which inequalities obtained in the literature thus far do not apply directly.

EXAMPLE 3.1. As an application of Theorem 2.3, we consider the nonlinear retarded integral equation

$$u^4(t) \leqslant t + \int_0^{\sqrt{t}} \left[s^2u^4(s) + su^2(s)\right]ds, \tag{3.1}$$

where u is defined as in Theorem 2.3 and we assume that every solution $u(t)$ of (3.1) exists on $\mathbb{R}_+$. Hence, by Theorem 2.3 and equation (3.1), we obtain

$$u(t) \leqslant \left\{t + \exp\left[\int_0^{\sqrt{t}}\left(r^2 + \frac{1}{2}r\right)dr\right] \\ \times \int_0^{\sqrt{t}}\left[s(s^2) + s\left(\frac{1}{2} + \frac{1}{2}(s)\right)\right] \\ \times \exp\left[-\int_0^s\left(r^2 + \frac{1}{2}r\right)dr\right]ds\right\}^{\frac{1}{4}}.$$

In Figure 1, we plot the graph of estimated bound of $u(t)$ for $0 \leqslant t \leqslant 1$.

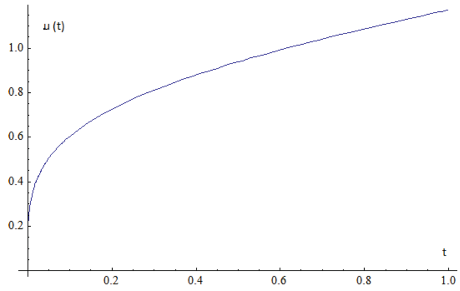

Figure 1: Graph of the estimated solution

EXAMPLE 3.2. Consider the following retarded integral equation:

$$u^p(t) = M\left(s, \int_0^t H(s, u^p(\alpha(s)), u(\alpha(s)))\right), \forall t \in \mathbb{R}_+, \tag{3.2}$$

where $M \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and $H \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, satisfy the following hypothesis:

$$|M(t,u)| \leqslant a(t) + b(t)|u| \qquad \forall t \in \mathbb{R}_+, \tag{3.3}$$

$$|H(t,u,w)| \leqslant g(t)|u| + w \qquad \forall t \in \mathbb{R}_+, \tag{3.4}$$

where u, a, b, g, α and p as defined in theorem (2.3). Using the conditions (3.3) and (3.4), from (3.2), we get

$$\begin{aligned} |u(t)|^p &\leqslant a(t) + b(t)\int_0^t \Big\{ g(s)u^p(\alpha(s)) + u(\alpha(s)) \Big\} ds \\ &\leqslant a(t) + b(t)\int_0^{\alpha(t)} \left\{ \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} u^p(s) + u(s) \right\} ds, \end{aligned}$$

for all $t \in \mathbb{R}_+$. Now a suitable application of Theorem 2.3 with $h(t) = 1$, yields

$$\begin{aligned} |u(t)| \leqslant \Bigg\{ & a(t) + \exp\left[\int_0^{\alpha(t)} b(r)\left(\frac{g(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + \frac{r}{p} \right) dr \right] \\ & \times \int_0^{\alpha(t)} \left[\frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} a(s) + \left(\frac{p-1}{p} + \frac{a(s)}{p} \right) \right] \\ & \times \exp\left[-\int_0^s b(r)\left(\frac{g(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + \frac{r}{p} \right) dr \right] ds \Bigg\}^{\frac{1}{p}}, \end{aligned} \tag{3.5}$$

for all $t \in \mathbb{R}_+$. Thus, the estimation in (3.5) implies the boundedness of the solution $u(t)$ of (3.2).

Conclusions

In this work, some new results as regards Gronwall-Bellman type inequalities, which provide explicit bounds on unknown functions, are included. The results can be useful in the study of the uniqueness of the solution for nonlinear retarded differential equations, integral equations or integro-differential equations. Some applications also presented to illustrate the benefits of our results.

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