

## COMMUTATORS OF MULTILINEAR CALDERÓN–ZYGmund OPERATORS WITH KERNELS OF DINI’S TYPE AND APPLICATIONS

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*Abstract.* Let  $T$  be a multilinear Calderón-Zygmund operator of type  $\omega$  with  $\omega(t)$  being non-decreasing and satisfying a kind of Dini’s type condition. Let  $T_{\vec{b}}$  be the iterated commutators of  $T$  with  $BMO$  functions. The weighted strong and weak  $L(\log L)$ -type endpoint estimates for  $T_{\vec{b}}$  with multiple weights are established. Some boundedness properties on weighted variable exponent Lebesgue spaces are also obtained. As applications, multiple weighted estimates for iterated commutators of bilinear pseudo-differential operators and paraproducts with mild regularity are given.

### 1. Introduction and Main Results

The study of multilinear Calderón-Zygmund theory goes back to the pioneering works of Coifman and Meyer in 1970s, see e.g. [1, 2]. This topic was then further investigated by many authors in the last few decades, see for example [8, 12, 13, 14, 15, 16, 17, 19, 20, 22, 23].

Let  $T$  be a multilinear Calderón-Zygmund operator with associated kernel satisfying the standard estimates as in [13] and [16]. For  $\vec{b} = (b_1, \dots, b_m) \in BMO^m$ , that is  $b_j \in BMO(\mathbb{R}^n)$  for  $j = 1, \dots, m$ , the  $m$ -linear commutator of  $T$  with  $\vec{b}$  is defined by

$$T_{\vec{b}}(f_1, \dots, f_m) = \sum_{j=1}^m T_{b_j}^j(\vec{f}),$$

where

$$T_{b_j}^j(\vec{f}) = b_j T(f_1, \dots, f_j, \dots, f_m) - T(f_1, \dots, b_j f_j, \dots, f_m). \quad (1.1)$$

The iterated commutators of  $T$  with  $\vec{b}$  is defined by

$$T_{\vec{b}}(\vec{f})(x) = [b_1, [b_2, \dots, [b_{m-1}, [b_m, T]_m]_{m-1} \dots]_1(\vec{f})(x). \quad (1.2)$$

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For an  $m$ -linear Calderón-Zygmund operator with associated kernel  $K(x, \vec{y})$ , the iterated commutator  $T_{\Pi\vec{b}}$  can also be given formally by

$$T_{\Pi\vec{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \left( \prod_{j=1}^m (b_j(x) - b_j(y_j)) \right) K(x, \vec{y}) f_1(y_1) \cdots f_m(y_m) d\vec{y}.$$

Here and in what follows,  $\vec{y} = (y_1, \dots, y_m)$ ,  $(x, \vec{y}) = (x, y_1, \dots, y_m)$  and  $d\vec{y} = dy_1 \cdots dy_m$ .

In 2009, Lerner et al. [16] developed a multiple weight theory that adapts to multilinear Calderón-Zygmund operators. They established multiple weighted norm inequalities for multilinear Calderón-Zygmund operators and their commutators  $T_{\Sigma\vec{b}}$ . Recently, Pérez et al. [22] studied the iterated commutators  $T_{\Pi\vec{b}}$  in products of Lebesgue spaces. Both strong type and weak type estimates with multiple weights are obtained.

The purpose of this paper is to consider weighted inequalities with multiple weights for iterated commutators of multilinear Calderón-Zygmund operators of type  $\omega$ . Some boundedness properties on weighted variable exponent Lebesgue spaces are also obtained. In addition, we will give some applications to the iterated commutators of bilinear pseudo-differential operators and paraproducts with mild regularity introduced by Maldonado and Naibo in [20].

We now recall the definition of multilinear Calderón-Zygmund operators of type  $\omega$ .

DEFINITION 1.1. Let  $\omega(t) : [0, \infty) \rightarrow [0, \infty)$  be a nondecreasing function. A locally integrable function  $K(x, y_1, \dots, y_m)$ , defined away from the diagonal  $x = y_1 = \dots = y_m$  in  $(\mathbb{R}^n)^{m+1}$ , is called an  $m$ -linear Calderón-Zygmund kernel of type  $\omega$  if, for some constants  $0 < \tau < 1$ , there exists a constant  $A > 0$  such that

$$|K(x, y_1, \dots, y_m)| \leq \frac{A}{(|x - y_1| + \dots + |x - y_m|)^{nm}}$$

for all  $(x, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$  with  $x \neq y_j$  for some  $j \in \{1, 2, \dots, m\}$ , and

$$\begin{aligned} & |K(x, y_1, \dots, y_m) - K(x', y_1, \dots, y_m)| \\ & \leq \frac{A}{(|x - y_1| + \dots + |x - y_m|)^{nm}} \omega \left( \frac{|x - x'|}{|x - y_1| + \dots + |x - y_m|} \right) \end{aligned}$$

whenever  $|x - x'| \leq \tau \max_{1 \leq j \leq m} |x - y_j|$ , and

$$\begin{aligned} & |K(x, y_1, \dots, y_j, \dots, y_m) - K(x, y_1, \dots, y'_j, \dots, y_m)| \\ & \leq \frac{A}{(|x - y_1| + \dots + |x - y_m|)^{nm}} \omega \left( \frac{|y_j - y'_j|}{|x - y_1| + \dots + |x - y_m|} \right) \end{aligned}$$

whenever  $|y_j - y'_j| \leq \tau \max_{1 \leq i \leq m} |x - y_i|$ .

We say  $T : \mathcal{S}(\mathbb{R}^n) \times \dots \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$  is an  $m$ -linear operator with an  $m$ -linear Calderón-Zygmund kernel of type  $\omega$ ,  $K(x, y_1, \dots, y_m)$ , if

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m$$

whenever  $x \notin \bigcap_{j=1}^m \text{supp } f_j$  and each  $f_j \in C_c^\infty(\mathbb{R}^n)$ ,  $j = 1, \dots, m$ .

If  $T$  can be extended to a bounded multilinear operator from  $L^{q_1}(\mathbb{R}^n) \times \dots \times L^{q_m}(\mathbb{R}^n)$  to  $L^{q,\infty}(\mathbb{R}^n)$  for some  $1 \leq q_1, \dots, q_m < \infty$  and  $1/q = 1/q_1 + \dots + 1/q_m$ , then  $T$  is called an  $m$ -linear Calderón-Zygmund operator of type  $\omega$ , abbreviated to  $m$ -linear  $\omega$ -CZO.

When  $\omega(t) = t^\varepsilon$  for some  $\varepsilon > 0$ , the  $m$ -linear  $\omega$ -CZO is exactly the multilinear Calderón-Zygmund operator studied in [13] and [16]. The linear  $\omega$ -CZO was studied by Yabuta [26].

DEFINITION 1.2. Let  $\omega(t) : [0, \infty) \rightarrow [0, \infty)$  be a nondecreasing function. For  $a > 0$ , we say that  $\omega$  satisfies the *Dini*( $a$ ) condition and write  $\omega \in \text{Dini}(a)$ , if

$$|\omega|_{\text{Dini}(a)} := \int_0^1 \frac{\omega^a(t)}{t} dt < \infty.$$

We would like to note that Maldonado and Naibo [20] studied the bilinear  $\omega$ -CZOs when  $\omega$  is a nondecreasing, concave function and belongs to *Dini*( $1/2$ ). Recently, Lu and Zhang [19] improve and extend their results by removing the hypothesis that  $\omega$  is concave and reducing the condition  $\omega \in \text{Dini}(1/2)$  to a weaker condition  $\omega \in \text{Dini}(1)$ .

THEOREM 1.1. ([19]) *Let  $\omega \in \text{Dini}(1)$  and  $T$  be an  $m$ -linear operator with an  $m$ -linear Calderón-Zygmund kernel of type  $\omega$ . Suppose that for some  $1 \leq q_1, \dots, q_m \leq \infty$  and some  $0 < q < \infty$  with  $1/q = 1/q_1 + \dots + 1/q_m$ ,  $T$  maps  $L^{q_1}(\mathbb{R}^n) \times \dots \times L^{q_m}(\mathbb{R}^n)$  into  $L^{q,\infty}(\mathbb{R}^n)$ . Then  $T$  can be extended to a bounded operator from  $L^1(\mathbb{R}^n) \times \dots \times L^1(\mathbb{R}^n)$  to  $L^{1/m,\infty}(\mathbb{R}^n)$ .*

Denote by  $A_{\vec{p}}$  the multiple weight classes introduced by Lerner et al. [16] (see Definition 2.2 below). The following result holds.

THEOREM 1.2. ([19]) *Let  $T$  be an  $m$ -linear  $\omega$ -CZO and  $\omega \in \text{Dini}(1)$ . Let  $\vec{P} = (p_1, \dots, p_m)$  with  $1/p = 1/p_1 + \dots + 1/p_m$  and  $\vec{w} \in A_{\vec{p}}$ .*

(1) *If  $1 < p_j < \infty$  for all  $j = 1, \dots, m$ , then*

$$\|T(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

(2) *If  $1 \leq p_j < \infty$  for all  $j = 1, \dots, m$ , and at least one of the  $p_j = 1$ , then*

$$\|T(\vec{f})\|_{L^{p,\infty}(v_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Our first result is the following multiple weighted strong-type estimates for the iterated commutator of  $m$ -linear  $\omega$ -CZO with *BMO* functions.

**THEOREM 1.3.** *Let  $T$  be an  $m$ -linear  $\omega$ -CZO and  $\vec{b} \in BMO^m$ . Suppose that  $\vec{w} \in A_{\vec{p}}$  with  $1/p = 1/p_1 + \dots + 1/p_m$  and  $1 < p_j < \infty$ ,  $j = 1, \dots, m$ . If  $\omega$  satisfies*

$$\int_0^1 \frac{\omega(t)}{t} \left(1 + \log \frac{1}{t}\right)^m dt < \infty, \tag{1.3}$$

*then there exists a constant  $C > 0$  such that*

$$\|T_{\Pi\vec{b}}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \left(\prod_{j=1}^m \|b_j\|_{BMO}\right) \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

**REMARK 1.1.** Since the commutator has more singularity, the more regular conditions imposed on the kernel is reasonable. In addition, although (1.3) is stronger than the *Dini*(1) condition but it is much weaker than the standard kernel  $\omega(t) = t^\varepsilon$ .

It is easy to check that if  $\omega$  satisfies (1.3), then  $\omega \in Dini(1)$  and

$$\sum_{k=1}^\infty k^m \cdot \omega(2^{-k}) \approx \int_0^1 \frac{\omega(t)}{t} \left(1 + \log \frac{1}{t}\right)^m dt < \infty. \tag{1.4}$$

For the multiple weighted weak-type estimate, we have the following result.

**THEOREM 1.4.** *Let  $T$  be an  $m$ -linear  $\omega$ -CZO and  $\vec{b} \in BMO^m$ . If  $\vec{w} \in A_{(1, \dots, 1)}$  and  $\omega$  satisfies (1.3), then there is a constant  $C > 0$  depending on  $\|b_j\|_{BMO}$ ,  $j = 1, \dots, m$ , such that for any  $\lambda > 0$ ,*

$$v_{\vec{w}}(\{x \in \mathbb{R}^n : |T_{\Pi\vec{b}}(\vec{f})(x)| > \lambda^m\}) \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(x)|}{\lambda}\right) w_j(x) dx\right)^{1/m},$$

*here and in the sequel,  $\Phi(t) = t(1 + \log^+ t)$  and  $\Phi^{(m)} = \underbrace{\Phi \circ \dots \circ \Phi}_m$ .*

**REMARK 1.2.** Pérez et al. [22] proved the same results as Theorems 1.3 and 1.4 when  $\omega(t) = t^\varepsilon$  for some  $\varepsilon > 0$ . We also note that similar results for  $T_{\Sigma\vec{b}}$  were proved in [19]. For commutators of the linear Calderón-Zygmund operator of type  $\omega$ , see [18] and [27].

Next, we study the boundedness of iterated commutators on weighted variable exponent Lebesgue spaces. We now recall some definition and notation.

A measurable function  $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$  is called a variable exponent. For any variable exponent  $p(\cdot)$ , let

$$p^- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p^+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

Denote by  $\mathcal{P}_0$  the set of all variable exponents  $p(\cdot)$  with  $0 < p^- \leq p^+ < \infty$  and by  $\mathcal{P}$  the collection of all variable exponents  $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$  with  $1 < p^- \leq p^+ < \infty$ .

Given  $p(\cdot) \in \mathcal{P}_0$ , the variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f \text{ measurable} : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some constant } \lambda > 0 \right\}.$$

The set  $L^{p(\cdot)}(\mathbb{R}^n)$  is a quasi-Banach space (Banach space if  $p(\cdot) \in \mathcal{P}$ ) with the quasi-norm (norm if  $p(\cdot) \in \mathcal{P}$ ) given by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

For  $p(\cdot) \in \mathcal{P}$ , we define the conjugate exponent  $p'(\cdot)$  by  $1/p(\cdot) + 1/p'(\cdot) = 1$  with  $1/\infty = 0$ .

DEFINITION 1.3. ([4]) A locally integrable function  $v$  with  $0 < v < \infty$  almost everywhere is called a weight. Given a variable exponent  $p(\cdot) \in \mathcal{P}$ , we say that  $v \in A_{p(\cdot)}$  if

$$[v]_{A(\cdot)} = \sup_B |B|^{-1} \|v\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|v^{-1}\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} < \infty,$$

where the supremum is taken over all balls  $B \subset \mathbb{R}^n$ .

The weighted variable exponent Lebesgue space  $L_v^{p(\cdot)}(\mathbb{R}^n)$  is defined to be the set of all measurable functions  $f$  such that  $f v \in L^{p(\cdot)}(\mathbb{R}^n)$ , and we define  $\|f\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} = \|f v\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ .

We say a variable exponent  $p(\cdot)$  satisfies the globally log-Hölder continuous condition, if there exist positive constants  $C_0, C_\infty$  and  $p_\infty$  such that

$$|p(x) - p(y)| \leq \frac{C_0}{-\log(|x - y|)}, \quad x, y \in \mathbb{R}^n, |x - y| \leq 1/2 \tag{1.5}$$

and

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}, \quad x \in \mathbb{R}^n. \tag{1.6}$$

For iterated commutator of  $m$ -linear  $\omega$ -CZO, we have the following result.

THEOREM 1.5. Let  $T$  be an  $m$ -linear  $\omega$ -CZO with  $\omega$  satisfying (1.3) and  $\vec{b} \in BMO^m$ . Suppose that  $p(\cdot) \in \mathcal{P}_0$  and  $p_1(\cdot), \dots, p_m(\cdot) \in \mathcal{P}$  so that  $1/p(\cdot) = 1/p_1(\cdot) + \dots + 1/p_m(\cdot)$ . If  $p(\cdot), p_1(\cdot), \dots, p_m(\cdot)$  satisfy (1.5) and (1.6),  $v_j \in A_{p_j(\cdot)}$ ,  $j = 1, \dots, m$ , and  $v = \prod_{j=1}^m v_j$ , then  $T_{\Pi\vec{b}}$  can be extended to a bounded operator from  $L_{v_1}^{p_1(\cdot)}(\mathbb{R}^n) \times \dots \times L_{v_m}^{p_m(\cdot)}(\mathbb{R}^n)$  to  $L_v^{p(\cdot)}(\mathbb{R}^n)$ , that is, there exists a positive constant  $C$  such that

$$\|T_{\Pi\vec{b}}(\vec{f})\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} \leq C \prod_{j=1}^m \|f_j\|_{L_{v_j}^{p_j(\cdot)}(\mathbb{R}^n)}.$$

This paper is arranged as follows. In Section 2, we recall some basic definitions and known results. Section 3 is devoted to proving Theorems 1.3 and 1.4. We give the proof of Theorem 1.5 in Section 4. In the last section, we apply our results to bilinear pseudo-differential operators and paraproducts with mild regularity.

### 2. Definitions and Preliminaries

As usual, for a cube  $Q$  and a locally integrable function  $f$ , we denote by  $f_Q = (f)_Q = \frac{1}{|Q|} \int_Q f(y) dy$ . The Hardy-Littlewood maximal function of  $f$  is defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

#### 2.1. Orlicz norms

For  $\Phi = t(1 + \log^+ t)$ , the  $\Phi$ -average of a function  $f$  on a cube  $Q$  is defined by

$$\|f\|_{L(\log L), Q} \equiv \|f\|_{\Phi, Q} := \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

The maximal function associated to  $\Phi(t) = t(1 + \log^+ t)$  is defined by

$$M_{L(\log L)}(f)(x) = \sup_{Q \ni x} \|f\|_{L(\log L), Q},$$

where the supremum is taken over all the cubes containing  $x$ .

It is not difficult to check the following pointwise equivalence (see (21) in [21])

$$M_{L(\log L)}(f)(x) \approx M^2 f(x), \text{ where } M^2 = M \circ M.$$

Let  $b \in BMO(\mathbb{R}^n)$ , by the generalized Hölder’s inequality in Orlicz spaces (see [24, page 58]) and John-Nirenberg’s inequality, we have (see also [16, (2.14)])

$$\frac{1}{|Q|} \int_Q |b(x) - b_Q| |f(x)| dx \leq C \|b\|_{BMO} \|f\|_{L(\log L), Q}. \tag{2.1}$$

Let  $t > 1$  and  $Q$  be a cube in  $\mathbb{R}^n$ . Denote by  $tQ$  the cube that is concentric with  $Q$  and whose side length is  $t$  times the side length of  $Q$ . Then, there exists a dimensional constant  $C_n$  such that for any  $b \in BMO(\mathbb{R}^n)$ , we have (see [11, page 166])

$$|b_Q - b_{tQ}| \leq C_n \log(t + 1) \|b\|_{BMO}. \tag{2.2}$$

#### 2.2. Sharp maximal function and $A_p$ weights

The sharp maximal function of Fefferman and Stein [9] is defined by

$$M^\sharp(f)(x) = \sup_{Q \ni x} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c| dy \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

For  $0 < \delta < \infty$ , we define the maximal functions  $M_\delta$  and  $M_\delta^\sharp$  by

$$M_\delta(f) = [M(|f|^\delta)]^{1/\delta} \quad \text{and} \quad M_\delta^\sharp(f) = [M^\sharp(|f|^\delta)]^{1/\delta}.$$

Let  $w$  be a nonnegative locally integrable function and  $1 < p < \infty$ . We say  $w \in A_p$  if there is a constant  $C > 0$  such that for any cube  $Q$ ,

$$\left(\frac{1}{|Q|} \int_Q w(x) dx\right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx\right)^{p-1} \leq C.$$

We say  $w \in A_1$  if there is a constant  $C > 0$  such that  $Mw(x) \leq Cw(x)$  almost everywhere. And we define  $A_\infty = \bigcup_{p \geq 1} A_p$ . See [10] or [25] for more information.

The following relationships between  $M_\delta^\sharp$  and  $M_\delta$  to be used is a version of the classical ones due to Fefferman and Stein [9], see also [16, page 1228].

LEMMA 2.1. (1) Let  $0 < p, \delta < \infty$  and  $w \in A_\infty$ . Then there exists a constant  $C > 0$  (depending on the  $A_\infty$  constant of  $w$ ) such that

$$\int_{\mathbb{R}^n} [M_\delta(f)(x)]^p w(x) dx \leq C \int_{\mathbb{R}^n} [M_\delta^\sharp(f)(x)]^p w(x) dx,$$

for every function  $f$  such that the left-hand side is finite.

(2) Let  $0 < \delta < \infty$  and  $w \in A_\infty$ . If  $\varphi : (0, \infty) \rightarrow (0, \infty)$  is doubling, then there exists a constant  $C > 0$  (depending on the  $A_\infty$  constant of  $w$  and the doubling condition of  $\varphi$ ) such that

$$\sup_{\lambda > 0} \varphi(\lambda) w(\{y \in \mathbb{R}^n : M_\delta(f)(y) > \lambda\}) \leq C \sup_{\lambda > 0} \varphi(\lambda) w(\{y \in \mathbb{R}^n : M_\delta^\sharp(f)(y) > \lambda\}),$$

for every function  $f$  such that the left-hand side is finite.

### 2.3. Multilinear maximal functions and multiple weights

The following multilinear maximal functions were introduced by Lerner et al. [16].

DEFINITION 2.1. ([16]) For all  $m$ -tuples  $\vec{f} = (f_1, \dots, f_m)$  of locally integrable functions and  $x \in \mathbb{R}^n$ , the multilinear maximal functions  $\mathcal{M}$  and  $\mathcal{M}_r$  are defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^m \frac{1}{|Q|} \int_Q |f_j(y_j)| dy_j,$$

and

$$\mathcal{M}_r(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^m \left(\frac{1}{|Q|} \int_Q |f_j(y_j)|^r dy_j\right)^{1/r}, \text{ for } r > 1.$$

The maximal functions related to function  $\Phi(t) = t(1 + \log^+ t)$  are defined by

$$\mathcal{M}_{L(\log L)}^i(\vec{f})(x) = \sup_{Q \ni x} \|f_i\|_{L(\log L), Q} \prod_{j=1(j \neq i)}^m \frac{1}{|Q|} \int_Q |f_j(y_j)| dy_j$$

and

$$\mathcal{M}_{L(\log L)}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^m \|f_j\|_{L(\log L), Q},$$

where the supremum is taken over all the cubes  $Q$  containing  $x$ .

Obviously, if  $r > 1$  then the following pointwise estimates hold

$$\mathcal{M}(\vec{f})(x) \leq C \mathcal{M}_{L(\log L)}^i(\vec{f})(x) \leq C' \mathcal{M}_{L(\log L)}(\vec{f})(x) \leq C'' \mathcal{M}_r(\vec{f})(x). \tag{2.3}$$

The following multiple  $A_{\vec{p}}$  conditions were introduced by Lerner et al. [16].

DEFINITION 2.2. ([16]) Let  $\vec{P} = (p_1, \dots, p_m)$  and  $1/p = 1/p_1 + \dots + 1/p_m$  with  $1 \leq p_1, \dots, p_m < \infty$ . Given  $\vec{w} = (w_1, \dots, w_m)$  with each  $w_j$  being nonnegative measurable, set

$$v_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}.$$

We say that  $\vec{w}$  satisfies the  $A_{\vec{p}}$  condition and write  $\vec{w} \in A_{\vec{p}}$ , if

$$\sup_Q \left( \frac{1}{|Q|} \int_Q v_{\vec{w}}(x) dx \right)^{1/p} \prod_{j=1}^m \left( \frac{1}{|Q|} \int_Q w_j(x)^{1-p'_j} dx \right)^{1/p'_j} < \infty,$$

where the supremum is taken over all cubes  $Q \subset \mathbb{R}^n$ , and the term  $(\frac{1}{|Q|} \int_Q w_j(x)^{1-p'_j} dx)^{1/p'_j}$  is understood as  $(\inf_Q w_j)^{-1}$  when  $p_j = 1$ .

The following characterization of the multiple weight classes  $A_{\vec{p}}$  was established in [16].

LEMMA 2.2. ([16]) Let  $\vec{w} = (w_1, \dots, w_m)$ ,  $\vec{P} = (p_1, \dots, p_m)$ ,  $1 \leq p_1, \dots, p_m < \infty$  and  $1/p = 1/p_1 + \dots + 1/p_m$ . Then  $\vec{w} \in A_{\vec{p}}$  if and only if

$$\begin{cases} w_j^{1-p'_j} \in A_{mp'_j}, & j = 1, \dots, m, \\ v_{\vec{w}} \in A_{mp}, \end{cases}$$

where the condition  $w_j^{1-p'_j} \in A_{mp'_j}$  in the case  $p_j = 1$  is understood as  $w_j^{1/m} \in A_1$ .

The following boundedness of  $\mathcal{M}_r$  is contained in the proof of [16, Theorem 3.18].

LEMMA 2.3. Let  $\vec{P} = (p_1, \dots, p_m)$ ,  $1 < p_1, \dots, p_m < \infty$  and  $1/p = 1/p_1 + \dots + 1/p_m$ . If  $\vec{w} \in A_{\vec{p}}$ , then there exists a constant  $r > 1$  such that  $\mathcal{M}_r$  is bounded from  $L^{p_1}(w_1) \times \dots \times L^{p_m}(w_m)$  to  $L^p(v_{\vec{w}})$ .

The maximal function  $\mathcal{M}_{L(\log L)}$  satisfies the following weak-type estimates.

LEMMA 2.4. ([22]) Let  $\vec{w} \in A_{(1, \dots, 1)}$ . Then there exists a constant  $C > 0$  such that

$$v_{\vec{w}}(\{x \in \mathbb{R}^n : \mathcal{M}_{L(\log L)}(\vec{f})(x) > t^m\}) \leq C \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \Phi^{(m)} \left( \frac{|f_j(x)|}{t} \right) w_j(x) dx \right)^{1/m}.$$



### 3. Proof of Theorems 1.3 and 1.4

Before proving the results, we first remark that the specific value of  $\tau \in (0, 1)$  in Definition 1.1 is immaterial. From now on, we may assume the constant  $\tau$  appearing in Definition 1.1 equals to  $1/2$  for simplicity. The argument with trivial modifications is also applicable to any specific value of  $\tau \in (0, 1)$ .

We now introduce some notation for convenience as in [22]. For positive integers  $m$  and  $j$  with  $1 \leq j \leq m$ , we denote by  $C_j^m$  the family of all finite subsets  $\sigma = \{\sigma(1), \dots, \sigma(j)\}$  of  $\{1, \dots, m\}$  of  $j$  different elements, where we always take  $\sigma(k) < \sigma(j)$  if  $k < j$ . For any  $\sigma \in C_j^m$ , we write  $\sigma' = \{1, \dots, m\} \setminus \sigma$  and  $C_0^m = \emptyset$ .

Given an  $m$ -tuple  $\vec{b} = (b_1, \dots, b_m) \in BMO^m$  and  $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$  with  $\sigma' = \{\sigma'(1), \dots, \sigma'(m-j)\}$ , we denote by  $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$  and  $\vec{b}_{\sigma'} = (b_{\sigma'(1)}, \dots, b_{\sigma'(m-j)})$ .

For  $\sigma \in C_j^m$  and  $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)}) \in BMO^j$ , similar to (1.2), we can define the following iterated commutator

$$T_{\Pi\vec{b}_\sigma}(\vec{f})(x) = [b_{\sigma(1)}, [b_{\sigma(2)}, \dots, [b_{\sigma(j-1)}, [b_{\sigma(j)}, T]_{\sigma(j)}]_{\sigma(j-1)} \dots]_{\sigma(2)}, b_{\sigma(1)}](\vec{f})(x).$$

This is, formally

$$T_{\Pi\vec{b}_\sigma}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \left( \prod_{i=1}^j (b_{\sigma(i)}(x) - b_{\sigma(i)}(y_{\sigma(i)})) \right) K(x, \vec{y}) f_1(y_1) \dots f_m(y_m) d\vec{y}.$$

Obviously,  $T_{\Pi\vec{b}_\sigma} = T_{\Pi\vec{b}}$  if  $\sigma = \{1, \dots, m\}$  and  $T_{\Pi\vec{b}_\sigma} = T_{b_j}^j$  if  $\sigma = \{j\}$ .

To prove Theorems 1.3 and 1.4, we first give the following pointwise estimates.

**PROPOSITION 3.1.** *Let  $T$  be an  $m$ -linear  $\omega$ -CZO with  $\omega$  satisfying (1.3) and  $\vec{b} \in BMO^m$ . Let  $0 < \delta < \varepsilon$  and  $0 < \delta < 1/m$ . Then, there exists a positive constant  $C$ , depending on  $\delta$  and  $\varepsilon$ , such that*

$$M_\delta^\#(T_{\Pi\vec{b}}(\vec{f}))(x) \leq C \left( \prod_{j=1}^m \|b_j\|_{BMO} \right) \left\{ \mathcal{M}_{L(\log L)}(\vec{f})(x) + M_\varepsilon(T(\vec{f}))(x) \right\} + C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left( \prod_{i=1}^j \|b_{\sigma(i)}\|_{BMO} \right) M_\varepsilon(T_{\Pi\vec{b}_{\sigma'}}(\vec{f}))(x), \tag{3.1}$$

for all bounded measurable functions  $f_1, \dots, f_m$  with compact support.

*Proof.* Some ideas of the proof are taken from [22]. For the sake of clarity and simplicity, we prove only the case  $m = 2$ . Fix  $b_1, b_2 \in BMO(\mathbb{R}^n)$ , for any constants  $\lambda_1$  and  $\lambda_2$ , as in the proof of Theorem 3.1 in [22], we have

$$T_{\Pi\vec{b}}(f_1, f_2) = -(b_1 - \lambda_1)(b_2 - \lambda_2)T(f_1, f_2) + (b_2 - \lambda_2)T_{b_1 - \lambda_1}^1(f_1, f_2) + (b_1(x) - \lambda_1)T_{b_2 - \lambda_2}^2(f_1, f_2) + T((b_1 - \lambda_1)f_1, (b_2 - \lambda_2)f_2).$$

Now we fix  $x \in \mathbb{R}^n$ . For any cube  $Q$  centered at  $x$ , set  $Q^* = 8\sqrt{n}Q$  and let  $\lambda_j = (b_j)_{Q^*}$  be the average of  $b_j$  on  $Q^*$ ,  $j = 1, 2$ . Since  $0 < \delta < 1/2$ , then, for any real number  $c$ ,

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q \left| |T_{\Pi\bar{b}}(f_1, f_2)(z)|^\delta - |c|^\delta \right| dz \right)^{1/\delta} &\leq \left(\frac{1}{|Q|} \int_Q |T_{\Pi\bar{b}}(f_1, f_2)(z) - c|^\delta dz \right)^{1/\delta} \\ &\leq C(I_1 + I_2 + I_3 + I_4), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} I_1 &= \left(\frac{1}{|Q|} \int_Q |(b_1(z) - \lambda_1)(b_2(z) - \lambda_2)T(f_1, f_2)(z)|^\delta dz \right)^{1/\delta}, \\ I_2 &= \left(\frac{1}{|Q|} \int_Q |(b_2(z) - \lambda_2)T_{b_1 - \lambda_1}^1(f_1, f_2)(z)|^\delta dz \right)^{1/\delta}, \\ I_3 &= \left(\frac{1}{|Q|} \int_Q |(b_1(z) - \lambda_1)T_{b_2 - \lambda_2}^2(f_1, f_2)(z)|^\delta dz \right)^{1/\delta}, \\ I_4 &= \left(\frac{1}{|Q|} \int_Q |T((b_1 - \lambda_1)f_1, (b_2 - \lambda_2)f_2)(z) - c|^\delta dz \right)^{1/\delta}. \end{aligned}$$

Following the same arguments as the ones to estimate *I, II* and *III* in [22], we obtain

$$\begin{aligned} I_1 &\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}M_\varepsilon(T(f_1, f_2))(x), \\ I_2 &\leq C\|b_2\|_{BMO}M_\varepsilon(T_{b_1}^1(f_1, f_2))(x) \end{aligned}$$

and

$$I_3 \leq C\|b_1\|_{BMO}M_\varepsilon(T_{b_2}^2(f_1, f_2))(x).$$

Now, we are in the position to consider the last term  $I_4$ . For each  $j = 1, 2$ , we decompose  $f_j$  as  $f_j = f_j^0 + f_j^\infty$ , where  $f_j^0 = f_j\chi_{Q^*}$  and  $f_j^\infty = f_j - f_j^0$ . Let  $c = \sum_{j=1}^2 c_j$ , where

$$\begin{aligned} c_1 &= T((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(x), \\ c_2 &= T((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^0)(x), \\ c_3 &= T((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty)(x). \end{aligned}$$

Then,

$$\begin{aligned} I_4 &\leq C \left(\frac{1}{|Q|} \int_Q |T((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^0)(z)|^\delta dz \right)^{1/\delta} \\ &\quad + C \left(\frac{1}{|Q|} \int_Q |T((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(z) - c_1|^\delta dz \right)^{1/\delta} \\ &\quad + C \left(\frac{1}{|Q|} \int_Q |T((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^0)(z) - c_2|^\delta dz \right)^{1/\delta} \end{aligned}$$

$$\begin{aligned}
 &+ C \left( \frac{1}{|Q|} \int_Q |T((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty)(z) - c_3|^\delta dz \right)^{1/\delta} \\
 &:= I_{4,1} + I_{4,2} + I_{4,3} + I_{4,4}.
 \end{aligned}$$

We first estimate  $I_{4,1}$ . Applying Kolmogorov’s inequality (see [10, page 485] or [16, (2.16)]) with  $p = \delta < 1/2$  and  $q = 1/2$ , Theorem 1.1 and (2.1), we have

$$\begin{aligned}
 I_{4,1} &= C|Q|^{-1/\delta} \|T((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^0)\|_{L^\delta(Q)} \\
 &\leq C|Q|^{-2} \|T((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^0)\|_{L^{1/2,\infty}(Q)} \\
 &\leq C|Q|^{-2} \|(b_1 - \lambda_1)f_1^0\|_{L^1(\mathbb{R}^n)} \|(b_2 - \lambda_2)f_2^0\|_{L^1(\mathbb{R}^n)} \\
 &\leq C \frac{1}{|Q|} \int_{Q^*} |b_1(z) - (b_1)_{Q^*}| |f_1^0(z)| dz \frac{1}{|Q|} \int_{Q^*} |b_2(z) - (b_2)_{Q^*}| |f_2^0(z)| dz \\
 &\leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L(\log L), Q^*} \|f_2\|_{L(\log L), Q^*} \\
 &\leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \mathcal{M}_{L(\log L)}(f_1, f_2)(x).
 \end{aligned}$$

Next, we consider the term  $I_{4,2}$ . For any  $z \in Q$ , we have

$$\begin{aligned}
 &|T((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(z) - c_1| \\
 &= |T((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(z) - T((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(x)| \\
 &\leq \int_{(\mathbb{R}^n)^2} |K(z, y_1, y_2) - K(x, y_1, y_2)| |(b_1(y_1) - \lambda_1)f_1^0(y_1)| |(b_2(y_2) - \lambda_2)f_2^\infty(y_2)| dy_1 dy_2 \\
 &\leq \int_{Q^*} |(b_1(y_1) - \lambda_1)f_1^0(y_1)| \left( \int_{\mathbb{R}^n \setminus Q^*} |K(z, y_1, y_2) \right. \\
 &\quad \left. - K(x, y_1, y_2)| |(b_2(y_2) - \lambda_2)f_2^\infty(y_2)| dy_2 \right) dy_1
 \end{aligned}$$

Note the following fact that, for any  $z \in Q$ ,  $y_1 \in Q^*$  and  $y_2 \in \mathcal{Q}_k := 2^{k+3}\sqrt{n}Q \setminus 2^{k+2}\sqrt{n}Q$ ,

$$\begin{aligned}
 |K(z, y_1, y_2) - K(x, y_1, y_2)| &\leq \frac{C}{(|x - y_1| + |x - y_2|)^{2n}} \omega\left(\frac{|z - x|}{|x - y_1| + |x - y_2|}\right) \\
 &\leq C \frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^2},
 \end{aligned} \tag{3.3}$$

we get

$$\begin{aligned}
 &|T((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(z) - c_1| \\
 &\leq C \int_{Q^*} |(b_1(y_1) - \lambda_1)f_1^0(y_1)| \left( \sum_{k=1}^\infty \int_{\mathcal{Q}_k} \frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^2} |(b_2(y_2) - \lambda_2)f_2^\infty(y_2)| dy_2 \right) dy_1 \\
 &\leq C \int_{Q^*} |(b_1(y_1) - \lambda_1)f_1^0(y_1)| \left( \sum_{k=1}^\infty \frac{\omega(2^{-k})}{|2^k Q^*|^2} \int_{2^k Q^*} |(b_2(y_2) - \lambda_2)f_2^\infty(y_2)| dy_2 \right) dy_1
 \end{aligned}$$

$$\begin{aligned} &\leq C \sum_{k=1}^{\infty} \omega(2^{-k}) \frac{1}{|2^k Q^*|} \int_{2^k Q^*} |(b_1(y_1) - \lambda_1) f_1(y_1)| dy_1 \times \\ &\quad \times \frac{1}{|2^k Q^*|} \int_{2^k Q^*} |(b_2(y_2) - \lambda_2) f_2(y_2)| dy_2. \end{aligned}$$

By (2.1) and (2.2), we have

$$\begin{aligned} &\frac{1}{|2^k Q^*|} \int_{2^k Q^*} |(b_j(y_j) - \lambda_j) f_j(y_j)| dy_j \\ &= \frac{1}{|2^k Q^*|} \int_{2^k Q^*} |(b_j(y_j) - (b_j)_{Q^*}) f_j(y_j)| dy_j \\ &\leq \frac{1}{|2^k Q^*|} \int_{2^k Q^*} |(b_j(y_j) - (b_j)_{2^k Q^*}) f_j(y_j)| dy_j \\ &\quad + \frac{|(b_j)_{2^k Q^*} - (b_j)_{Q^*}|}{|2^k Q^*|} \int_{2^k Q^*} |f_j(y_j)| dy_j \\ &\leq C \|b_j\|_{BMO} \|f_j\|_{L(\log L), 2^k Q^*} + Ck \|b_j\|_{BMO} \|f_j\|_{L(\log L), 2^k Q^*} \\ &\leq Ck \|b_j\|_{BMO} \|f_j\|_{L(\log L), 2^k Q^*}. \end{aligned} \tag{3.4}$$

Then by (1.4),

$$\begin{aligned} I_{4,2} &\leq \frac{C}{|Q|} \int_Q |T((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^\infty)(z) - c_1| dz \\ &\leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \sum_{k=1}^{\infty} k^2 \omega(2^{-k}) \|f_1\|_{L(\log L), 2^k Q^*} \|f_2\|_{L(\log L), 2^k Q^*} \\ &\leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \mathcal{M}_{L(\log L)}(f_1, f_2)(x). \end{aligned}$$

Similarly to  $I_{4,2}$ , we can estimate

$$I_{4,3} \leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \mathcal{M}_{L(\log L)}(f_1, f_2)(x).$$

Finally, we consider the term  $I_{4,4}$ . For any  $(y_1, y_2) \in (2^{k+3} \sqrt{n} Q)^2 \setminus (2^{k+2} \sqrt{n} Q)^2$  and  $z \in Q$ , similar to (3.3) we have

$$|K(z, y_1, y_2) - K(x, y_1, y_2)| \leq C \frac{\omega(2^{-k})}{|2^{k+3} \sqrt{n} Q|^2}.$$

This together with (3.4) gives

$$\begin{aligned} &|T((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^\infty)(z) - c_3| \\ &\leq \int_{(\mathbb{R}^n \setminus Q^*)^2} |K(z, y_1, y_2) - K(x, y_1, y_2)| \left( \prod_{j=1}^2 |(b_j(y_j) - \lambda_j) f_j^\infty(y_j)| \right) dy_1 dy_2 \\ &\leq \sum_{k=1}^{\infty} \int_{(2^{k+3} \sqrt{n} Q)^2 \setminus (2^{k+2} \sqrt{n} Q)^2} |K(z, y_1, y_2) - K(x, y_1, y_2)| \times \end{aligned}$$

$$\begin{aligned} & \times \left( \prod_{j=1}^2 |(b_j(y_j) - \lambda_j) f_j^\infty(y_j)| \right) dy_1 dy_2 \\ \leq & C \sum_{k=1}^\infty \int_{(2^{k+3}\sqrt{n}Q)^2} \frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^2} \left( \prod_{j=1}^2 |(b_j(y_j) - \lambda_j) f_j(y_j)| \right) dy_1 dy_2 \\ \leq & C \|b_1\|_{BMO} \|b_2\|_{BMO} \sum_{k=1}^\infty k^2 \omega(2^{-k}) \|f_1\|_{L(\log L), 2^k Q^*} \|f_2\|_{L(\log L), 2^k Q^*} \\ \leq & C \|b_1\|_{BMO} \|b_2\|_{BMO} \mathcal{M}_{L(\log L)}(f_1, f_2)(x), \end{aligned}$$

which concludes that

$$\begin{aligned} I_{4,4} &= C \left( \frac{1}{|Q|} \int_Q |T((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^\infty)(z) - c_3|^\delta dz \right)^{1/\delta} \\ &\leq C \frac{1}{|Q|} \int_Q |T((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^\infty)(z) - c_3| dz \\ &\leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \mathcal{M}_{L(\log L)}(f_1, f_2)(x). \end{aligned}$$

Thus, we have

$$I_4 \leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \mathcal{M}_{L(\log L)}(f_1, f_2)(x).$$

This, together with (3.2) and the estimates of  $I_1, I_2$  and  $I_3$ , gives

$$\begin{aligned} M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))(x) &\leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \left\{ \mathcal{M}_{L(\log L)}(f_1, f_2)(x) + M_\varepsilon(T(f_1, f_2))(x) \right\} \\ &\quad + C \|b_2\|_{BMO} M_\varepsilon(T_{b_1^1}(f_1, f_2))(x) + C \|b_1\|_{BMO} M_\varepsilon(T_{b_2^2}(f_1, f_2))(x). \end{aligned}$$

The proof of Proposition 3.1 is now complete.

REMARK 3.1. We can also obtain analogous estimates to (3.1) for iterated commutators involving  $j < m$  functions in  $BMO$ . More precisely, for  $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ , we have

$$\begin{aligned} M_\delta^\sharp(T_{\Pi\vec{b}_\sigma}(\vec{f}))(x) &\leq C \left( \prod_{k=1}^j \|b_k\|_{BMO} \right) \left\{ \mathcal{M}_{L(\log L)}(\vec{f})(x) + M_\varepsilon(T(\vec{f}))(x) \right\} \\ &\quad + C \sum_{k=1}^{j-1} \sum_{\eta \in C_k^m} \left( \prod_{i=1}^k \|b_{\eta(i)}\|_{BMO} \right) M_\varepsilon(T_{\Pi\vec{b}_{\eta'}}(\vec{f}))(x). \end{aligned} \tag{3.5}$$

From the pointwise estimates obtained above, we can get the following strong and weak type estimates for the iterated commutator  $T_{\Pi\vec{b}}$ .

PROPOSITION 3.2. Let  $T$  be an  $m$ -linear  $\omega$ -CZO with  $\omega$  satisfying (1.3) and  $\vec{b} \in BMO^m$ . Suppose that  $0 < p < \infty$  and  $w \in A_\infty$ . Then, there exist positive constant

$C_w$  (depending on the  $A_\infty$  constant of  $w$ , but independent of  $\vec{b}$ ) and  $C_{w,\vec{b}}$  (depending on  $w$  and  $\vec{b}$ ) such that

$$\int_{\mathbb{R}^n} |T_{\Pi\vec{b}}(\vec{f})(x)|^p w(x) dx \leq C_w \left( \prod_{j=1}^m \|b_j\|_{BMO} \right)^p \int_{\mathbb{R}^n} [\mathcal{M}_{L(\log L)}(\vec{f})(x)]^p w(x) dx \tag{3.6}$$

and

$$\begin{aligned} & \sup_{t>0} \frac{1}{\Phi^{(m)}(1/t)} w(\{y \in \mathbb{R}^n : |T_{\Pi\vec{b}}(\vec{f})(y)| > t^m\}) \\ & \leq C_{w,\vec{b}} \sup_{t>0} \frac{1}{\Phi^{(m)}(1/t)} w(\{y \in \mathbb{R}^n : \mathcal{M}_{L(\log L)}(\vec{f})(y) > t^m\}), \end{aligned} \tag{3.7}$$

for all bounded measurable functions  $f_1, \dots, f_m$  with compact support.

To prove Proposition 3.2, we need the following results obtained in [19].

LEMMA 3.1. ([19]) *Let  $T$  be an  $m$ -linear  $\omega$ -CZO with  $\omega \in \text{Dini}(1)$ ,  $0 < p < \infty$  and  $w \in A_\infty$ . Then there exists a constant  $C > 0$  such that*

$$\|T(\vec{f})\|_{L^p(w)} \leq C \|\mathcal{M}(\vec{f})\|_{L^p(w)}$$

for all bounded measurable functions  $f_1, \dots, f_m$  with compact support.

We remark that the authors of [19] proved Lemma 3.1 for  $1/m \leq p < \infty$ . Indeed, we can extend the range of  $p$  from  $1/m \leq p < \infty$  to  $0 < p < \infty$  by using the same argument as the ones of the proof of Corollary 3.8 in [16].

LEMMA 3.2. ([19]) *Let  $T$  be an  $m$ -linear  $\omega$ -CZO and  $\vec{b} \in BMO^m$ . Suppose that  $1 \leq j \leq m$  is an integer and  $T_{b_j}^j$  is the  $j$ -th entry commutator of  $T$  with  $\vec{b}$  defined by (1.1). If  $0 < p < \infty$ ,  $w \in A_\infty$  and  $\omega$  satisfies*

$$\int_0^1 \frac{\omega(t)}{t} \left(1 + \log \frac{1}{t}\right) dt < \infty,$$

then, there exists a constant  $C > 0$ , depending on the  $A_\infty$  constant of  $w$ , such that

$$\int_{\mathbb{R}^n} |T_{b_j}^j(\vec{f})(x)|^p w(x) dx \leq C \|b_j\|_{BMO}^p \int_{\mathbb{R}^n} [\mathcal{M}_{L(\log L)}^j(\vec{f})(x)]^p w(x) dx$$

for all bounded measurable functions  $f_1, \dots, f_m$  with compact support.

*Proof of Proposition 3.2.* From Proposition 3.1, Lemma 3.1 and Lemma 3.2, by applying the same arguments as the ones in the proof of Theorem 3.19 in [16] and the proof of Theorem 3.2 in [22], we can get (3.6) and (3.7). We omit the details.

*Proof of Theorem 1.3.* It suffices to prove Theorem 1.3 for  $f_1, \dots, f_m$  being bounded with compact support. For  $\vec{w} \in A_{\vec{p}}$ , Lemma 2.2 implies  $v_{\vec{w}} \in A_\infty$ . It follows from (3.6) that

$$\|T_{\Pi\vec{b}}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \left( \prod_{j=1}^m \|b_j\|_{BMO} \right) \|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(v_{\vec{w}})}.$$

By (2.3) and Lemma 2.3, for some  $r > 1$ ,

$$\begin{aligned} \|T_{\Pi\vec{b}}(\vec{f})\|_{L^p(v_{\vec{w}})} &\leq C \left( \prod_{j=1}^m \|b_j\|_{BMO} \right) \|\mathcal{M}_r(\vec{f})\|_{L^p(v_{\vec{w}})} \\ &\leq C \left( \prod_{j=1}^m \|b_j\|_{BMO} \right) \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}. \end{aligned}$$

This concludes the proof of Theorem 1.3.

*Proof of Theorem 1.4.* By homogeneity, we can assume  $\lambda = 1$  and  $\|b_j\|_{BMO} = 1$  for  $j = 1, \dots, m$ , and hence it is enough to prove

$$v_{\vec{w}}(\{x \in \mathbb{R}^n : |T_{\Pi\vec{b}}(\vec{f})(x)| > 1\}) \leq C \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \Phi^{(m)}(|f_j(x)|) w_j(x) dx \right)^{1/m}.$$

Note that  $v_{\vec{w}} \in A_\infty$  when  $\vec{w} \in A_{(1, \dots, 1)}$ , it follows from (3.7) and Lemma 2.4 that

$$\begin{aligned} &v_{\vec{w}}(\{x \in \mathbb{R}^n : |T_{\Pi\vec{b}}(\vec{f})(x)| > 1\}) \\ &\leq \sup_{t>0} \frac{1}{\Phi^{(m)}(1/t)} v_{\vec{w}}(\{y \in \mathbb{R}^n : |T_{\Pi\vec{b}}(\vec{f})(y)| > t^m\}) \\ &\leq C \sup_{t>0} \frac{1}{\Phi^{(m)}(1/t)} v_{\vec{w}}(\{y \in \mathbb{R}^n : \mathcal{M}_{L(\log L)}(\vec{f})(y) > t^m\}) \\ &\leq C \sup_{t>0} \frac{1}{\Phi^{(m)}(1/t)} \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(x)|}{t}\right) w_j(x) dx \right)^{1/m} \\ &\leq C \sup_{t>0} \frac{1}{\Phi^{(m)}(1/t)} \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \Phi^{(m)}(|f_j(x)|) w_j(x) \Phi^{(m)}(1/t) dx \right)^{1/m} \\ &\leq C \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \Phi^{(m)}(|f_j(x)|) w_j(x) dx \right)^{1/m}. \end{aligned}$$

So complete the proof of Theorem 1.4.

#### 4. Proof of Theorem 1.5

We first recall some facts on variable exponent Lebesgue spaces.

LEMMA 4.1. ([5]) *Given  $p(\cdot) \in \mathcal{P}_0$ , then for all  $s > 0$ , we have*

$$\| |f|^s \|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{L^{sp(\cdot)}(\mathbb{R}^n)}^s.$$

The following generalized Hölder’s inequality holds (see [3] or [7]).

LEMMA 4.2. *Let  $q(\cdot), q_1(\cdot), \dots, q_m(\cdot) \in \mathcal{P}_0$  so that  $1/q(\cdot) = 1/q_1(\cdot) + \dots + 1/q_m(\cdot)$ . Then for any  $f_j \in L^{q_j(\cdot)}(\mathbb{R}^n), j = 1, \dots, m,$*

$$\|f_1 \cdots f_m\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C \|f_1\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \cdots \|f_m\|_{L^{q_m(\cdot)}(\mathbb{R}^n)},$$

where the constant  $C$  depends only on  $q_1(\cdot), \dots, q_m(\cdot)$ .

LEMMA 4.3. ([4]) *Let  $p(\cdot) \in \mathcal{P}$  and satisfy (1.5) and (1.6), then for any  $v \in A_{p(\cdot)},$*

$$\|Mf\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L_v^{p(\cdot)}(\mathbb{R}^n)}.$$

The following extrapolation theorem is due to Cruz-Uribe and Wang [6, Theorem 2.25].

LEMMA 4.4. ([6]) *Given a family  $\mathcal{F}$  of ordered pairs of measurable functions. Suppose that for some  $0 < p_0 < \infty$  and every  $w \in A_\infty,$  the following inequality holds for all  $(f, g) \in \mathcal{F},$*

$$\int_{\mathbb{R}^n} |f(x)|^{p_0} w(x) dx \leq C_0 \int_{\mathbb{R}^n} |g(x)|^{p_0} w(x) dx.$$

Let  $p(\cdot) \in \mathcal{P}_0,$  if there exists  $s \leq p^-$  such that  $v^s \in A_{p(\cdot)/s}$  and  $M$  is bounded on  $L_{v^{-s}}^{(p(\cdot)/s)'}(\mathbb{R}^n),$  then there is a positive constant  $C$  such that

$$\|f\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} \leq C \|g\|_{L_v^{p(\cdot)}(\mathbb{R}^n)}$$

for every pair  $(f, g) \in \mathcal{F}$  such that the left-hand side is finite.

To prove our result, we will need the following density property, see [6, Lemma 3.1].

LEMMA 4.5. ([6]) *Given  $p(\cdot) \in \mathcal{P}$  and a weight  $v \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n),$  then  $L_c^\infty(\mathbb{R}^n),$  the set of all bounded functions with compact support, is dense in  $L_v^{p(\cdot)}(\mathbb{R}^n).$*

The following monotone convergence theorem is due to [5, Lemma 2.5].

LEMMA 4.6. ([5]) *Suppose  $p(\cdot) \in \mathcal{P}_0.$  Given a sequence  $\{f_k\}$  of  $L^{p(\cdot)}(\mathbb{R}^n)$  functions that increases pointwise almost everywhere to a function  $f,$  we have*

$$\lim_{k \rightarrow \infty} \|f_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

We also need the following result for weights with variable exponents.

LEMMA 4.7. *Let  $p(\cdot) \in \mathcal{P}_0$  and  $p_1(\cdot), \dots, p_m(\cdot) \in \mathcal{P}$  with  $1/p(\cdot) = 1/p_1(\cdot) + \dots + 1/p_m(\cdot).$  For  $v_j \in A_{p_j(\cdot)}, j = 1, \dots, m,$  let  $v = \prod_{j=1}^m v_j.$  Then  $v^{1/m} \in A_{mp(\cdot)}.$*



*Proof.* Since  $p(\cdot) \in \mathcal{P}_0$ ,  $p_1(\cdot), \dots, p_m(\cdot) \in \mathcal{P}$  and  $1/p(\cdot) = 1/p_1(\cdot) + \dots + 1/p_m(\cdot)$  then  $mp(\cdot) \in \mathcal{P}$  and

$$\frac{1}{(mp(\cdot))'} = \sum_{j=1}^m \frac{1}{mp'_j(\cdot)}.$$

By the generalized Hölder’s inequality (Lemma 4.2) and Lemma 4.1, we have

$$\begin{aligned} & \left\| |B|^{-1} \left\| v^{1/m} \chi_B \right\|_{L^{mp(\cdot)}(\mathbb{R}^n)} \left\| v^{-1/m} \chi_B \right\|_{L^{(mp(\cdot))'}(\mathbb{R}^n)} \right\| \\ & \leq C |B|^{-1} \prod_{j=1}^m \left\| v_j^{1/m} \chi_B \right\|_{L^{mp_j(\cdot)}(\mathbb{R}^n)} \prod_{j=1}^m \left\| v_j^{-1/m} \chi_B \right\|_{L^{mp'_j(\cdot)}(\mathbb{R}^n)} \\ & \leq C |B|^{-1} \prod_{j=1}^m \left\| v_j \chi_B \right\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}^{1/m} \prod_{j=1}^m \left\| v_j^{-1} \chi_B \right\|_{L^{p'_j(\cdot)}(\mathbb{R}^n)}^{1/m} \\ & = C \prod_{j=1}^m \left( \left\| |B|^{-1} v_j \chi_B \right\|_{L^{p_j(\cdot)}(\mathbb{R}^n)} \left\| v_j^{-1} \chi_B \right\|_{L^{p'_j(\cdot)}(\mathbb{R}^n)} \right)^{1/m} \\ & < \infty, \end{aligned}$$

where the last step follows from  $v_j \in A_{p_j(\cdot)}$ ,  $j = 1, \dots, m$ . This concludes the proof.

Now, we have all the ingredients to prove Theorem 1.5.

*Proof of Theorem 1.5.* By Lemma 4.5 it suffices to prove Theorem 1.5 for all bounded functions  $f_1, \dots, f_m$  with compact support.

We define a sequence of operators  $\{T_k\}_{k \in \mathbb{N}}$ , where  $T_k(\vec{f}) = \min\{|T_{\Pi\vec{b}}(\vec{f})|, k\} \chi_{B(0,k)}$  and  $T_k(\vec{f}) = 0$  when  $x \notin B(0,k)$ , and consider the following family

$$\mathcal{F} = \left\{ (T_k(\vec{f}), \mathcal{M}_{L(\log L)}(\vec{f})) : \vec{f} = (f_1, \dots, f_m) \in (L_c^\infty(\mathbb{R}^n))^m, k \in \mathbb{N} \right\}.$$

It follows from Proposition 3.2 that, for any  $0 < p_0 < \infty$  and every  $w \in A_\infty$ ,

$$\int_{\mathbb{R}^n} |T_{\Pi\vec{b}}(\vec{f})(x)|^{p_0} w(x) dx \leq C \int_{\mathbb{R}^n} [\mathcal{M}_{L(\log L)}(\vec{f})(x)]^{p_0} w(x) dx$$

holds for all bounded functions  $f_1, \dots, f_m$  with compact support.

Since  $0 \leq T_k(\vec{f})(x) \leq |T_{\Pi\vec{b}}(\vec{f})(x)|$ , then for any  $0 < p_0 < \infty$  and every  $w \in A_\infty$ ,

$$\int_{\mathbb{R}^n} [T_k(\vec{f})(x)]^{p_0} w(x) dx \leq C \int_{\mathbb{R}^n} [\mathcal{M}_{L(\log L)}(\vec{f})(x)]^{p_0} w(x) dx$$

holds for every ordered pair  $(T_k(\vec{f}), \mathcal{M}_{L(\log L)}(\vec{f}))$  in  $\mathcal{F}$ .

By Lemma 4.7 we have  $v^{1/m} \in A_{mp(\cdot)}$ , which implies  $v^{-1/m} \in A_{(mp(\cdot))'}$ . Since  $p(\cdot)$  satisfies (1.5) and (1.6) then  $mp(\cdot)$  and  $(mp(\cdot))'$  also satisfy (1.5) and (1.6). Note that  $mp(\cdot) \in \mathcal{P}$ , then, it follows from Lemma 4.3 that  $M$  is bounded on  $L_{v^{-1/m}}^{(mp(\cdot))'}(\mathbb{R}^n)$ .

So, to use Lemma 4.4 for all ordered pairs in  $\mathcal{F}$ , we need to check that  $\|T_k(\vec{f})\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} < \infty$  for every  $\vec{f} = (f_1, \dots, f_m) \in (L_c^\infty(\mathbb{R}^n))^m$ . It is obviously always the case. Indeed,

since  $v^{1/m} \in A_{mp(\cdot)}$  then  $v^{1/m} \in L_{\text{loc}}^{mp(\cdot)}(\mathbb{R}^n)$ . This together with Lemma 4.1 gives  $v \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n)$ , which implies  $\|v\chi_{B(0,k)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$  for each integer  $k$ . Thus

$$\|T_k(\vec{f})\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} = \|T_k(\vec{f})v\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq k\|v\chi_{B(0,k)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

Now, we can apply Lemma 4.4 for  $s = 1/m$  to each pair in  $\mathcal{F}$  and get

$$\|T_k(\vec{f})\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} \leq C\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L_v^{p(\cdot)}(\mathbb{R}^n)}. \tag{4.1}$$

Recall the pointwise equivalence  $M_{L(\log L)}(g)(x) \approx M^2(g)(x)$  for any locally integrable function  $g$  (see (21) in [21]), we have

$$\mathcal{M}_{L(\log L)}(\vec{f})(x) \leq \prod_{j=1}^m M_{L(\log L)}(f_j)(x) \leq C \prod_{j=1}^m M^2(f_j)(x).$$

Then, it follows from (4.1) and the generalized Hölder’s inequality (Lemma 4.2) that

$$\|T_k(\vec{f})\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} \leq C\left\|\prod_{j=1}^m M^2(f_j)\right\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} \leq C\prod_{j=1}^m \|M^2(f_j)\|_{L_{v_j}^{p_j(\cdot)}(\mathbb{R}^n)}.$$

Since  $p_j(\cdot) \in \mathcal{P}$  and satisfies (1.5) and (1.6), and  $v_j \in A_{p_j(\cdot)}$  for  $j = 1, \dots, m$ , then, by applying Lemma 4.3 twice, we have

$$\|T_k(\vec{f})\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} \leq C\prod_{j=1}^m \|f_j\|_{L_{v_j}^{p_j(\cdot)}(\mathbb{R}^n)}. \tag{4.2}$$

Note that  $\{T_k(\vec{f})(x)v(x)\}$  increases pointwise almost everywhere to  $T_{\Pi\vec{b}}(\vec{f})(x)v(x)$  and  $T_k(\vec{f})v \in L^{p(\cdot)}(\mathbb{R}^n)$ , then, Lemma 4.6 together with (4.2) gives

$$\|T_{\Pi\vec{b}}(\vec{f})\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} = \|T_{\Pi\vec{b}}(\vec{f})v\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \lim_{k \rightarrow \infty} \|T_k(\vec{f})v\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\prod_{j=1}^m \|f_j\|_{L_{v_j}^{p_j(\cdot)}(\mathbb{R}^n)}.$$

So complete the proof of Theorem 1.5.

### 5. Applications

In 2009, Maldonado and Naibo [20] studied some bilinear pseudo-differential operators and paraproducts with mild regularity. In this section, we give some applications of our results to such kind of operators.

**5.1. Bilinear pseudo-differential operators with mild regularity**

Let  $m \in \mathbb{R}$ ,  $0 \leq \delta, \rho \leq 1$  and  $\alpha, \beta, \gamma \in \mathbb{Z}_+^n$ . A bilinear pseudo-differential operator  $T_\sigma$  with a bilinear symbol  $\sigma(x, \xi, \eta)$ , a priori defined from  $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$  to  $\mathcal{S}'(\mathbb{R}^n)$ , is given by

$$T_\sigma(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix \cdot (\xi + \eta)} \sigma(x, \xi, \eta) \hat{f}_1(\xi) \hat{f}_2(\eta) d\xi d\eta.$$

We say that a symbol  $\sigma(x, \xi, \eta)$  belongs to the bilinear Hörmander class  $BS_{\rho, \delta}^m$  if

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \leq C_{\alpha, \beta} (1 + |\xi| + |\eta|)^{m + \delta|\alpha| - \rho(|\beta| + |\gamma|)}, \quad x, \xi, \eta \in \mathbb{R}^n.$$

for all multi-indices  $\alpha, \beta$  and  $\gamma$  and some constant  $C_{\alpha, \beta}$ .

For  $\Omega, \theta : [0, \infty) \rightarrow [0, \infty)$  and  $0 \leq \rho \leq 1$ , we say that a symbol  $\sigma \in BS_{\rho, \theta, \Omega}^m$  if

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x, \xi, \eta)| \leq C_{\alpha, \beta} (1 + |\xi| + |\eta|)^{m - \rho(|\alpha| + |\beta|)}$$

and

$$\begin{aligned} &|\partial_\xi^\alpha \partial_\eta^\beta (\sigma(x + h, \xi, \eta) - \sigma(x, \xi, \eta))| \\ &\leq C_{\alpha, \beta} \theta(|h|) \Omega(|\xi| + |\eta|) (1 + |\xi| + |\eta|)^{m - \rho(|\alpha| + |\beta|)} \end{aligned}$$

for all  $x, \xi, \eta \in \mathbb{R}^n$ . Obviously,  $BS_{\rho, 0}^m \subset BS_{\rho, \theta, \Omega}^m$ .

The following result was proved by Maldonado and Naibo in [20, Theorem 4.3].

LEMMA 5.1. ([20]) *Let  $a \in (0, 1)$ ,  $\theta$  be concave with  $\theta \in \text{Dini}(a/2)$  and  $\Omega : [0, \infty) \rightarrow [0, \infty)$  be nondecreasing such that*

$$\sup_{0 < t < 1} \theta^{1-a}(t) \Omega(1/t) < \infty. \tag{5.1}$$

*If  $\sigma \in BS_{1, \theta, \Omega}^0$  with  $|\alpha| + |\beta| \leq 4n + 4$ , then  $T_\sigma$  is a bilinear Calderón-Zygmund operator of type  $\omega$  with  $\omega(t) = \theta^a(t)$  and  $\tau = 1/3$ .*

Let  $\vec{b} = (b_1, b_2) \in BMO^2$ , the iterated commutator of bilinear pseudo-differential operator  $T_\sigma$  with  $\vec{b} = (b_1, b_2)$  is defined by

$$\begin{aligned} T_{\sigma, \Pi \vec{b}}(f_1, f_2)(x) &= [b_1, [b_2, T_\sigma]_2]_1(f_1, f_2)(x) \\ &= b_1(x) b_2(x) T_\sigma(f_1, f_2)(x) - b_2(x) T_\sigma(b_1 f_1, f_2)(x) \\ &\quad - b_1(x) T_\sigma(f_1, b_2 f_2)(x) + T_\sigma(b_1 f_1, b_2 f_2)(x). \end{aligned}$$

Lemma 5.1 together with Theorems 1.3 – 1.5 gives the follows reesults.

THEOREM 5.1. *Let  $a \in (0, 1)$ ,  $\theta$  be concave with  $\theta \in \text{Dini}(a/2)$  and satisfy*

$$\int_0^1 \frac{\theta^\alpha(t)}{t} \left(1 + \log \frac{1}{t}\right)^2 dt < \infty.$$

Let  $\Omega : [0, \infty) \rightarrow [0, \infty)$  be nondecreasing such that (5.1) holds. Suppose that  $\sigma \in BS_{1, \theta, \Omega}^0$  with  $|\alpha| + |\beta| \leq 4n + 4$ . If  $\vec{b} \in BMO^2$  and  $\vec{v}_w \in A_{\vec{p}}$  with  $1 < p_1, p_2 < \infty$  and  $1/p = 1/p_1 + 1/p_2$ , then there exists a constant  $C > 0$  such that

$$\|T_{\sigma, \Pi \vec{b}}(f_1, f_2)\|_{L^p(v_{\vec{w}})} \leq C \|\vec{b}\|_{BMO^2} \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}.$$

**THEOREM 5.2.** Let  $a, \theta, \Omega$  and  $\sigma$  be the same as in Theorem 5.1. If  $\vec{b} \in BMO^2$  and  $\vec{w} \in A_{(1,1)}$ , then there is a constant  $C > 0$ , depending on  $\|\vec{b}\|_{BMO^2}$ , such that for any  $\lambda > 0$ ,

$$v_{\vec{w}} \left( \left\{ x \in \mathbb{R}^n : |T_{\sigma, \Pi \vec{b}}(f_1, f_2)(x)| > \lambda^2 \right\} \right) \leq C \prod_{j=1}^2 \left( \int_{\mathbb{R}^n} \Phi^{(2)} \left( \frac{|f_j(x)|}{\lambda} \right) w_j(x) dx \right)^{1/2}.$$

**THEOREM 5.3.** Let  $a, \theta, \Omega$  and  $\sigma$  be the same as in Theorem 5.1 and  $\vec{b} \in BMO^2$ . Suppose that  $p(\cdot) \in \mathcal{P}_0$  and  $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$  so that  $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$ . If  $p(\cdot), p_1(\cdot)$  and  $p_2(\cdot)$  satisfy (1.5) and (1.6), then, for  $v_i \in A_{p_i(\cdot)}, i = 1, 2$ , and  $v = v_1 v_2$ , there exists a constant  $C > 0$  such that

$$\|T_{\sigma, \Pi \vec{b}}(f_1, f_2)\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f_1\|_{L_{v_1}^{p_1(\cdot)}(\mathbb{R}^n)} \|f_2\|_{L_{v_2}^{p_2(\cdot)}(\mathbb{R}^n)}.$$

### 5.2. Paraproducts with mild regularity

For  $v \in \mathbb{Z}$  and  $\kappa = (k_1, \dots, k_n) \in \mathbb{Z}^n$ , let  $P_{v\kappa}$  be the dyadic cube

$$P_{v\kappa} := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : k_i \leq 2^v x_i < k_i + 1, i = 1, \dots, n \right\}.$$

The lower left-corner of  $P := P_{v\kappa}$  is  $x_P = x_{v\kappa} := 2^{-v\kappa}$  and the Lebesgue measure of  $P$  is  $|P| = 2^{-vn}$ . We set

$$\mathcal{D} = \{P_{v\kappa} : v \in \mathbb{Z}, \kappa \in \mathbb{Z}^n\}$$

as the collection of all dyadic cubes.

**DEFINITION 5.1.** ([20]) Let  $\theta : [0, \infty) \rightarrow [0, \infty)$  be a nondecreasing and concave function. An  $\theta$ -molecule associated to a dyadic cube  $P = P_{v\kappa}$  is a function  $\phi_P = \phi_{v\kappa} : \mathbb{R}^n \rightarrow \mathbb{C}$  such that, for some  $A_0 > 0$  and  $N > n$ , it satisfies the decay condition

$$|\phi_P(x)| \leq \frac{A_0 2^{vn/2}}{(1 + 2^v|x - x_P|)^N}, \quad x \in \mathbb{R}^n,$$

and the mild regularity condition

$$|\phi_P(x) - \phi_P(y)| \leq A_0 2^{vn/2} \theta(2^v|x - y|) \left[ \frac{1}{(1 + 2^v|x - x_P|)^N} + \frac{1}{(1 + 2^v|y - x_P|)^N} \right]$$

for all  $x, y \in \mathbb{R}^n$ .

DEFINITION 5.2. ([20]) Given three families of  $\theta$ -molecules  $\{\phi_Q^j\}_{Q \in \mathcal{D}}$ ,  $j = 1, 2, 3$ , the paraproduct  $\Pi(f, g)$  associated to these families is defined by

$$\Pi(f, g) = \sum_{Q \in \mathcal{D}} |Q|^{-1/2} \langle f, \phi_Q^1 \rangle \langle g, \phi_Q^2 \rangle \phi_Q^3, \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

In [20], some sufficient conditions on  $\theta$  were given so that the paraproducts defined above can be realized as bilinear  $\omega$ -CZO. The following result was proved in [20, Theorem 5.3] when  $\theta \in Dini(1/2)$ . Indeed, the condition  $\theta \in Dini(1/2)$  can be reduced to  $\theta \in Dini(1)$ , see Lemmas 8.2 and 8.3 in [19] for details.

LEMMA 5.2. Let  $\theta$  be concave and  $\theta \in Dini(1)$ , and let  $\{\phi_Q^j\}_{Q \in \mathcal{D}}$ ,  $j = 1, 2, 3$ , be three families of  $\theta$ -molecules with decay  $N > 10n$  and such that at least two of them, say  $j = 1, 2$ , enjoy the following cancelation property

$$\int_{\mathbb{R}^n} \phi_Q^j(x) dx = 0, \quad Q \in \mathcal{D}, \quad j = 1, 2,$$

then  $\Pi$  is a bilinear Calderón-Zygmund operator of type  $\omega$  with  $\omega(t) = A_0^3 A_N \theta(C_N t)$  and  $\tau = 1/2$ , where  $A_N$  and  $C_N$  are constants depending on  $N$ .

Let  $\vec{b} = (b_1, b_2) \in BMO^2$ , the iterated commutator of  $\Pi$  with  $\vec{b}$  is defined by

$$\begin{aligned} \Pi_{\Pi \vec{b}}(f_1, f_2)(x) &= [b_1, [b_2, T]_2, ]_1(f_1, f_2)(x) \\ &= b_1(x)b_2(x)\Pi(f_1, f_2)(x) - b_2(x)\Pi(b_1 f_1, f_2)(x) \\ &\quad - b_1(x)\Pi(f_1, b_2 f_2)(x) + \Pi(b_1 f_1, b_2 f_2)(x). \end{aligned}$$

Note that if  $\theta \in Dini(1)$  and satisfies

$$\int_0^1 \frac{\theta(t)}{t} \left(1 + \log \frac{1}{t}\right)^2 dt < \infty \tag{5.2}$$

then  $\omega(t) = A_0^3 A_N \theta(C_N t)$  also belongs to  $Dini(1)$  and satisfies (5.2). So, the following estimates for iterated commutator  $\Pi_{\Pi \vec{b}}$  are direct consequences of Theorems 1.3, 1.4 and 1.5 and Lemma 5.2.

THEOREM 5.4. Let  $\theta$  and  $\phi_Q^j$  be the same as in Lemma 5.2. Assume that  $\theta$  satisfy (5.2). If  $\vec{b} \in BMO^2$  and  $\vec{w} \in A_{\vec{b}}$  with  $1 < p_1, p_2 < \infty$  and  $1/p = 1/p_1 + 1/p_2$ , then there exists a constant  $C > 0$  such that

$$\|\Pi_{\Pi \vec{b}}(f_1, f_2)\|_{L^p(v_{\vec{w}})} \leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}.$$

THEOREM 5.5. Let  $\theta$  and  $\phi_Q^j$  be the same as in Theorem 5.4. If  $\vec{b} \in BMO^2$  and  $\vec{w} \in A_{(1,1)}$ , then there is a constant  $C > 0$  depending on  $\|\vec{b}\|_{BMO^2}$ , such that for all  $\lambda > 0$ ,

$$v_{\vec{w}}(\{x \in \mathbb{R}^n : |\Pi_{\Pi \vec{b}}(f_1, f_2)(x)| > \lambda^2\}) \leq C \prod_{j=1}^2 \left( \int_{\mathbb{R}^n} \Phi^{(2)}\left(\frac{|f_j(x)|}{\lambda}\right) w_j(x) dx \right)^{1/2}.$$

**THEOREM 5.6.** *Let  $\theta$  and  $\phi_Q^j$  be the same as in Theorem 5.4 and  $\vec{b} \in BMO^2$ . Suppose that  $p(\cdot) \in \mathcal{P}_0$  and  $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$  so that  $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$ . If  $p(\cdot)$ ,  $p_1(\cdot)$  and  $p_2(\cdot)$  satisfy (1.5) and (1.6), then, for  $v_i \in A_{p_i(\cdot)}$ ,  $i = 1, 2$ , and  $v = v_1 v_2$ , there exists a positive constant  $C$  such that*

$$\left\| \Pi_{\Pi \vec{b}}(f_1, f_2) \right\|_{L_v^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f_1\|_{L_{v_1}^{p_1(\cdot)}(\mathbb{R}^n)} \|f_2\|_{L_{v_2}^{p_2(\cdot)}(\mathbb{R}^n)}.$$

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