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REFINEMENTS OF CAUCHY–SCHWARZ NORM INEQUALITY

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Abstract. In this paper, we utilize the convexity of the function $f(\nu)=\left|\left|\left|\,|A^{\nu}XB^{1-\nu}|^{r}\right|\right|\right|\cdot\left|\left|\left|\,|A^{1-\nu}XB^{\nu}|^{r}\right|\right|\right|$ and the Hermite-Hadamard inequality to obtain a family of new refinements of Cauchy-Schwarz norm inequality for operators, which extends the related results.

1. Introduction

A capital letter H stands for a complex Hilbert space and $B(H)$ denotes the algebra of all bounded linear operators on H. Let $|||\,.\,|||$ denote a unitarily invariant norm defined on a norm ideal associated with it. Having considered the convexity of the function $f(\nu)=|||\,|A^{\nu}XB^{1-\nu}|^{r}|||\cdot|||\,|A^{1-\nu}XB^{\nu}|^{r}|||$ for $A,B,X\in B(H)$ such that A,B are positive, every real number $r>0$ and every unitarily invariant norm, Hiai and Zhan [7] proved that , then $f(\nu)$ is convex on the interval $[0,1]$ and attained its minimum at $\nu=\frac{1}{2}$, its maximum at $\nu=0$ and $\nu=1$. Moreover, $f(\nu)=f(1-\nu)$. With the help of the convexity of the function $f(\nu)$, they obtained the following refinement of Cauchy-Schwarz norm inequality for operators

$$\left|\left|\left|\,\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|^{r}\right|\right|\right|^{2}\leqslant\left|\left|\left|\,\left|A^{\nu}XB^{1-\nu}\right|^{r}\right|\right|\right|\cdot\left|\left|\left|\,\left|A^{1-\nu}XB^{\nu}\right|^{r}\right|\right|\right|\leqslant|||\,|AX|^{r}|||\cdot|||\,|XB|^{r}|||.$$

Recently, Burqan [5] obtained several refinements of Cauchy-Schwarz norm inequality with the help of the well-known Hermite-Hadamard inequality as follows.

THEOREM A. *Let $A,B,X\in B(H)$ such that A,B are positive. Then for $0\leqslant\mu\leqslant1$, $r>0$, and for every unitarily invariant norm, we have*

$$\begin{aligned}\left|\left|\left|\,\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|^{r}\right|\right|\right|^{2}&\leqslant\left|\left|\left|\,\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}\right|^{r}\right|\right|\right|\cdot\left|\left|\left|\,\left|A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|^{r}\right|\right|\right|\\&\leqslant\frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu}\left|\left|\left|\,\left|A^{\nu}XB^{1-\nu}\right|^{r}\right|\right|\right|\cdot\left|\left|\left|\,\left|A^{1-\nu}XB^{\nu}\right|^{r}\right|\right|\right|d\nu\right|\\&\leqslant\frac{1}{2}\left[\left|\left|\left|\,\left|A^{\mu}XB^{1-\mu}\right|^{r}\right|\right|\right|\cdot\left|\left|\left|\,\left|A^{1-\mu}XB^{\mu}\right|^{r}\right|\right|\right|+\left|\left|\left|\,\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|^{r}\right|\right|\right|^{2}\right]\\&\leqslant\left|\left|\left|\,\left|A^{\mu}XB^{1-\mu}\right|^{r}\right|\right|\right|\cdot\left|\left|\left|\,\left|A^{1-\mu}XB^{\mu}\right|^{r}\right|\right|\right|\\&\leqslant|||\,|AX|^{r}|||\cdot|||\,|XB|^{r}|||.\end{aligned}$$

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The following Heinz inequality due to Bhatia and Davis [3] holds

$$2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \leqslant |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}||| \leqslant |||AX+XB|||.$$

As a refinement of the above Heinz inequality, Abbas and Mourad [1] got the following result.

THEOREM B. *Let $A,B,X \in B(H)$ such that A,B are positive and let n be a positive integer. Then for $0 \leqslant \mu \leqslant 1$, and for every unitarily invariant norm, we have*

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\leqslant \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||\,d\nu\right| \\ &\leqslant \frac{1}{2n}\left[(2n-1)\,|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}||| + 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right] \\ &\leqslant |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||. \end{aligned}$$

In this paper, we utilize the convexity of the function $f(\nu)$ and the Hermite-Hadamard inequality to obtain a family of new refinements of Cauchy-Schwarz norm inequality and Heiz inequality. Accordingly Theorem A and Theorem B are special cases of this new family.

2. Main Results

In this section, We start by the well-known Hermite-Hadamard integral inequality [4] which includes a basic property of convex functions and plays a central role in our investigation to obtain a further series of refinements of the Cauchy-Schwarz norm inequalities.

If f is a real-valued function which is convex on the interval $[a,b]$, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}.$$

For the first and the second inequality in the above Hermite-Hadamard integral inequality, Burqan and Feng constructed the following refinement respectively.

LEMMA 2.1. *([5]) Let f be a real valued function which is convex on the interval $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}. \tag{2.1}$$

LEMMA 2.2. *([6]) Let f be a real valued function which is convex on the interval $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right] \leqslant \frac{f(a)+f(b)}{2}. \tag{2.2}$$

We present a refinement of the above Hermite-Hadamard inequality (2.1) and (2.2) as follows.

LEMMA 2.3. *Let f be a real valued function which is convex on the interval $[a,b]$ and let n be a positive integer. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \frac{1}{2}\left\{f\left[\frac{(2^n+1)a+(2^n-1)b}{2^{n+1}}\right]+f\left[\frac{(2^n-1)a+(2^n+1)b}{2^{n+1}}\right]\right\} \\ &\leqslant \frac{1}{b-a}\int_a^b f(t)dt \\ &\leqslant \frac{1}{2(n+1)}\left[nf(a)+2f\left(\frac{a+b}{2}\right)+nf(b)\right] \\ &\leqslant \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.3}$$

Proof. Since f is convex on $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2}.$$

Thus

$$\begin{aligned} \frac{1}{2(n+1)}\left[nf(a)+2f\left(\frac{a+b}{2}\right)+nf(b)\right] &\leqslant \frac{1}{2(n+1)}[nf(a)+f(a)+f(b)+nf(b)] \\ &= \frac{f(a)+f(b)}{2}. \end{aligned}$$

This completes the proof of the fourth inequality. Using convexity of f, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left[\frac{1}{2}\cdot\frac{(2^n+1)a+(2^n-1)b}{2^{n+1}}+\frac{1}{2}\cdot\frac{(2^n-1)a+(2^n+1)b}{2^{n+1}}\right] \\ &\leqslant \frac{1}{2}\left\{f\left[\frac{(2^n+1)a+(2^n-1)b}{2^{n+1}}\right]+f\left[\frac{(2^n-1)a+(2^n+1)b}{2^{n+1}}\right]\right\}. \end{aligned}$$

This completes the proof of the first inequality.

To prove the second and the third inequality, it is only need to prove the following inequalities by Lemma 2.1 and Lemma 2.2.

$$f\left[\frac{(2^n+1)a+(2^n-1)b}{2^{n+1}}\right]+f\left[\frac{(2^n-1)a+(2^n+1)b}{2^{n+1}}\right] \leqslant f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right) \tag{2.4}$$

and

$$\frac{1}{2}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right] \leqslant \frac{1}{n+1}\left[nf(a)+2f\left(\frac{a+b}{2}\right)+nf(b)\right]. \tag{2.5}$$

Next, we prove inequality (2.4) by induction. By Lemma 2.1, we have

$$f(a)+f(b) \geqslant f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right). \tag{2.6}$$

So (2.4) holds trivially for the case $n=1$. Now suppose the assertion (2.4) holds for the case $n=k$. By the induction hypothesis, we have

$$
\begin{aligned}
& f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right) \\
& \geqslant f\left[\frac{(2^k+1)a+(2^k-1)b}{2^{k+1}}\right]+f\left[\frac{(2^k-1)a+(2^k+1)b}{2^{k+1}}\right] \\
& \geqslant f\left[\frac{3\cdot\frac{(2^k+1)a+(2^k-1)b}{2^{k+1}}+\frac{(2^k-1)a+(2^k+1)b}{2^{k+1}}}{4}\right] \\
& \qquad +f\left[\frac{\frac{(2^k+1)a+(2^k-1)b}{2^{k+1}}+3\cdot\frac{(2^k-1)a+(2^k+1)b}{2^{k+1}}}{4}\right] \qquad \text{(by (2.6))} \\
& = f\left[\frac{(2^{k+1}+1)a+(2^{k+1}-1)b}{2^{(k+1)+1}}\right]+f\left[\frac{(2^{k+1}-1)a+(2^{k+1}+1)b}{2^{(k+1)+1}}\right],
\end{aligned}
$$

and so (2.4) holds for the case $n=k+1$. Hence (2.4) holds by induction. Similarly, the inequality (2.5) holds by induction.

Given all that, the proof of Lemma 2.3 is complete.

Applying Lemma 2.3 to the function $f(\nu)=\left|\left|\left|\,|A^{\nu}XB^{1-\nu}|^r\right|\right|\right|\cdot\left|\left|\left|\,|A^{1-\nu}XB^{\nu}|^r\right|\right|\right|$ on the interval $[\mu,1-\mu]$ when $0\leqslant\mu<\frac{1}{2}$, on the interval $[1-\mu,\mu]$ when $\frac{1}{2}<\mu\leqslant1$, we obtain our first refinement of Cauchy-Schwarz norm inequality in [5].

THEOREM 2.1. *Let $A,B,X\in B(H)$ such that A,B are positive and let n be a positive integer. Then for $0\leqslant\mu\leqslant1$, $r>0$, and for every unitarily invariant norm,*

$$
\begin{aligned}
\left|\left|\left|\,\left|A^{\frac12}XB^{\frac12}\right|^r\right|\right|\right|^2 &\leqslant \left|\left|\left|\,\left|A^{\frac{2\mu+2^n-1}{2^{n+1}}}XB^{\frac{2^n+1-2\mu}{2^{n+1}}}\right|^r\right|\right|\right|\cdot\left|\left|\left|\,\left|A^{\frac{2^n+1-2\mu}{2^{n+1}}}XB^{\frac{2\mu+2^n-1}{2^{n+1}}}\right|^r\right|\right|\right| \\
&\leqslant \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu}\left|\left|\left|\,|A^{\nu}XB^{1-\nu}|^r\right|\right|\right|\cdot\left|\left|\left|\,|A^{1-\nu}XB^{\nu}|^r\right|\right|\right|d\nu\right| \\
&\leqslant \frac{1}{n+1}\left[n\left|\left|\left|\,|A^{\mu}XB^{1-\mu}|^r\right|\right|\right|\cdot\left|\left|\left|\,|A^{1-\mu}XB^{\mu}|^r\right|\right|\right|+\left|\left|\left|\,\left|A^{\frac12}XB^{\frac12}\right|^r\right|\right|\right|^2\right] \\
&\leqslant \left|\left|\left|\,|A^{\mu}XB^{1-\mu}|^r\right|\right|\right|\cdot\left|\left|\left|\,|A^{1-\mu}XB^{\mu}|^r\right|\right|\right| \\
&\leqslant |||\,|AX|^r|||\cdot|||\,|XB|^r|||. \qquad (2.7)
\end{aligned}
$$

Proof. We first consider the case $0\leqslant\mu<\frac{1}{2}$. Applying Lemma 2.3 to the function

$f(\nu) = \left|\left|\left| |A^{\nu}XB^{1-\nu}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{1-\nu}XB^{\nu}|^r \right|\right|\right|$ on the interval $[\mu, 1-\mu]$, we obtain

$$\begin{aligned}
f\left(\frac{1}{2}\right) &\leqslant \frac{1}{2}\left\{ f\left[\frac{(2^n+1)\mu+(2^n-1)(1-\mu)}{2^{n+1}}\right] + f\left[\frac{(2^n-1)\mu+(2^n+1)(1-\mu)}{2^{n+1}}\right]\right\} \\
&= \left[f\left(\frac{2\mu+2^n-1}{2^{n+1}}\right) + f\left(\frac{2^n+1-2\mu}{2^{n+1}}\right)\right] \\
&\leqslant \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} f(t)dt \\
&\leqslant \frac{1}{2(n+1)}\left[nf(\mu) + 2f\left(\frac{1}{2}\right) + nf(1-\mu)\right] \\
&= \frac{1}{n+1}\left[nf(\mu) + f\left(\frac{1}{2}\right)\right] \\
&\leqslant f(\mu).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\left|\left| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r \right|\right|\right|^2 &\leqslant \left|\left|\left| \left|A^{\frac{2\mu+2^n-1}{2^{n+1}}}XB^{\frac{2^n+1-2\mu}{2^{n+1}}}\right|^r \right|\right|\right| \cdot \left|\left|\left| \left|A^{\frac{2^n+1-2\mu}{2^{n+1}}}XB^{\frac{2\mu+2^n-1}{2^{n+1}}}\right|^r \right|\right|\right| \\
&\leqslant \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} \left|\left|\left| |A^{\nu}XB^{1-\nu}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{1-\nu}XB^{\nu}|^r \right|\right|\right| d\nu \\
&\leqslant \frac{1}{n+1}\left[n\left|\left|\left| |A^{\mu}XB^{1-\mu}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{1-\mu}XB^{\mu}|^r \right|\right|\right| + \left|\left|\left| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r \right|\right|\right|^2\right] \\
&\leqslant \left|\left|\left| |A^{\mu}XB^{1-\mu}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{1-\mu}XB^{\mu}|^r \right|\right|\right| \qquad (2.8)
\end{aligned}$$

Similarly, when $\frac{1}{2} < \mu \leqslant 1$, applying Lemma 2.3 to the function $f(\nu)$ on the interval $[1-\mu, \mu]$, we get

$$\begin{aligned}
\left|\left|\left| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r \right|\right|\right|^2 &\leqslant \left|\left|\left| \left|A^{\frac{2\mu+2^n-1}{2^{n+1}}}XB^{\frac{2^n+1-2\mu}{2^{n+1}}}\right|^r \right|\right|\right| \cdot \left|\left|\left| \left|A^{\frac{2^n+1-2\mu}{2^{n+1}}}XB^{\frac{2\mu+2^n-1}{2^{n+1}}}\right|^r \right|\right|\right| \\
&\leqslant \frac{1}{2\mu-1}\int_{1-\mu}^{\mu} \left|\left|\left| |A^{\nu}XB^{1-\nu}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{1-\nu}XB^{\nu}|^r \right|\right|\right| d\nu \\
&\leqslant \frac{1}{n+1}\left[n\left|\left|\left| |A^{\mu}XB^{1-\mu}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{1-\mu}XB^{\mu}|^r \right|\right|\right| + \left|\left|\left| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r \right|\right|\right|^2\right] \\
&\leqslant \left|\left|\left| |A^{\mu}XB^{1-\mu}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{1-\mu}XB^{\mu}|^r \right|\right|\right| \qquad (2.9)
\end{aligned}$$

Since

$$\left|\left|\left| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r \right|\right|\right|^2 = \lim_{\mu\to\frac{1}{2}} \frac{1}{|2\mu-1|}\left|\int_{1-\mu}^{\mu} \left|\left|\left| |A^{\nu}XB^{1-\nu}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{1-\nu}XB^{\nu}|^r \right|\right|\right| d\nu\right|,$$

the inequalities in (2.7) follow by combining the inequalities (2.8) and (2.9). So the required result is proved.

In view of the fact that the function $f(v)=|||A^vXB^{1-v}|^r|||\cdot|||A^{1-v}XB^v|^r|||$ is decreasing on the interval $[0,\frac{1}{2}]$ and increasing on the interval $[\frac{1}{2},1]$, applying Lemma 2.3 to the function $f(v)$ on the interval $[0,\mu]$ when $0<\mu\leqslant\frac{1}{2}$, on the interval $[\mu,1]$ when $\frac{1}{2}\leqslant\mu\leqslant 1$, we obtain the following refinement of Cauchy-Schwarz norm inequality.

THEOREM 2.2. *Let $A,B,X\in B(H)$ such that A,B are positive and let n be a positive integer. Then*

(1) For $0\leqslant\mu\leqslant\frac{1}{2}$, $r>0$, and for every unitarily invariant norm,

$$\begin{aligned}
&\left|\left|\left||A^\mu XB^{1-\mu}|^r\right|\right|\right|\cdot\left|\left|\left||A^{1-\mu}XB^\mu|^r\right|\right|\right| \\
&\leqslant \left|\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}\right|^r\right|\right|\right|\cdot\left|\left|\left|\left|A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|^r\right|\right|\right| \\
&\leqslant \frac{1}{\mu}\int_0^\mu \left|\left|\left||A^vXB^{1-v}|^r\right|\right|\right|\cdot\left|\left|\left||A^{1-v}XB^v|^r\right|\right|\right|dv \\
&\leqslant \frac{1}{2(n+1)}\Bigg[n\,|||AX|^r|||\cdot|||XB|^r|||+2\left|\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}\right|^r\right|\right|\right|\cdot\left|\left|\left|\left|A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|^r\right|\right|\right| \\
&\qquad\qquad +n\left|\left|\left||A^\mu XB^{1-\mu}|^r\right|\right|\right|\cdot\left|\left|\left||A^{1-\mu}XB^\mu|^r\right|\right|\right|\Bigg] \\
&\leqslant \frac{1}{2}\left[|||AX|^r|||\cdot|||XB|^r|||+\left|\left|\left||A^\mu XB^{1-\mu}|^r\right|\right|\right|\cdot\left|\left|\left||A^{1-\mu}XB^\mu|^r\right|\right|\right|\right] \\
&\leqslant |||AX|^r|||\cdot|||XB|^r|||.
\end{aligned}\tag{2.10}$$

(2) For $\frac{1}{2}\leqslant\mu\leqslant 1$, $r>0$, and for every unitarily invariant norm,

$$\begin{aligned}
&\left|\left|\left||A^\mu XB^{1-\mu}|^r\right|\right|\right|\cdot\left|\left|\left||A^{1-\mu}XB^\mu|^r\right|\right|\right| \\
&\leqslant \left|\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}\right|^r\right|\right|\right|\cdot\left|\left|\left|\left|A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|^r\right|\right|\right| \\
&\leqslant \frac{1}{1-\mu}\int_\mu^1 \left|\left|\left||A^vXB^{1-v}|^r\right|\right|\right|\cdot\left|\left|\left||A^{1-v}XB^v|^r\right|\right|\right|dv \\
&\leqslant \frac{1}{2(n+1)}\Bigg[n\,|||AX|^r|||\cdot|||XB|^r|||+2\left|\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}\right|^r\right|\right|\right|\cdot\left|\left|\left|\left|A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|^r\right|\right|\right| \\
&\qquad\qquad +n\left|\left|\left||A^\mu XB^{1-\mu}|^r\right|\right|\right|\cdot\left|\left|\left||A^{1-\mu}XB^\mu|^r\right|\right|\right|\Bigg] \\
&\leqslant \frac{1}{2}\left[|||AX|^r|||\cdot|||XB|^r|||+\left|\left|\left||A^\mu XB^{1-\mu}|^r\right|\right|\right|\cdot\left|\left|\left||A^{1-\mu}XB^\mu|^r\right|\right|\right|\right] \\
&\leqslant |||AX|^r|||\cdot|||XB|^r|||.
\end{aligned}\tag{2.11}$$

REMARK 2.1. When $n=1$, it is easy to see that Theorem 2, Theorem 3 and Theorem 4 in [5] are special cases of Theorem 2.1, and Theorem 2.2 respectively.

On the other hand, it has been proved, in [2], that the function

$$g(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$$

is also convex on $[0,1]$ with symmetry about $\nu = \frac{1}{2}$, and attains its minimum at $\nu = \frac{1}{2}$, its maximum at $\nu = 0$ and $\nu = 1$.

Notice that when $n = 2k-1$, Lemma 2.3 just is Lemma 1 in [1] as follows.

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &\leqslant \frac{1}{b-a}\int_a^b g(t)dt \leqslant \frac{1}{4k}\left[(2k-1)g(a) + 2g\left(\frac{a+b}{2}\right) + (2k-1)g(b)\right] \\ &\leqslant \frac{g(a)+g(b)}{2}. \end{aligned}$$

Hence, applying Lemma 2.3 to the function $g(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ on the interval $[\mu, 1-\mu]$ when $0 \leqslant \mu < \frac{1}{2}$, on the interval $[1-\mu, \mu]$ when $\frac{1}{2} < \mu \leqslant 1$, we obtain the following refinement of Heinz inequality in [1].

THEOREM 2.3. *Let $A, B, X \in B(H)$ such that A, B are positive and let n be a positive integer. Then for $0 \leqslant \mu \leqslant 1$, and for every unitarily invariant norm,*

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\leqslant \left|\left|\left|A^{\frac{2\mu+2^n-1}{2^{n+1}}}XB^{\frac{2^n+1-2\mu}{2^{n+1}}} + A^{\frac{2^n+1-2\mu}{2^{n+1}}}XB^{\frac{2\mu+2^n-1}{2^{n+1}}}\right|\right|\right| \\ &\leqslant \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||\,d\nu\right| \\ &\leqslant \frac{1}{n+1}\left[n|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}||| + 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right] \\ &\leqslant |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||. \end{aligned}$$

Applying Lemma 2.3 to the function $g(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ on the interval $[\mu, \frac{1}{2}]$ when $0 \leqslant \mu < \frac{1}{2}$, on the interval $[\frac{1}{2}, \mu]$ when $\frac{1}{2} \leqslant \mu \leqslant 1$, we obtain the following refinement of Heinz inequality in [1].

THEOREM 2.4. *Let $A, B, X \in B(H)$ such that A, B are positive and let n be a positive integer. Then for $0 \leqslant \mu \leqslant 1$, and for every unitarily invariant norm,*

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\leqslant \left|\left|\left|A^{\frac{1+2\mu}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{1+2\mu}{4}}\right|\right|\right| \\ &\leqslant \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||\,d\nu\right| \end{aligned}$$

$$\leqslant \frac{1}{2(n+1)}\Bigg[n|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2n\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|$$
$$+2\left|\left|\left|A^{\frac{1+2\mu}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{1+2\mu}{4}}\right|\right|\right|\Bigg]$$
$$\leqslant \frac{1}{2}\left[|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right]$$
$$\leqslant |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||.$$

Similarly, applying Lemma 2.3 to the function $g(\nu)=|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||$ on the interval $[0,\mu]$ when $0<\mu\leqslant\frac{1}{2}$, on the interval $[\mu,1]$ when $\frac{1}{2}\leqslant\mu\leqslant 1$, we obtain the following refinement of Heinz inequality in [1].

THEOREM 2.5. *Let $A,B,X\in B(H)$ such that A,B are positive and let n be a positive integer. Then*

(1) For $0\leqslant\mu\leqslant\frac{1}{2}$, and for every unitarily invariant norm,

$$|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||$$
$$\leqslant \left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}+A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right|$$
$$\leqslant \frac{1}{\mu}\int_0^{\mu}|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||\,d\nu$$
$$\leqslant \frac{1}{2(n+1)}\Bigg[n|||AX+XB|||+2\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}+A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right|$$
$$+n\left|\left|\left|A^{\mu}XB^{1-\mu}\right|^{r}\right|\right|\cdot\left|\left|\left|A^{1-\mu}XB^{\mu}\right|^{r}\right|\right|\Bigg]$$
$$\leqslant \frac{1}{2}\left[|||AX+XB|||+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right]$$
$$\leqslant |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||.$$

(2) For $\frac{1}{2}\leqslant\mu\leqslant 1$, and for every unitarily invariant norm,

$$|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||$$
$$\leqslant \left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right|$$
$$\leqslant \frac{1}{1-\mu}\int_{\mu}^{1}|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||\,d\nu$$
$$\leqslant \frac{1}{2(n+1)}\Bigg[n|||AX+XB|||+2\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right|$$

$$
\begin{aligned}
&\qquad\qquad + n\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|\Bigg] \\
&\leqslant \frac{1}{2}\left[|||AX+XB|||+\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|\right] \\
&\leqslant |||AX+XB|||.
\end{aligned}
$$

REMARK 2.2. When $n = 1$, $n = 3$ and $n = 5$, the refinements of Heinz inequalities obtained in [6, 10, 9] are special cases of the above Theorem 2.3, Theorem 2.4 and Theorem 2.5 respectively.

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