

COMPLETE REFINEMENTS OF THE BEREZIN NUMBER INEQUALITIES

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Abstract. In this paper, several refinements of the Berezin number inequalities are obtained. We generalize inequalities involving powers of the Berezin number for product of two operators acting on a reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ and also improve them. Among other inequalities, it is shown that if $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$, f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ ($t \geq 0$), then

$$\mathbf{ber}^p(AB) \leq r^p(B) \times \left(\mathbf{ber} \left(\frac{1}{\alpha} f^{\alpha p}(|A|) + \frac{1}{\beta} g^{\beta p}(|A^*|) \right) - r_0 \left(\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\alpha p/4} - \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\beta p/4} \right)^2 \right)$$

for every $p \geq 1, \alpha \geq \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1, \beta p \geq 2$ and $r_0 = \min\{\frac{1}{\alpha}, \frac{1}{\beta}\}$.

1. Introduction

Throughout this paper, a reproducing kernel Hilbert space (RKHS for short) $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of complex valued functions on a (nonempty) set Ω , which has the property that point evaluations are continuous i.e. for each $\lambda \in \Omega$ the map $f \mapsto f(\lambda)$ is a continuous linear functional on \mathcal{H} . The Riesz representation theorem ensure that for each $\lambda \in \Omega$ there is a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$, for all $f \in \mathcal{H}$. The collection $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of \mathcal{H} . If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space \mathcal{H} , then the reproducing kernel of \mathcal{H} is given by $k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z)$; (see [12, problem 37]). For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of \mathcal{H} . For a bounded linear operator A on \mathcal{H} , the function \tilde{A} defined on Ω by $\tilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$ is the Berezin symbol of A , which firstly have been introduced by Berezin [4, 5]. The Berezin set and the Berezin number of the operator A are defined by

$$\mathbf{Ber}(A) := \{\tilde{A}(\lambda) : \lambda \in \Omega\} \quad \text{and} \quad \mathbf{ber}(A) := \sup\{|\tilde{A}(\lambda)| : \lambda \in \Omega\},$$

respectively(see [13]). The Berezin number of operators A and B satisfies the property $\mathbf{ber}(\alpha A) = |\alpha| \mathbf{ber}(A)$ ($\alpha \in \mathbb{C}$) and $\mathbf{ber}(A+B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$ and $\mathbf{ber}(A) \leq \|A\|$,

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where $\| \cdot \|$ is the operator norm. The spectral radius of $A \in \mathcal{B}(\mathcal{H})$ is defined by $r(A) := \sup\{|\eta| : \eta \in \text{sp}(A)\}$. Let

$$l(A) = \inf \{ \|Ax\| : x \in \mathcal{H}, \|x\| = 1 \} = \inf \{ |\langle Ax, y \rangle| : x, y \in \mathcal{H}, \|x\| = \|y\| = 1 \}.$$

In [18], Kittaneh estimated a spectral radius inequality for any $A, B \in \mathcal{B}(\mathcal{H})$ as follows:

$$r(AB) \leq \frac{1}{4} \left(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4m(A, B)} \right), \tag{1}$$

where $m(A, B) = \min\{\|A\|\|BAB\|, \|B\|\|ABA\|\}$. Also, he showed

$$\|A^{1/2}B^{1/2}\| \leq \|AB\|^{1/2} \tag{2}$$

and

$$\|A + B\| \leq \frac{1}{2} (\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \min(\|AB\|, \|BA\|)}). \tag{3}$$

Namely, the Berezin transform have been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is widely applied in the various questions of analysis and uniquely determines the operator (i.e., for all $\lambda \in \Omega$, $\tilde{A}(\lambda) = \tilde{B}(\lambda)$ implies $A = B$). For further information about Berezin symbol we refer the reader to [2, 14, 15, 21] and references therein. Recently in [3, 9, 10, 11, 22, 23] have studied about the inequalities for the Berezin number and the numerical radius of operators. Also, some Berezin number inequalities were obtained by using the Hardy types inequalities (see [7, 8, 24]).

2. Main results

To prove our Berezin number inequalities, we need several well known lemmas.

LEMMA 1. [17] *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$. If f and g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t (t \geq 0)$, then*

$$|\langle ABx, y \rangle| \leq r(B) \|f(|A|)x\| \|g(|A^*|)y\| \tag{4}$$

for every $x, y \in \mathcal{H}$.

LEMMA 2. [1] *Let $A \in \mathcal{B}(\mathcal{H})$ be positive. Then*

$$\begin{aligned} |\langle Ax, x \rangle|^{2p} &\leq [\langle A^p x, x \rangle - \langle |A - \langle Ax, x \rangle I|^p x, x \rangle] \times [\langle A^p y, y \rangle - \langle |A - \langle Ay, y \rangle I|^p y, y \rangle] \\ &\leq \langle A^p x, x \rangle \langle A^p y, y \rangle \end{aligned} \tag{5}$$

for all $p \geq 2$ and any $x, y \in \mathcal{H}$.

Now, we show some Berezin number inequalities.

THEOREM 1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$. If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t (t \geq 0)$, then*

$$\mathbf{ber}(AB) \leq \frac{1}{2}r(B) \mathbf{ber}[f^2(|A|) + g^2(|A^*|)].$$

In particular for $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, we have

$$\mathbf{ber}(AB) \leq \frac{1}{2}r(B) \mathbf{ber}(|A|^{2\alpha} + |A^*|^{2(1-\alpha)}).$$

Proof. If we put $x = \hat{k}_\lambda$ in (4), we have

$$\begin{aligned} |\langle AB\hat{k}_\lambda, \hat{k}_\lambda \rangle| &\leq r(B) \|f(|A|)\hat{k}_\lambda\| \|g(|A^*|)\hat{k}_\lambda\| \\ &= r(B) \langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2} \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2} \\ &\leq \frac{1}{2}r(B) (\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle + \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle) \\ &= \frac{1}{2}r(B) \langle (f^2(|A|) + g^2(|A^*|))\hat{k}_\lambda, \hat{k}_\lambda \rangle \\ &\leq \frac{1}{2}r(B) \mathbf{ber}[f^2(|A|) + g^2(|A^*|)]. \end{aligned} \tag{6}$$

By taking the supremum over $\lambda \in \Omega$ we have

$$\mathbf{ber}(AB) \leq \frac{1}{2}r(B) \mathbf{ber}[f^2(|A|) + g^2(|A^*|)].$$

REMARK 1. Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$. If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t (t \geq 0)$, then from (3) and (1) for $A = I$, we have

$$\begin{aligned} \mathbf{ber}(AB) &\leq \frac{1}{2}r(B) \mathbf{ber}[f^2(|A|) + g^2(|A^*|)] \\ &\leq \frac{1}{8}(\|B\| + \|B^2\|^{1/2}) [\|f^2(|A|)\| + \|g^2(|A^*|)\| \\ &\quad + \sqrt{(\|f^2(|A|)\| - \|g^2(|A^*|)\|)^2 + 4\|f(|A|) \cdot g(|A^*|)\|^2}]. \end{aligned}$$

THEOREM 2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$. If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t (t \geq 0)$, then*

$$\mathbf{ber}^p(AB) \leq r^p(B) \mathbf{ber} \left[\frac{1}{\alpha} f^{\alpha p}(|A|) + \frac{1}{\beta} g^{\beta p}(|A^*|) \right]$$

for every $p \geq 1, \alpha \geq \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\beta p \geq 2$.

Proof. Let $\hat{k}_\lambda \in \mathcal{H}$. We have

$$\begin{aligned} |\langle AB\hat{k}_\lambda, \hat{k}_\lambda \rangle|^p &\leq r^p(B) \|f(|A|)\hat{k}_\lambda\|^p \|g(|A^*|)\hat{k}_\lambda\|^p \\ &= r^p(B) \langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{p/2} \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{p/2} \\ &\leq r^p(B) \left[\frac{1}{\alpha} \langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\alpha p/2} + \frac{1}{\beta} \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\beta p/2} \right] \\ &\leq r^p(B) \left[\frac{1}{\alpha} \langle f^{\alpha p}(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle + \frac{1}{\beta} \langle g^{\beta p}(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle \right] \\ &= r^p(B) \langle \left[\frac{1}{\alpha} f^{\alpha p}(|A|) + \frac{1}{\beta} g^{\beta p}(|A^*|) \right] \hat{k}_\lambda, \hat{k}_\lambda \rangle. \\ &\leq r^p(B) \mathbf{ber} \left(\frac{1}{\alpha} f^{\alpha p}(|A|) + \frac{1}{\beta} g^{\beta p}(|A^*|) \right). \end{aligned}$$

By taking the supremum over $\lambda \in \Omega$ we get the desired result.

In the following by using of refinements of the Cauchy-Schwarz inequality, we have an upper bound for product two operators.

THEOREM 3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$. If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t(t \geq 0)$, then*

$$\begin{aligned} |\langle AB\hat{k}_\lambda, \hat{k}_\mu \rangle| &\leq r(B) \sqrt[2p]{\langle f^{2p}(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle - \langle |f^2(|A|) - \langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle I|^p \hat{k}_\lambda, \hat{k}_\lambda \rangle} \\ &\quad \times \sqrt[2p]{\langle g^{2p}(|A^*|)\hat{k}_\mu, \hat{k}_\mu \rangle - \langle |g^2(|A^*|) - \langle g^2(|A^*|)\hat{k}_\mu, \hat{k}_\mu \rangle I|^p \hat{k}_\mu, \hat{k}_\mu \rangle} \\ &\leq r(B) \sqrt[2p]{\langle f^{2p}(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle} \sqrt[2p]{\langle g^{2p}(|A^*|)\hat{k}_\mu, \hat{k}_\mu \rangle} \end{aligned} \tag{7}$$

for all $p \geq 2$ and any $\hat{k}_\lambda, \hat{k}_\mu \in \mathcal{H}$.

Proof. Let $\hat{k}_\lambda, \hat{k}_\mu \in \mathcal{H}$. Applying (4) and (5), we have

$$\begin{aligned} |\langle AB\hat{k}_\lambda, \hat{k}_\mu \rangle| &\leq r(B) \|f(|A|)\hat{k}_\lambda\| \|g(|A^*|)\hat{k}_\mu\| \\ &\leq r(B) \langle f(|A|)\hat{k}_\lambda^2, \hat{k}_\lambda \rangle^{1/2} \langle g(|A^*|)\hat{k}_\mu^2, \hat{k}_\mu \rangle^{1/2} \\ &\leq r(B) \sqrt[2p]{\langle f(|A|)\hat{k}_\lambda^{2p}, \hat{k}_\lambda \rangle - \langle |f^2(|A|) - \langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle I|^p \hat{k}_\lambda, \hat{k}_\lambda \rangle} \\ &\quad \times \sqrt[2p]{\langle g(|A^*|)\hat{k}_\mu^{2p}, \hat{k}_\mu \rangle - \langle |g^2(|A^*|) - \langle g^2(|A^*|)\hat{k}_\mu, \hat{k}_\mu \rangle I|^p \hat{k}_\mu, \hat{k}_\mu \rangle}. \end{aligned}$$

We get the result.

COROLLARY 1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$, $p \geq 2$ and $0 \leq \alpha \leq 1$. Then*

$$\begin{aligned} |\langle AB\hat{k}_\lambda, \hat{k}_\mu \rangle| &\leq r(B) \sqrt[2p]{\langle |A|^{2p\alpha} \hat{k}_\lambda, \hat{k}_\lambda \rangle - \langle |A|^{2\alpha} - \langle |A|^{2\alpha} \hat{k}_\lambda, \hat{k}_\lambda \rangle I|^p \hat{k}_\lambda, \hat{k}_\lambda \rangle} \\ &\quad \times \sqrt[2p]{\langle |A^*|^{2p(1-\alpha)} \hat{k}_\mu, \hat{k}_\mu \rangle - \langle |A^*|^{2(1-\alpha)} - \langle |A^*|^{2(1-\alpha)} \hat{k}_\mu, \hat{k}_\mu \rangle I|^p \hat{k}_\mu, \hat{k}_\mu \rangle} \\ &\leq r(B) \sqrt[2p]{\langle |A|^{2p\alpha} \hat{k}_\lambda, \hat{k}_\lambda \rangle} \sqrt[2p]{\langle |A^*|^{2p(1-\alpha)} \hat{k}_\mu, \hat{k}_\mu \rangle}. \end{aligned} \tag{8}$$

Proof. By putting $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ ($0 \leq \alpha \leq 1$) in (7), we get the result. The next result gives an upper bound for the product of two operators based on the refinement of the Cauchy-Schwarz inequality.

THEOREM 4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$. If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ ($t \geq 0$). Then*

$$\begin{aligned} \mathbf{ber}(AB) &\leq \frac{1}{2}(\|B\| + \|B^2\|^{1/2}) [\mathbf{ber}(f^{2p}(|A|)) - l(\|[f^2(|A|) - \|f(|A|)\|^2]^p)]^{\frac{1}{2p}} \\ &\quad \times [\mathbf{ber}(g^{2p}(|A^*|)) - l(\|[g^2(|A^*|) - \|g(|A^*|)\|^2]^p)]^{\frac{1}{2p}} \end{aligned}$$

for all $p \geq 2$.

Proof. If $\hat{k}_\lambda, \hat{k}_\mu \in \mathcal{H}$, then (7) implies that

$$\begin{aligned} |\langle AB\hat{k}_\lambda, \hat{k}_\mu \rangle|^{2p} &\leq r^{2p}(B) [\langle f^{2p}(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle - \langle f^2(|A|) - \langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle I^p \hat{k}_\lambda, \hat{k}_\lambda \rangle] \\ &\quad \times [\langle g^{2p}(|A^*|)\hat{k}_\mu, \hat{k}_\mu \rangle - \langle g^2(|A^*|) - \langle g^2(|A^*|)\hat{k}_\mu, \hat{k}_\mu \rangle I^p \hat{k}_\mu, \hat{k}_\mu \rangle] \\ &\leq r^{2p}(B) [\mathbf{ber}(f^{2p}(|A|)) - \langle f^2(|A|) - \langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle I^p \hat{k}_\lambda, \hat{k}_\lambda \rangle] \\ &\quad \times [\mathbf{ber}(g^{2p}(|A^*|)) - \langle g^2(|A^*|) - \langle g^2(|A^*|)\hat{k}_\mu, \hat{k}_\mu \rangle I^p \hat{k}_\mu, \hat{k}_\mu \rangle]. \end{aligned}$$

Now, let $\hat{k}_\lambda = \hat{k}_\mu$ and taking supremum over $\lambda \in \Omega$, we have

$$\begin{aligned} \mathbf{ber}^{2p}(AB) &\leq r^{2p}(B) [\mathbf{ber}(f^{2p}(|A|)) - l(\|[f^2(|A|) - \|f(|A|)\|^2]^p)] \\ &\quad \times [\mathbf{ber}(g^{2p}(|A^*|)) - l(\|[g^2(|A^*|) - \|g(|A^*|)\|^2]^p)]. \end{aligned}$$

Now inequality (1) implies the statement.

Through following we state some refinements of Theorems 1 and 2, which based on a refinement the Young inequality that is shown in [19] by Kittaneh as follows:

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1 - \alpha)b - r_0(a^{1/2} - b^{1/2})^2 \tag{9}$$

for any $a, b > 0$, $0 \leq \alpha \leq 1$ and $r_0 = \min\{\alpha, 1 - \alpha\}$.

THEOREM 5. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$. If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ ($t \geq 0$), then*

$$\begin{aligned} \mathbf{ber}(AB) &\leq \frac{1}{2}r(B) \\ &\quad \times (\mathbf{ber}[f^2(|A|) + g^2(|A^*|)] - (\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2} - \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2})^2). \end{aligned}$$

In particular, for $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, which $0 \leq \alpha \leq 1$, we have

$$\begin{aligned} \mathbf{ber}(AB) &\leq \frac{1}{2}r(B) \\ &\quad \times (\mathbf{ber}(|A|^{2\alpha} + |A^*|^{2(1-\alpha)}) - (\langle |A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2} - \langle |A^*|^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2})^2). \end{aligned}$$

Proof. If we put $x = y = \hat{k}_\lambda$ in (4) and applying (9), we have

$$|\langle AB\hat{k}_\lambda, \hat{k}_\lambda \rangle| \tag{10}$$

$$\begin{aligned} &\leq r(B)\|f(|A|)\hat{k}_\lambda\|\|g(|A^*|)\hat{k}_\lambda\| \\ &= r(B)\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2}\langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2} \\ &\leq \frac{1}{2}r(B)(\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle + \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle \\ &\quad - (\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2} - \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2})^2) \\ &= \frac{1}{2}r(B)(\langle (f^2(|A|) + g^2(|A^*|))\hat{k}_\lambda, \hat{k}_\lambda \rangle - (\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2} - \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2})^2) \\ &\leq \frac{1}{2}r(B)(\mathbf{ber}[f^2(|A|) + g^2(|A^*|)] - (\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2} - \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{1/2})^2). \end{aligned} \tag{11}$$

By taking the supremum over $\lambda \in \Omega$ we get the desired inequality.

THEOREM 6. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$. If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t (t \geq 0)$, then*

$$\mathbf{ber}^p(AB) \leq r^p(B) \left[\mathbf{ber}\left(\frac{1}{\alpha}f^{\alpha p}(|A|) + \frac{1}{\beta}g^{\beta p}(|A^*|)\right) - r_0(\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\alpha p/4} - \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\beta p/4})^2 \right].$$

for every $p \geq 1, \alpha \geq \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1, \beta p \geq 2$ and $r_0 = \min\{\frac{1}{\alpha}, \frac{1}{\beta}\}$.

Proof. Let $\hat{k}_\lambda \in \mathcal{H}$. We have

$$\begin{aligned} &|\langle AB\hat{k}_\lambda, \hat{k}_\lambda \rangle|^p \\ &\leq r^p(B)\|f(|A|)\hat{k}_\lambda\|^p\|g(|A^*|)\hat{k}_\lambda\|^p \\ &= r^p(B)\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{p/2}\langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{p/2} \\ &\leq r^p(B) \left[\frac{1}{\alpha}\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\alpha p/2} + \frac{1}{\beta}\langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\beta p/2} \right] \\ &\leq r^p(B) \left[\frac{1}{\alpha}\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\alpha p/2} + \frac{1}{\beta}\langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\beta p/2} \right. \\ &\quad \left. - r_0(\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\alpha p/4} - \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\beta p/4})^2 \right] \\ &\leq r^p(B) \left[\frac{1}{\alpha}\langle f^{\alpha p}(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle + \frac{1}{\beta}\langle g^{\beta p}(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle \right. \\ &\quad \left. - r_0(\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\alpha p/4} - \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\beta p/4})^2 \right] \\ &\leq r^p(B) \left[\frac{1}{\alpha}\langle f^{\alpha p}(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle + \frac{1}{\beta}\langle g^{\beta p}(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle \right. \\ &\quad \left. - r_0(\langle f^2(|A|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\alpha p/4} - \langle g^2(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda \rangle^{\beta p/4})^2 \right] \end{aligned}$$

$$\begin{aligned}
 &= r^p(B) \left(\left\langle \frac{1}{\alpha} f^{\alpha p}(|A|) + \frac{1}{\beta} g^{\beta p}(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right. \\
 &\quad \left. - r_0(\langle f^2(|A|) \hat{k}_\lambda, \hat{k}_\lambda \rangle^{\alpha p/4} - \langle g^2(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \rangle^{\beta p/4})^2 \right) \\
 &\leq r^p(B) \left[\mathbf{ber} \left(\frac{1}{\alpha} f^{\alpha p}(|A|) + \frac{1}{\beta} g^{\beta p}(|A^*|) \right) \right. \\
 &\quad \left. - r_0(\langle f^2(|A|) \hat{k}_\lambda, \hat{k}_\lambda \rangle^{\alpha p/4} - \langle g^2(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \rangle^{\beta p/4})^2 \right].
 \end{aligned}$$

By taking the supremum over $\lambda \in \Omega$ we get the desired result.

3. Number Berezin inequalities involving off diagonal matrices

In this section, we improve and extend some Berezin number inequalities for 2×2 off diagonal matrices by nonnegative increasing convex functions. We recall that the polarization identity says that,

$$\langle x, y \rangle = \frac{1}{4} \sum_{i=1}^3 i^k \|x + i^k y\|^2 \quad (x, y \in \mathcal{H}). \tag{12}$$

For our goals, we need to the following lemmas.

LEMMA 3. [2] *Let $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in \mathcal{B}(\mathcal{H}_2)$. Then the following statements hold:*

- (a) $\mathbf{ber} \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) \leq \max\{\mathbf{ber}(A), \mathbf{ber}(D)\};$
- (b) $\mathbf{ber} \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{2}(\|B\| + \|C\|);$
- (c) $\mathbf{ber}(A) = \sup_{\theta \in \mathbb{R}} \mathbf{ber}(Re(e^{i\theta} A)).$

LEMMA 4. [16] *Let h be a nonnegative nondecreasing convex function on $[0, \infty)$ and let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$h \left(\left\| \frac{A+B}{2} \right\| \right) \leq \left\| \frac{h(A) + h(B)}{2} \right\|.$$

THEOREM 7. *Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$ and f, g be nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$). Then*

$$h(\mathbf{ber}(T)) \leq \frac{1}{4} \|h(f^2(|C|)) + h(g^2(|C|))\| + \frac{1}{4} \|h(f^2(|B|)) + h(g^2(|B|))\|. \tag{13}$$

Proof. Let $B = U|B|$ and $C = V|C|$ be the polar decomposition of operators B and C . Then $T = W|T| = \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} \begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}$ is the polar decomposition of T .

For any $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{k}_{(\lambda_1, \lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel in $\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2)$. Then

$$\begin{aligned} & \langle \operatorname{Re} e^{i\theta} T \hat{k}_{(\lambda_1, \lambda_2)}, \hat{k}_{(\lambda_1, \lambda_2)} \rangle \\ &= \operatorname{Re} \langle e^{i\theta} W |T| \hat{k}_{(\lambda_1, \lambda_2)}, \hat{k}_{(\lambda_1, \lambda_2)} \rangle \\ &= \operatorname{Re} \langle e^{i\theta} W f(|T|) g(|T|) \hat{k}_{(\lambda_1, \lambda_2)}, \hat{k}_{(\lambda_1, \lambda_2)} \rangle \\ &= \operatorname{Re} \langle e^{i\theta} g(|T|) \hat{k}_{(\lambda_1, \lambda_2)}, f(|T|) W^* \hat{k}_{(\lambda_1, \lambda_2)} \rangle \\ &= \operatorname{Re} \left\langle e^{i\theta} \begin{bmatrix} g(|C|) & 0 \\ 0 & g(|B|) \end{bmatrix} \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}, \begin{bmatrix} f(|C|) & 0 \\ 0 & f(|B|) \end{bmatrix} \begin{bmatrix} 0 & V^* \\ U^* & 0 \end{bmatrix} \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix} \right\rangle \\ &= \operatorname{Re} \langle e^{i\theta} (g(|C|)k_{\lambda_1}, g(|B|)k_{\lambda_2}), (f(|C|)V^*k_{\lambda_2}, f(|B|)U^*k_{\lambda_1}) \rangle \\ &= \operatorname{Re} (\langle e^{i\theta} g(|C|)k_{\lambda_1}, f(|C|)V^*k_{\lambda_2} \rangle + \langle e^{i\theta} g(|B|)k_{\lambda_2}, f(|B|)U^*k_{\lambda_1} \rangle) \\ &= \frac{1}{4} \left(\|e^{i\theta} g(|C|)k_{\lambda_1} + f(|C|)V^*k_{\lambda_2}\|^2 - \|e^{i\theta} g(|C|)k_{\lambda_1} - f(|C|)V^*k_{\lambda_2}\|^2 \right) \\ &+ \frac{1}{4} \left(\|e^{i\theta} g(|B|)k_{\lambda_2} + f(|B|)U^*k_{\lambda_1}\|^2 - \|e^{i\theta} g(|B|)k_{\lambda_2} - f(|B|)U^*k_{\lambda_1}\|^2 \right) \\ & \quad \text{(by (12))} \\ &\leq \frac{1}{4} \|e^{i\theta} g(|C|)k_{\lambda_1} + f(|C|)V^*k_{\lambda_2}\|^2 + \frac{1}{4} \|e^{i\theta} g(|B|)k_{\lambda_2} + f(|B|)U^*k_{\lambda_1}\|^2 \\ &= \frac{1}{4} \left\| [e^{i\theta} g(|C|)f(|C|)V^*] \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix} \right\|^2 + \frac{1}{4} \left\| [e^{i\theta} g(|B|)f(|B|)U^*] \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix} \right\|^2 \\ &\leq \frac{1}{4} \| [e^{i\theta} g(|C|)f(|C|)V^*] \|^2 + \frac{1}{4} \| [e^{i\theta} g(|B|)f(|B|)U^*] \|^2 \\ &= \frac{1}{4} \left\| [e^{i\theta} g(|C|)f(|C|)V^*] \begin{bmatrix} e^{i\theta} g(|C|) \\ V f(|C|) \end{bmatrix} \right\| + \frac{1}{4} \left\| [e^{i\theta} g(|B|)f(|B|)U^*] \begin{bmatrix} U f(|B|) \\ e^{-i\theta} g(|B|) \end{bmatrix} \right\| \\ &= \frac{1}{4} \|g^2(|C|) + f(|C|)V^*V f(|C|)\| + \frac{1}{4} \|f(|B|)U^*U f(|B|) + g^2(|B|)\| \\ &= \frac{1}{4} \|f^2(|C|) + g^2(|C|)\| + \frac{1}{4} \|f^2(|B|) + g^2(|B|)\|. \end{aligned}$$

By taking the supremum over all $\lambda \in \Omega$, Lemma 3(c) and applying Lemma 4 for any nondecreasing convex function h , we get the desired result.

COROLLARY 2. Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$. Then for any $\alpha \in [0, 1]$ and $p \geq 1$,

$$\operatorname{ber}^p \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{4} \| |B|^{2p\alpha} + |B|^{2p(1-\alpha)} \| + \frac{1}{4} \| |C|^{2p\alpha} + |C|^{2p(1-\alpha)} \|.$$

Proof. By putting $h(t) = t^p$, $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ in inequality (13), we get the desired inequality.

4. Berezin number and Cartesian decomposition

In this section, our purpose is to give an upper bound for Berezin number in terms of the Cartesian decomposition of operators on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$. Before giving the results, we need several well known lemmas.

LEMMA 5. [20] *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then for $x \in \mathcal{H}$*

- (i) $\langle A^p x, x \rangle \geq \|x\|^{2(1-p)} \langle Ax, x \rangle^p$, if $p \geq 1$;
- (ii) $\langle A^p x, x \rangle \leq \|x\|^{2(1-p)} \langle Ax, x \rangle^p$, if $0 < p < 1$.

LEMMA 6. [17] *Let $A \in \mathcal{B}(\mathcal{H})$ and $0 \leq p \leq 1$. Then for $x, y \in \mathcal{H}$*

$$|\langle Ax, y \rangle|^2 \leq \langle |A|^{2p} x, x \rangle \langle |A^*|^{2(1-p)} y, y \rangle.$$

LEMMA 7. [6] *Let x_n be a positive real number, $1 \leq n \leq k$. Then for each $p \geq 1$*

$$\left(\sum_{n=1}^k x_n \right)^p \leq k^{p-1} \sum_{n=1}^k x_n^p.$$

Now, we are ready to give our results.

THEOREM 8. *Let $A_n \in \mathcal{B}(\mathcal{H})$ have the Cartesian decomposition $A_n = B_n + iC_n$ for $n = 1, \dots, k$ and $p \geq 1$. Then*

$$\mathbf{ber}^p \left(\sum_{n=1}^k A_n \right) \leq (\sqrt{2k})^{p-1} \sup_{\lambda \in \Omega} \left[\sum_{n=1}^k \left(\widetilde{|B_n|^{2p}}(\lambda) + \widetilde{|C_n|^{2p}}(\lambda) \right)^{\frac{1}{2}} \right]^p$$

for $\lambda \in \Omega$.

Proof. Let $\widehat{k}_\lambda \in \mathcal{H}(\Omega)$. Then

$$\left| \left\langle \sum_{n=1}^k A_n \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^p \leq \left(\sum_{n=1}^k \left(\langle B_n \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^2 + \langle C_n \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^2 \right)^{\frac{1}{2}} \right)^p$$

for $\lambda \in \Omega$. Applying Lemma 6 for $\alpha = 1$, we get

$$\left| \left\langle \sum_{n=1}^k A_n \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^p \leq \left(\sum_{n=1}^k \left(\langle |B_n|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \langle |C_n|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right)^{\frac{1}{2}} \right)^p$$

for $\lambda \in \Omega$. Using Lemma 7 and Lemma 5, we obtain

$$\begin{aligned} \left| \left\langle \sum_{n=1}^k A_n \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^p &\leq k^{p-1} \sum_{n=1}^k \left(\langle |B_n|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \langle |C_n|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right)^{\frac{p}{2}} \\ &\leq (\sqrt{2}k)^{p-1} \sum_{n=1}^k \left(\langle |B_n|^{2p} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^p + \langle |C_n|^{2p} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^p \right)^{\frac{1}{2}} \\ &\leq (\sqrt{2}k)^{p-1} \sum_{n=1}^k \left(\langle |B_n|^{2p} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \langle |C_n|^{2p} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right)^{\frac{1}{2}} \end{aligned}$$

for $\lambda \in \Omega$. Taking supremum over $\lambda \in \Omega$, we have

$$\sup_{\lambda \in \Omega} \left| \left\langle \sum_{n=1}^k A_n \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^p \leq (\sqrt{2}k)^{p-1} \sup_{\lambda \in \Omega} \left[\sum_{n=1}^k \left(\langle |B_n|^{2p} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \langle |C_n|^{2p} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right)^{\frac{1}{2}} \right]$$

and so

$$\mathbf{ber}^p \left(\sum_{n=1}^k A_n \right) \leq (\sqrt{2}k)^{p-1} \sup_{\lambda \in \Omega} \left[\sum_{n=1}^k \left(\widetilde{|B_n|^{2p}}(\lambda) + \widetilde{|C_n|^{2p}}(\lambda) \right)^{\frac{1}{2}} \right].$$

THEOREM 9. Let $A_n \in \mathcal{B}(\mathcal{H})$ have the Cartesian decomposition $A_n = B_n + iC_n$ for $n = 1, \dots, k$ and $p \geq 1$. Then

$$\mathbf{ber}^p \left(\sum_{n=1}^k A_n \right) \leq k^{p-1} 2^{\frac{p}{2}-1} \sup_{\lambda \in \Omega} \left[\sum_{n=1}^k \left(\widetilde{|B_n + C_n|^{2p}}(\lambda) + \widetilde{|B_n - C_n|^{2p}}(\lambda) \right)^{\frac{1}{2}} \right]$$

for $\lambda \in \Omega$.

Proof. Let $\widehat{k}_\lambda \in \mathcal{H}(\Omega)$. We have

$$\begin{aligned} \left| \left\langle \sum_{n=1}^k A_n \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^p &\leq \left(\sum_{n=1}^k \left(\langle B_n \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^2 + \langle C_n \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^2 \right)^{\frac{1}{2}} \right)^p \\ &\leq \left(\sum_{n=1}^k \left(\frac{1}{2} \left(\langle (B_n + C_n) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^2 + \langle (B_n - C_n) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^2 \right) \right)^{\frac{1}{2}} \right)^p \end{aligned}$$

for $\lambda \in \Omega$. Using Lemma 7 and Lemma 6, respectively, we have

$$\begin{aligned} \left| \left\langle \sum_{n=1}^k A_n \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^p &\leq k^{p-1} 2^{-\frac{p}{2}} \sum_{n=1}^k \left(\langle (B_n + C_n) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^2 + \langle (B_n - C_n) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^2 \right)^{\frac{p}{2}} \\ &\leq k^{p-1} 2^{-\frac{p}{2}} \sum_{n=1}^k \left(\langle |B_n + C_n|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \langle |B_n - C_n|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right)^{\frac{p}{2}} \end{aligned}$$

for $\lambda \in \Omega$. Then, applying Lemma 7 and Lemma 5, we obtain

$$\begin{aligned} \left| \left\langle \sum_{n=1}^k A_n \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^p &\leq k^{p-1} 2^{\frac{p}{2}-1} \sum_{n=1}^k \left(\left\langle |B_n + C_n|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^p + \left\langle |B_n - C_n|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^p \right)^{\frac{1}{2}} \\ &\leq k^{p-1} 2^{\frac{p}{2}-1} \sum_{n=1}^k \left(\left\langle |B_n + C_n|^{2p} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle |B_n - C_n|^{2p} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right)^{\frac{1}{2}} \end{aligned}$$

for $\lambda \in \Omega$. Taking supremum over $\lambda \in \Omega$, we reach that

$$\mathbf{ber}^p \left(\sum_{n=1}^k A_n \right) \leq k^{p-1} 2^{\frac{p}{2}-1} \sup_{\lambda \in \Omega} \left[\sum_{n=1}^k \left(\widetilde{|B_n + C_n|^{2p}}(\lambda) + \widetilde{|B_n - C_n|^{2p}}(\lambda) \right)^{\frac{1}{2}} \right].$$

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