

SOME RESULTS FOR THE COMMUTATORS OF GENERALIZED HAUSDORFF OPERATOR

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Abstract. In this paper, we study commutator of generalized Hausdorff operator on function spaces. We mainly discuss the continuity criteria for such commutator operator when the symbol functions are either from central-*BMO* or Lipschitz class of functions.

1. Introduction

To study the continuity properties of Hausdorff operators on function spaces, an appropriate point for opening discussion is the one dimensional Hausdorff operator:

$$\mathcal{H}_{\Phi}f(x) = \int_0^{\infty} \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt, \quad x \in \mathbb{R}, \quad (1.1)$$

where $\Phi \in L^1(\mathbb{R})$. Also, in the interest of convenience, it is usually assumed that functions f are initially in Schwartz class $\mathcal{S}(\mathbb{R})$. In fact, there are many popular and important operators in analysis which become special cases of \mathcal{H}_{Φ} , if the function Φ is suitably chosen. Among many others, here we mention a few namely, the Cesàro operator [21], the Hardy operator and its adjoint operator [8]. The survey articles [6, 24] may serve as an important source of information about history and recent advances in the study of this operator.

In the case of n -dimensional Euclidian space $\mathbb{R}^n (n \geq 2)$, an extension of (1.1) was studied in [22] and is given by

$$H_{\Phi,A}f(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f(A(y)x) dy, \quad (1.2)$$

where $A(y)$ is an n -th order square matrix, which satisfy $\det A(y) \neq 0$ almost everywhere in the support of $\Phi \in L^1_{\text{loc}}(\mathbb{R}^n)$. Using duality approach, Lerner and Liflayand [22] obtained the boundedness of $H_{\Phi,A}$ on real Hardy space $H^1(\mathbb{R}^n)$, after which the same problem was reconsidered in [7, 23, 31].

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In the above definition (1.2), if one considers $A(y) = \text{diag}[1/|y|, 1/|y|, \dots, 1/|y|]$, then we obtain a natural n -dimensional extension of (1.1) given by

$$H_{\Phi}f(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy. \quad (1.3)$$

Like one dimensional Hausdorff operator [25, 26], the operators H_{Φ} and $H_{\Phi,A}$ also enjoy a variety of estimates on different function spaces; see [3, 4, 5, 10, 11, 14, 15, 16, 17, 20, 32, 33] and the references therein.

Likewise the study of boundedness properties of commutators operators on function spaces is another most important problem in analysis. The study can be utilized for the characterization of function spaces, in the regularity theory to special kind of partial differential equations (PDEs) and in the well posedness problems of solution to many PDEs. Therefore, the study of boundedness results for the commutators of Hausdorff operators is as important as the study of Hausdorff operators itself. In an exploratory research, one can find a very few papers discussing the boundedness of commutators of various Hausdorff operators [9, 18, 19, 20, 34], except that of $H_{\Phi,A}$. Recently, in [17], the author and his collaborator defined the commutator generated by generalized Hausdorff operator $H_{\Phi,A}$ and locally integrable function b as:

$$H_{\Phi,A}^b(f)(x) = b(x)H_{\Phi,A}(f)(x) - H_{\Phi,A}(bf)(x),$$

and constructed weighted estimates for it on central Morrey spaces, when $b \in \dot{CMO}(\mathbb{R}^n)$. Also, they raised an open question regarding $L^p(\mathbb{R}^n)$ boundedness of $H_{\Phi,A}^b$. Here, we give partially positive answer to this question by establishing $L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ estimates when the symbol function belongs to the Lipschitz class of functions. However, in the case $b \in \dot{CMO}(\mathbb{R}^n)$, the question of $L^p(\mathbb{R}^n)$ boundedness of $H_{\Phi,A}^b$ still remains open.

In this paper, we give Lipschitz estimates for the commutators of $H_{\Phi,A}$ on Lebesgue, Morrey and Herz-Type spaces and thus generalize some results presented in [11, 18]. In addition, when $b \in \dot{\Lambda}_{\beta}(\mathbb{R}^n)$, we obtain $L^p \rightarrow \dot{F}_p^{\beta,\infty}$ boundedness for $H_{\Phi,A}^b$, where $\dot{F}_p^{\beta,\infty}$ is the homogeneous Triebel-Lizorkin space. Also, we estimate $H_{\Phi,A}^b$ on Herz-type spaces when $b \in \dot{CMO}(\mathbb{R}^n)$. The significance of our results lies in the fact that Herz-Type spaces are used in the characterization of multiplier on the Hardy spaces [1] and in the study of certain kind of PDEs [27].

The plot of this paper is as follows. The second section contains some basic definitions and notations likewise some necessary lemmas which will be used in the succeeding sections. Lipschitz estimates for $H_{\Phi,A}^b$ are established in the third section. While results regarding central- BMO estimates for $H_{\Phi,A}^b$ on Herz-type spaces are stated and proved in the last section.

2. Preliminaries

In 1938, Morrey [29] studied the local behavior of second order elliptic and parabolic PDEs and introduced a function space what is called Morrey space.

DEFINITION 2.1. Let $1 \leq p < \infty$, $0 \leq \lambda \leq n$. The Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is the set of all functions f

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

satisfying

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0, x_0 \in \mathbb{R}^n} \left(\frac{1}{r^\lambda} \int_{Q(x_0,r)} |f(x)|^p dx \right)^{1/p},$$

where $Q = Q(x_0, r)$ denotes the cube centered at x_0 with side length r along the coordinate axes.

It is easy to see that $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. If $n < \lambda$, then we have $L^{p,\lambda}(\mathbb{R}^n) = 0$. Hence, we only consider the case $0 < \lambda < n$.

Let $B_k := \{x \in \mathbb{R}^n : |x| < 2^k\}$, $C_k = B_k/B_{k-1}$ for $k \in \mathbb{Z}$. Then we will consider the following definition of homogeneous Herz space.

DEFINITION 2.2. Let $\alpha \in \mathbb{R}, 0 < p, q < \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) := \left\{ f \in L^q_{\text{loc}}(\mathbb{H}^n / \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\|_{L^q(C_k)}^p \right\}^{1/p}.$$

It is easy to verify that $\dot{K}_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. Hence, Herz space can be considered as an extension of Lebesgue space $L^p(\mathbb{R}^n)$.

Similarly the homogenous Herz-Morrey space $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ is given by the following definition.

DEFINITION 2.3. Let $\alpha \in \mathbb{R}, 0 < p, q < \infty, \lambda \geq 0$. The homogeneous Herz-Morrey space $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) := \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n / \{0\}) : \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\|_{L^q(C_k)}^p \right\}^{1/p}.$$

Obviously, $M\dot{K}_{p,q}^{\alpha,0}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $L^{q,\lambda}(\mathbb{R}^n) \subset M\dot{K}_{q,q}^{0,\lambda}(\mathbb{R}^n)$.

DEFINITION 2.4. Let $1 < q < \infty$. A function $f \in L^q_{loc}(\mathbb{R}^n)$ is said to belong to the space $\dot{C}MO^q(\mathbb{R}^n)$ if

$$\|f\|_{\dot{C}MO^q(\mathbb{R}^n)} = \sup_{r>0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx \right)^{1/q} < \infty,$$

where $f_B = \frac{1}{|B|} \int_B f(x) dx$ denote the average value of f over B . Obviously, we have $BMO(\mathbb{R}^n) \subset \dot{C}MO^q(\mathbb{R}^n)$ for all $1 \leq q < \infty$, and $\dot{C}MO^q(\mathbb{R}^n) \subset \dot{C}MO^p(\mathbb{R}^n)$, $1 \leq p < q < \infty$.

DEFINITION 2.5. Let $0 < \beta < 1$. The Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ is defined by

$$\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} = \sup_{x,h \in \mathbb{R}^n} \frac{|f(x+h) - f(x)|}{|h|^\beta} < \infty.$$

Next, for a non-singular matrix B , we consider the following definition of norm

$$\|B\| = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{|Bx|}{|x|}, \tag{2.1}$$

which implies that

$$\|B\|^{-n} \leq |\det(B^{-1})| \leq \|B^{-1}\|^n. \tag{2.2}$$

Finally, for $0 \leq \beta < n$, we define the fractional maximal function as

$$M_\beta f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\beta}{n}}} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q containing x . Notice that when $\beta = 0$, we obtain the usual Hardy Littlewood maximal function $M = M_0$.

This finishes the streak of definitions concerning function spaces, norm of a matrix and maximal function. We are now in position to state some lemmas which will be used to prove our main results.

LEMMA 2.6. ([30]) Let $0 < \beta < 1$ and $f \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then for any cube $Q \subset \mathbb{R}^n$,

$$\sup_{x \in Q} |f(x) - f_Q| \leq C |Q|^{\frac{\beta}{n}} \|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}.$$

LEMMA 2.7. Let $0 < \beta < 1$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then

$$\begin{aligned} &M(H_{\Phi,A}^b f)(x) \\ &\leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{1, |\det A^{-1}(y)|^{\beta/n}\} (1 + \|A(y)\|^\beta) M_\beta(f)(A(y)x) dy. \end{aligned}$$

Proof: Let us consider a cube $Q \subset \mathbb{R}^n$ such that $x \in Q$ and

$$\begin{aligned} \frac{1}{|Q|} \int_Q |H_{\Phi, A}^b f(z)| dz &= \frac{1}{|Q|} \int_Q \left| \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} (b(z) - b(A(y)z)) f(A(y)z) dy \right| dz \\ &\leq \frac{1}{|Q|} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \int_Q |(b(z) - b(A(y)z)) f(A(y)z)| dz dy \\ &\leq \frac{1}{|Q|} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \int_Q |(b(z) - b_Q) f(A(y)z)| dz dy \\ &\quad + \frac{1}{|Q|} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \int_Q |(b_Q - b_{A(y)Q}) f(A(y)z)| dz dy \\ &\quad + \frac{1}{|Q|} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \int_Q |(b(A(y)z) - b_{A(y)Q}) f(A(y)z)| dz dy \\ &= I_1 + I_2 + I_3. \end{aligned}$$

In the approximation of I_1 , we use Lemma 2.6 to obtain

$$\begin{aligned} I_1 &\leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\frac{1}{|Q|^{1-\beta/n}} \int_Q |f(A(y)z)| dz \right) dy \\ &\leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\frac{1}{|A(y)Q|^{1-\beta/n}} \int_{A(y)Q} |f(z)| dz \right) |\det A^{-1}(y)|^{\beta/n} dy \\ &\leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{\beta/n} M_\beta(f)(A(y)x) dy. \end{aligned}$$

In order to estimate I_2 , we have to approximate $|b_Q - b_{A(y)Q}|$. For $0 < \beta < 1$, we obtain

$$\begin{aligned} |b_Q - b_{A(y)Q}| &\leq \frac{1}{|Q|} \int_Q |b(z) - b_{A(y)Q}| dz \\ &\leq \frac{1}{|Q|} \frac{1}{|A(y)Q|} \int_Q \int_{A(y)Q} |b(z) - b(t)| dt dz \\ &\leq \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left(\frac{1}{|Q|} \int_Q |x|^\beta dz + \frac{1}{|A(y)Q|} \int_{A(y)Q} |t|^\beta dt \right) \\ &\leq C |Q|^{\beta/n} \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} (1 + \|A(y)\|^\beta) \end{aligned}$$

Therefore,

$$\begin{aligned} I_2 &\leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} (1 + \|A(y)\|^\beta) \left(\frac{1}{|Q|^{1-\beta/n}} \int_Q |f(xA(y))| dz \right) dy \\ &\leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} (1 + \|A(y)\|^\beta) |\det A^{-1}(y)|^{\beta/n} M_\beta(f)(A(y)x) dy. \end{aligned}$$

It remains to estimate I_3 . By virtue of Lemma 2.6, we get

$$\begin{aligned} I_3 &= \frac{1}{|Q|} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \int_Q |(b(A(y)z) - b_{A(y)Q})f(A(y)z)| dz dy \\ &= \frac{1}{|Q|} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A(y)|^{-1} \int_{A(y)Q} |b(z) - b_{A(y)Q}| |f(z)| dz dy \\ &\leq C \|b\|_{\dot{\lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\frac{1}{|A(y)Q|^{1-\beta/n}} \int_{A(y)Q} |f(z)| dz \right) dy \\ &\leq C \|b\|_{\dot{\lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} M_\beta(f)(A(y)x) dy. \end{aligned}$$

We combine the estimates for I_1 , I_2 and I_3 , to have

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |H_{\Phi,A}^b f(z)| dz \\ &\leq C \|b\|_{\dot{\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \max\{1, |\det A^{-1}(y)|^{\beta/n}\} (1 + \|A(y)\|^\beta) M_\beta(f)(A(y)x). \end{aligned}$$

Finally, taking supremum over all Q such that $x \in Q$, we get the desired result.

LEMMA 2.8. ([2]) *Let $0 < \beta < n$, $1 < p < n/\beta$, $0 < \lambda < n - \beta p$ and $1/q = 1/p - \beta/(n - \lambda)$. Then M_β is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

LEMMA 2.9. ([28]) *Let $0 < p_1 \leq \infty, p_1 \leq p_2 \leq \infty, 0 < \beta < 1 < q_1 < n/\beta$ and $1/q_1 - 1/q_2 = \beta/n$. If $-n/q_1 + \beta < \alpha < n(1 - 1/q_1)$, then*

$$\|M_\beta f\|_{K_{q_2}^{\alpha,p_2}(\mathbb{R}^n)} \leq C \|f\|_{K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}.$$

LEMMA 2.10. ([13]) *Let $0 < p_1 \leq \infty, p_1 \leq p_2 \leq \infty, 0 < \beta < 1 < q_1 < n/\beta, 1/q_1 - 1/q_2 = \beta/n$ and $\lambda > 0$. If $-n/q_1 + \beta + \lambda < \alpha < n(1 - 1/q_1) + \lambda$, then*

$$\|M_\beta f\|_{MK_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \leq C \|f\|_{MK_{p_1,q_1}^{\alpha,\lambda}(\mathbb{R}^n)}.$$

LEMMA 2.11. ([30]) *Let $0 < \beta < 1 < p < \infty$, then*

$$\|f\|_{F_p^{\beta,\infty}(\mathbb{R}^n)} \approx \left\| \sup_{Q \ni x} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(z) - f_Q| dz \right\|_{L^p(\mathbb{R}^n)}.$$

3. Lipschitz Estimates for $H_{\Phi,A}^b$ on Function Spaces

As we stated in the introduction, this section is centered on obtaining estimates for $H_{\Phi,A}^b$ on function spaces. In this regard, our main results are as below.

3.1. Main Results

THEOREM 3.1. *Let $0 < \beta < 1 < p < n/\beta$, $0 < \lambda < n - \beta p$ and $1/q = 1/p - \beta/(n - \lambda)$. If $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then*

$$\|H_{\Phi,A}^b f\|_{L^{q,\lambda}(\mathbb{R}^n)} \leq CK_1 \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)},$$

where K_1 is

$$\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{|\det A^{-1}(y)|^{1/q-\lambda/q}, |\det A^{-1}(y)|^{\beta/n+1/q-\lambda/q}\}(1 + \|A(y)\|^\beta) dy.$$

THEOREM 3.2. *Let $0 < \beta < 1 < p < n/\beta$, and $1/q = 1/p - \beta/n$. If $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then*

$$\|H_{\Phi,A}^b f\|_{L^q(\mathbb{R}^n)} \leq CK_2 \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)},$$

where K_2 is

$$\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{|\det A^{-1}(y)|^{1/q}, |\det A^{-1}(y)|^{1/p}\}(1 + \|A(y)\|^\beta) dy,$$

THEOREM 3.3. *Let $0 < p_1 \leq \infty, p_1 \leq p_2 \leq \infty, 0 < \beta < 1 < q_1 < n/\beta, 1/q_1 - 1/q_2 = \beta/n$ and $\lambda > 0$. If $-n/q_1 + \beta + \lambda < \alpha < n(1 - 1/q_1) + \lambda$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then*

$$\|H_{\Phi,A}^b f\|_{MK_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \leq CK_3 \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{MK_{p_1,q_1}^{\alpha,\lambda}(\mathbb{R}^n)},$$

where K_3 is

$$\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{|\det A^{-1}(y)|^{1/q_2}, |\det A^{-1}(y)|^{1/q_1}\}(1 + \|A(y)\|^\beta) G_{\alpha,\lambda}(y) dy,$$

and

$$G_{\alpha,\lambda}(y) = \begin{cases} 1 + \log_2(\|A(y)\| \|A^{-1}(y)\|), & \alpha = \lambda, \\ \|A^{-1}(y)\|^{\alpha-\lambda}, & \alpha > \lambda, \\ \|A(y)\|^{\lambda-\alpha}, & \alpha < \lambda. \end{cases} \tag{3.1}$$

THEOREM 3.4. *Let $0 < p_1 \leq \infty, p_1 \leq p_2 \leq \infty, 0 < \beta < 1 < q_1 < n/\beta$ and $1/q_1 - 1/q_2 = \beta/n$. If $-n/q_1 + \beta < \alpha < n(1 - 1/q_1)$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then*

$$\|H_{\Phi,A}^b f\|_{\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)} \leq CK_4 \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)},$$

where K_4 is

$$\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{|\det A^{-1}(y)|^{1/q_2}, |\det A^{-1}(y)|^{1/q_1}\}(1 + \|A(y)\|^\beta) \tilde{G}_\alpha(y) dy,$$

and

$$\tilde{G}_\alpha(y) = \begin{cases} 1 + \log_2(\|A(y)\| \|A^{-1}(y)\|), & \alpha = 0, \\ \|A^{-1}(y)\|^\alpha, & \alpha > 0, \\ \|A(y)\|^{-\alpha}, & \alpha < 0. \end{cases} \tag{3.2}$$

THEOREM 3.5. *Let $0 < \beta < 1 < p < \infty$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then*

$$\|H_{\Phi,A}^b f\|_{F_p^{\beta,\infty}(\mathbb{R}^n)} \leq CK_5 \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)},$$

where K_5 is

$$\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{|\det A(y)|^{-1/p}, |\det A(y)|^{1+\beta/n-1/p}\} (1 + \|A(y)\|^\beta) dy.$$

3.2. Proof of the Main Results

Since Lemma 2.7 will be used frequently in the proofs of our main results for this section, therefore, we sometimes use the following notation

$$\phi(y) = \frac{|\Phi(y)|}{|y|^n} \max\{1, |\det A^{-1}(y)|^{\beta/n}\} (1 + \|A(y)\|^\beta),$$

for our convenience.

Proof of Theorem 3.1: In view of Lemma 2.7 and the Minkowski inequality, we have

$$\|M(H_{\Phi,A}^b f)(\cdot)\|_{L^{q,\lambda}(\mathbb{R}^n)} \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y) \|M_\beta(f)(A(y)\cdot)\|_{L^{q,\lambda}(\mathbb{R}^n)} dy.$$

Using the scaling argument and the fact that $|H_{\Phi,A}^b f(x)| \leq M(H_{\Phi,A}^b f)(x)$ a.e., we get

$$\|H_{\Phi,A}^b f\|_{L^{q,\lambda}(\mathbb{R}^n)} \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_\beta f\|_{L^{p,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y) |\det A^{-1}(y)|^{1/q-\lambda/q} dy.$$

Lastly, we apply Lemma 2.8 to obtain the desired result.

Proof of Theorem 3.2: Following the same procedure as followed in the proof of Theorem 3.1, the proof of this Theorem can be easily obtained.

Proof of Theorem 3.3: Making use of Lemma 2.7 and the Minkowski inequality, one has

$$\begin{aligned}
 & \|M(H_{\Phi,A}^b f)(\cdot)\|_{MK_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \\
 & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y) \|M_\beta f(A(y)\cdot)\|_{MK_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} dy \\
 & = C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y) \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|M_\beta f(A(y)\cdot)\|_{L^q(C_k)}^p \right\}^{1/p} dy \\
 & = C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y) |\det A^{-1}(y)|^{1/q_2} \\
 & \quad \times \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|M_\beta f(\cdot)\|_{L^q(A(y)C_k)}^p \right\}^{1/p} dy. \tag{3.3}
 \end{aligned}$$

In order to estimate $\|M_\beta f(\cdot)\|_{L^q(A(y)C_k)}$, we follow the method used in [33]. Thus, by definition of C_k and (2.2) it is easy to see that

$$A(y)C_k \subset \{x : \|A^{-1}(y)\|^{-1}2^{k-1} < |x| < \|A(y)\|2^k\}.$$

Next, for any y in the support of Φ there exist an integer l such that

$$2^l < \|A^{-1}(y)\|^{-1} < 2^{l+1}. \tag{3.4}$$

Furthermore, the relation $\|A^{-1}(y)\|^{-1} \leq \|A(y)\|$ implies the existence of non-negative integer m such that

$$2^{l+m} < \|A(y)\| < 2^{l+m+1}. \tag{3.5}$$

Inequalities (3.4) and (3.5) define the bounds for m , that is

$$\log_2(\|A(y)\| \|A^{-1}(y)\|/2) < m < \log_2(2\|A(y)\| \|A^{-1}(y)\|),$$

and lead us to have

$$A(y)C_k \subset \{x : 2^{l+k-1} < |x| < 2^{k+l+m+1}\}.$$

Hence,

$$\|M_\beta f(\cdot)\|_{L^{q_2}(A(y)C_k)} \leq \sum_{j=l}^{l+m+1} \|M_\beta f(\cdot)\|_{L^{q_2}(C_{k+j})}. \tag{3.6}$$

Incorporating the inequality (3.6) into (3.3), we obtain

$$\begin{aligned}
 & \|M(H_{\Phi,A}^b f)(\cdot)\|_{MK_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \\
 & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y) |\det A^{-1}(y)|^{1/q_2} \\
 & \quad \times \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{j=l}^{l+m+1} 2^{k\alpha} \|M_\beta f(\cdot)\|_{L^{q_2}(C_{k+j})} \right)^p \right\}^{1/p} dy \\
 & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y) |\det A^{-1}(y)|^{1/q_2} \\
 & \quad \times \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \sum_{j=l}^{l+m+1} 2^{-j\alpha} \left\{ \sum_{k=-\infty}^{k_0+j} 2^{k\alpha p} \|M_\beta f(\cdot)\|_{L^{q_2}(C_k)}^p \right\}^{1/p} dy \\
 & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_\beta f\|_{MK_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y) |\det A^{-1}(y)|^{1/q_2} \sum_{j=l}^{l+m+1} 2^{j(\lambda-\alpha)} dy. \tag{3.7}
 \end{aligned}$$

Now, for $\alpha = \lambda$, we have

$$\sum_{j=l}^{l+m+1} 2^{j(\lambda-\alpha)} = m + 2 \leq C(1 + \log_2(\|A(y)\| \|A^{-1}(y)\|)), \tag{3.8}$$

and otherwise

$$\begin{aligned}
 \sum_{j=l}^{l+m+1} 2^{j(\lambda-\alpha)} & = 2^{l(\lambda-\alpha)} \frac{1 - 2^{(l+m+1)(\lambda-\alpha)}}{1 - 2^{(\lambda-\alpha)}} \\
 & \leq C \begin{cases} \|A^{-1}(y)\|^{\alpha-\lambda}, & \alpha > \lambda, \\ \|A(y)\|^{\lambda-\alpha}, & \alpha < \lambda. \end{cases} \tag{3.9}
 \end{aligned}$$

Hence, (3.7), (3.8) and (3.9) together yield

$$\begin{aligned}
 & \|M(H_{\Phi,A}^b f)(\cdot)\|_{MK_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \\
 & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_\beta f\|_{MK_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y) |\det A^{-1}(y)|^{1/q_2} G_{\alpha,\lambda}(y) dy.
 \end{aligned}$$

Making use of the fact that $|H_{\Phi,A}^b f(x)| \leq M(H_{\Phi,A}^b f)(x)$ a.e. and Lemma 2.10 we get the desired result.

Proof of Theorem 3.4: Following the same arguments as given in the proof of Theorem 3.3, we get

$$\begin{aligned}
 & \|M(H_{\Phi,A}^b f)(\cdot)\|_{K_{q_2}^{\alpha,p_2}(\mathbb{R}^n)} \\
 & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_\beta f\|_{K_{q_2}^{\alpha,p_2}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y) |\det A^{-1}(y)|^{1/q_2} \tilde{G}_\alpha(y) dy.
 \end{aligned}$$

However, in contrast with Theorem 3.3, here we use Lemma 2.9 to fulfill the assertion made in the statement of this Theorem.

Proof of Theorem 3.5: For $x \in Q$, it is easy to see that

$$\begin{aligned} & \frac{1}{|Q|^{1+\beta/n}} \int_Q |H_{\Phi,A}^b f(z) - (H_{\Phi,A}^b)_Q| dz \\ & \leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |H_{\Phi,A}^b f(z)| dz \\ & \leq \frac{1}{|Q|^{1+\beta/n}} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \int_Q |(b(z) - b_Q)f(A(y)z)| dz dy \\ & + \frac{1}{|Q|^{1+\beta/n}} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \int_Q |(b_Q - b_{A(y)Q})f(A(y)z)| dz dy \\ & + \frac{1}{|Q|^{1+\beta/n}} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \int_Q |(b(A(y)z) - b_{A(y)Q})f(A(y)z)| dz dy \\ & =: J_1 + J_2 + J_3. \end{aligned} \tag{3.10}$$

Comparing J_1, J_2 and J_3 with I_1, I_2 and I_3 estimated in the proof of Lemma 2.7, one can easily estimate J_1, J_2 and J_3 by adjusting the factor $|Q|^{\beta/n}$. Hence, we have

$$\begin{aligned} J_1 & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} M(f)(A(y)x) dy. \\ J_2 & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} (1 + \|A(y)\|^\beta) M(f)(A(y)x) dy. \\ J_3 & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A(y)|^{1+\beta/n} M(f)(A(y)x) dy. \end{aligned}$$

In view of these estimates, inequality (3.10) assumes the following form

$$\begin{aligned} & \frac{1}{|Q|^{1+\beta/n}} \int_Q |H_{\Phi,A}^b f(z) - (H_{\Phi,A}^b)_Q| dz \\ & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} (1 + \|A(y)\|^\beta) \max\{1, |\det A(y)|^{1+\beta/n}\} M(f)(A(y)x) dy. \end{aligned}$$

Applying $L^p(\mathbb{R}^n)$ norm on both sides and using Lemma 2.11, we obtain

$$\begin{aligned} \|H_{\Phi,A}^b f\|_{\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)} & \leq C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \\ & \times \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} (1 + \|A(y)\|^\beta) \max\{1, |\det A(y)|^{1+\beta/n}\} \|Mf(A(y)\cdot)\|_{L^p(\mathbb{R}^n)} dy. \end{aligned}$$

Finally by scaling argument and boundedness of M on $L^p(\mathbb{R}^n)$, (see [12]) we have

$$\|H_{\Phi,A}^b f\|_{\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)} \leq CK_5 \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)},$$

which is as required.

4. Central-BMO Estimates for $H_{\Phi,A}^b$ on Herz-type Space

4.1. Main Results

THEOREM 4.1. *Let $1 < p, q_1, q_2 < \infty, 1/q_2 = 1/q + 1/q_1, \lambda > 0$ and $\alpha_2 \in \mathbb{R}$. If $\alpha_1 = n/q + \alpha_2$ and $b \in \dot{CMO}^q(\mathbb{R}^n)$, then*

$$\|H_{\Phi,A}^b f\|_{\dot{MK}_{p,q_2}^{\alpha_2,\lambda}(\mathbb{R}^n)} \leq CK_6 \|b\|_{\dot{CMO}^q(\mathbb{R}^n)} \|f\|_{\dot{MK}_{p,q_1}^{\alpha_1,\lambda}(\mathbb{R}^n)},$$

where

$$K_6 = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} G_{\alpha_1,\lambda}(y) \times \left(\log \frac{1}{\|A(y)\|} \chi_{\{\|A(y)\| < 1\}} + \log \|A(y)\| \chi_{\{\|A(y)\| \geq 1\}} + \frac{\|A(y)\|^n}{|\det A(y)|} \right) dy,$$

and $G_{\alpha_1,\lambda}(y)$ is the same function as given in (3.1) with α is replaced by α_1 .

THEOREM 4.2. *Let $1 < p, q_1, q_2 < \infty, 1/q_2 = 1/q + 1/q_1$ and $\alpha_2 \in \mathbb{R}$. If $\alpha_1 = n/q + \alpha_2$ and $b \in \dot{CMO}^q(\mathbb{R}^n)$, then*

$$\|H_{\Phi,A}^b f\|_{\dot{K}_{q_2}^{\alpha_2,p}(\mathbb{R}^n)} \leq CK_7 \|b\|_{\dot{CMO}^q(\mathbb{R}^n)} \|f\|_{\dot{K}_{q_1}^{\alpha_1,p}(\mathbb{R}^n)},$$

where

$$K_7 = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \tilde{G}_{\alpha_1}(y) \times \left(\log \frac{1}{\|A(y)\|} \chi_{\{\|A(y)\| < 1\}} + \log \|A(y)\| \chi_{\{\|A(y)\| \geq 1\}} + \frac{\|A(y)\|^n}{|\det A(y)|} \right) dy,$$

and $\tilde{G}_{\alpha_1}(y)$ is the same function as given in (3.2) with α is replaced by α_1 .

4.2. Proof of Theorem 4.1.

Here, we decompose $\|H_{\Phi, A}^b f\|_{L^{q_2}(C_k)}$ as:

$$\begin{aligned} \|H_{\Phi, A}^b f\|_{L^{q_2}(C_k)} &= \left(\int_{C_k} \left| \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} (b(x) - b(A(y)x)) f(A(y)x) dy \right|^{q_2} dx \right)^{1/q_2} \\ &\leq \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\int_{C_k} |(b(x) - b(A(y)x)) f(A(y)x)|^{q_2} dx \right)^{1/q_2} dy \\ &\leq \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\int_{C_k} |(b(x) - b_{B_k}) f(A(y)x)|^{q_2} dx \right)^{1/q_2} dy \\ &\quad + \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\int_{C_k} |(b_{B_k} - b_{A(y)B_k}) f(A(y)x)|^{q_2} dx \right)^{1/q_2} dy \\ &\quad + \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\int_{C_k} |(b(A(y)x) - b_{A(y)B_k}) f(A(y)x)|^{q_2} dx \right)^{1/q_2} dy \\ &= L_1 + L_2 + L_3, \end{aligned}$$

By Hölder inequality and change of variables it is simple to have

$$\begin{aligned} L_1 &\leq \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\int_{C_k} |b(x) - b_{B_k}|^q dx \right)^{1/q} \left(\int_{C_k} |f(A(y)x)|^{q_1} dx \right)^{1/q_1} dy \\ &\leq |B_k|^{1/q} \|b\|_{CMO^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} dy. \end{aligned}$$

In order to estimate L_2 , we rewrite it as

$$\begin{aligned} L_2 &= \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_2} \|f\|_{L^{q_2}(A(y)C_k)} |b_{B_k} - b_{A(y)B_k}| dy \\ &\leq \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_2} |A(y)B_k|^{1/q} \|f\|_{L^{q_1}(A(y)C_k)} |b_{B_k} - b_{A(y)B_k}| dy \\ &= |B_k|^{1/q} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} |b_{B_k} - b_{A(y)B_k}| dy \\ &= |B_k|^{1/q} \int_{\|A(y)\| < 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} |b_{B_k} - b_{A(y)B_k}| dy \\ &\quad + |B_k|^{1/q} \int_{\|A(y)\| \geq 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} |b_{B_k} - b_{A(y)B_k}| dy \\ &=: |B_k|^{1/q} (L_{21} + L_{22}). \end{aligned} \tag{4.1}$$

Thus, for $\|A(y)\| < 1$, we have

$$\begin{aligned} L_{21} &= \sum_{j=0}^{\infty} \int_{2^{-j-1} \leq \|A(y)\| < 2^{-j}} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} \\ &\quad \times \left\{ \sum_{i=1}^j |b_{2^{-i}B_k} - b_{2^{-i+1}B_k}| + |b_{2^{-j}B_k} - b_{A(y)B_k}| \right\} dy. \end{aligned}$$

A use of Hölder inequality yields

$$\begin{aligned} |b_{2^{-i}B_k} - b_{2^{-i+1}B_k}| &\leq \frac{1}{|2^{-i}B_k|} \int_{2^{-i}B_k} |b(y) - b_{2^{-i+1}B_k}| dy \\ &\leq \frac{|2^{-j}B_k|}{|2^{-i}B_k|} \|b\|_{CMO^q(\mathbb{R}^n)} \\ &\leq C \|b\|_{CMO^q(\mathbb{R}^n)}. \end{aligned}$$

Similarly,

$$\begin{aligned} |b_{2^{-j}B_k} - b_{A(y)B_k}| &\leq \frac{1}{|A(y)B_k|} \int_{A(y)B_k} |b(y) - b_{2^{-j}B_k}| dy \\ &\leq \frac{2^{-jn}}{|\det A(y)| |2^{-j}B_k|} \int_{2^{-j}B_k} |b(y) - b_{2^{-j}B_k}| dy \\ &\leq C \frac{\|A(y)\|^n}{|\det A(y)|} \|b\|_{CMO^q(\mathbb{R}^n)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} L_{21} &\leq C \|b\|_{CMO^q(\mathbb{R}^n)} \\ &\times \sum_{j=0}^{\infty} \int_{2^{-j-1} \leq \|A(y)\| < 2^{-j}} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} \left\{ j + \frac{\|A(y)\|^n}{|\det A(y)|} \right\} dy \\ &\leq C \|b\|_{CMO^q(\mathbb{R}^n)} \\ &\times \int_{\|A(y)\| < 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} \left(\log \frac{1}{\|A(y)\|} + \frac{\|A(y)\|^n}{|\det A(y)|} \right) dy. \end{aligned}$$

Similar arguments result in the following estimation of L_{22} .

$$\begin{aligned} L_{22} &\leq C \|b\|_{CMO^q(\mathbb{R}^n)} \\ &\times \int_{\|A(y)\| \geq 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} \left(\log \|A(y)\| + \frac{\|A(y)\|^n}{|\det A(y)|} \right) dy. \end{aligned}$$

Having these estimates of L_{21} and L_{22} , (4.1) assumes the following form

$$\begin{aligned} L_2 &\leq C |B_k|^{1/q} \|b\|_{CMO^q(\mathbb{R}^n)} \\ &\times \left[\int_{\|A(y)\| < 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} \left(\log \frac{1}{\|A(y)\|} + \frac{\|A(y)\|^n}{|\det A(y)|} \right) dy \right. \\ &\left. + \int_{\|A(y)\| \geq 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} \left(\log \|A(y)\| + \frac{\|A(y)\|^n}{|\det A(y)|} \right) dy \right]. \end{aligned}$$

It remains to estimate L_3 . For this purpose we take advantage of Hölder inequality

to obtain

$$\begin{aligned} & \| (b(A(y)\cdot) - b_{A(y)B_k})f(A(y)\cdot) \|_{L^{q_2}(C_k)} \\ &= \left(\int_{C_k} | (b(A(y)x) - b_{A(y)B_k})f(A(y)x) |^{q_2} dx \right)^{1/q_2} \\ &= |\det A^{-1}(y)|^{1/q_2} \left(\int_{A(y)C_k} | (b(x) - b_{A(y)B_k})f(x) |^{q_2} dx \right)^{1/q_2} \\ &\leq |\det A^{-1}(y)|^{1/q_2} \| b - b_{A(y)B_k} \|_{L^q(A(y)C_k)} \| f \|_{L^{q_1}(A(y)C_k)} \\ &\leq |\det A^{-1}(y)|^{1/q_2} |A(y)B_k|^{1/q} \| b \|_{CMO^q(\mathbb{R}^n)} \| f \|_{L^{q_1}(A(y)C_k)} \\ &= |B_k|^{1/q} |\det A^{-1}(y)|^{1/q_1} \| b \|_{CMO^q(\mathbb{R}^n)} \| f \|_{L^{q_1}(A(y)C_k)}. \end{aligned}$$

Hence,

$$L_3 \leq |B_k|^{1/q} \| b \|_{CMO^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \| f \|_{L^{q_1}(A(y)C_k)} dy.$$

Combining the estimates for L_1 , L_2 , and L_3 we obtain

$$\begin{aligned} \| H_{\Phi,A}^b f \|_{L^{q_2}(C_k)} &\leq C |B_k|^{1/q} \| b \|_{CMO^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \| f \|_{L^{q_1}(A(y)C_k)} \\ &\quad \left(\log \frac{1}{|A(y)|} \chi_{\{\|A(y)\| < 1\}} + \log \|A(y)\| \chi_{\{\|A(y)\| \geq 1\}} + \frac{\|A(y)\|^n}{|\det A(y)|} \right) dy. \end{aligned}$$

Again, to make our calculation convenient, we use the following notation

$$\begin{aligned} \varphi(y) &= \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \\ &\quad \times \left(\log \frac{1}{|A(y)|} \chi_{\{\|A(y)\| < 1\}} + \log \|A(y)\| \chi_{\{\|A(y)\| \geq 1\}} + \frac{\|A(y)\|^n}{|\det A(y)|} \right). \end{aligned}$$

Thus, we rewrite above inequality as

$$\| H_{\Phi,A}^b f \|_{L^{q_2}(C_k)} \leq C |B_k|^{1/q} \| b \|_{CMO^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} \varphi(y) \| f \|_{L^{q_1}(A(y)C_k)} dy. \tag{4.2}$$

Still we have to approximate $\| f \|_{L^{q_1}(A(y)C_k)}$. For this end we infer from (3.6) that

$$\| f \|_{L^{q_1}(A(y)C_k)} \leq \sum_{j=l}^{l+m+1} \| f \|_{L^{q_1}(C_{k+j})}.$$

By this inequality, (4.2) becomes

$$\| H_{\Phi,A}^b f \|_{L^{q_2}(C_k)} \leq C |B_k|^{1/q} \| b \|_{CMO^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} \varphi(y) \sum_{j=l}^{l+m+1} \| f \|_{L^{q_1}(C_{k+j})} dy. \tag{4.3}$$

Next by definition of Morrey-Herz space $M\dot{K}_{p,q_2}^{\alpha_2,\lambda}(\mathbb{R}^n)$, (4.3), Minkowski inequality and the condition $\alpha_1 = \alpha_2 + n/q$, we get

$$\begin{aligned} & \|H_{\Phi,A}^b f\|_{M\dot{K}_{p,q_2}^{\alpha_2,\lambda}(\mathbb{R}^n)} \\ &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p} \|H_{\Phi,A}^b f\|_{L^{q_2}(C_k)}^p \right\}^{1/p} \\ &\leq C \|b\|_{CMO^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} \varphi(y) \\ &\quad \times \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{j=l}^{l+m+1} 2^{k(\alpha_2+n/q)} \|f\|_{L^{q_1}(C_{k+j})} \right)^p \right\}^{1/p} dy \\ &\leq C \|b\|_{CMO^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} \varphi(y) \\ &\quad \times \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \sum_{j=l}^{l+m+1} 2^{-j\alpha_1} \left\{ \sum_{k=-\infty}^{k_0+j} 2^{k\alpha_1 p} \|f\|_{L^{q_1}(C_k)}^p \right\}^{1/p} dy \\ &\leq C \|b\|_{CMO^q(\mathbb{R}^n)} \|f\|_{M\dot{K}_{p,q_2}^{\alpha_1,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \varphi(y) \sum_{j=l}^{l+m+1} 2^{j(\lambda-\alpha_1)} dy \\ &\leq C \|b\|_{CMO^q(\mathbb{R}^n)} \|f\|_{M\dot{K}_{p,q_2}^{\alpha_1,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \varphi(y) G_{\alpha_1,\lambda}(y) dy, \end{aligned}$$

where $G_{\alpha_1,\lambda}(y)$ is the same function as given in (3.1) with α is replaced by α_1 . This finishes the proof of Theorem 4.1.

4.3. Proof of Theorem 4.2.

Since, the the proof of Theorem 4.1 and 4.2 are symmetrical. So, by definition of Herz space $\dot{K}_{q_2}^{\alpha_2,p}(\mathbb{R}^n)$, (4.3), Minkowski inequality and the condition $\alpha_1 = \alpha_2 + n/q$, we get

$$\|H_{\Phi,A}^b f\|_{\dot{K}_{q_2}^{\alpha_2,p}(\mathbb{R}^n)} \leq C \|b\|_{CMO^q(\mathbb{R}^n)} \|f\|_{\dot{K}_{q_1}^{\alpha_1,p}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \varphi(y) \tilde{G}_{\alpha_1}(y) dy,$$

where $\tilde{G}_{\alpha_1}(y)$ is the same function as given in (3.2) with α is replaced by α_1 . This completes the proof of Theorem 4.2.

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